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A Study of the Pressure Term in the Navier-Stokes Equations

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Dedication

to my family and friends

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Abstract

In this paper we consider the Cauchy problem for the 3D Navier-Stokes equations for incompressible flows, and their solutions. We will discuss the results of a paper by Otto Kreiss and Jens Lorenz on the role of the pressure term in the Navier-Stokes equations, and its relationship to the fluid field $u(x,t)$. The focus here is to concentrate on solutions to the equation where the fluid field u lies in the space $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and not necessarily in $L^2(\mathbb{R}^3)$. If $u(x,0) = f(x)$, where $f \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ we will consider the solutions for all t in time interval $0 \le t < T(f)$. In the original paper, estimates for the *derivatives* of the pressure were proved, but the definition of the pressure proved unsatisfactory due to the possibility of the divergence of the pressure term. The main object of this paper is to use the theory of singular integrals and the space of functions of Bounded Mean Oscillation to properly address the pressure. In doing so, we will provide estimates on pressure term itself. This will allow us to strengthen the results of the original paper, and rigorously extend all results from the original paper to *any* function $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

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Glossary

ln	natural logarithm
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
x	point in $\mathbb{R}^n$: $x \in \mathbb{R}^n$ is given by $x = (x_1, x_2, \ldots, x_n)$
$\langle x, y \rangle$	inner product: $\langle x, y \rangle = x \cdot y = \sum_{i=1}^n x_i y_i$
$\|x\|$	Euclidean norm: $\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$
$B(x, r)$	the open ball centered at x of radius r: $\{y \in \mathbb{R}^n : \|x - y\| < r\}$
$S(x, r)$	sphere centered at x of radius r: $\{y \in \mathbb{R}^n : \|x - y\| = r\}$
$\overline{B(x, r)}$	the closed ball centered at x of radius r: $\{y \in \mathbb{R}^n : \|x - y\| \leq r\}$ $(B(x, r) \cup S(x, r))$
$\|x\|_\infty$	max norm: $\|x\|_\infty = \max \|x_1\|, \|x_2\|, \ldots, \|x_n\|$
$\|f\|_p$	L^p norm: $\|f\|_p = \|f\|_{L^p} = \left(\left(\int \|f(x)\|^p \, dx \right)^{\frac{1}{p}} \right)$, where f is a measurable function.
$\|f\|_\infty$	L^∞ norm: $L^\infty = \inf\{C > 0 : \|f(x)\| \leq C \quad \text{a.e.}\}$
a.e	almost everywhere

Glossary

$C(\mathbb{R}^n)$	set of continuous functions defined on $\mathbb{R}^n$, n can be $1, 2, \ldots$, etc.
$C_0(A)$	set of continuous functions on a set A with compact support.
$C^k(\mathbb{R}^n)$	set of continuous functions with continuous derivatives up to order k on $\mathbb{R}^n$
$C^\infty(\mathbb{R}^n)$	set of continuous functions with continuous derivatives of all orders on $\mathbb{R}^n$
L^p	space of L^p functions: $L^p = \{f(x) : \|f\|_p < \infty\}$, where f is a measurable function.
L^∞	space of L^∞ functions: $L^\infty = \{f(x) : \|f\|_\infty < \infty\}$ where f is a measurable function.
L^1_{loc}	space of locally integrable functions: $f \in L^1_{\text{loc}}$ if $\int_K \|f\| < \infty$ for all compact sets K, and f is a measurable function.
$\mathbb{S}^2$	unit sphere in $\mathbb{R}^3$: $\{x \in \mathbb{R}^3 : \|x\| = 1\}$
∂_t	partial derivative with respect to t
$\|\alpha\|$	$\alpha_1 + \alpha_2 + \alpha_3$ for a multi–index$\alpha = (\alpha_1, \alpha_2, \alpha_3)$
D^α	$D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}, \quad D_j = \partial/\partial x_j$
NS	Navier-Stokes
BMO	the space of functions of Bounded Mean Oscillation
f_Q	the mean value of a function on a cube (or ball) Q

Chapter 1

Introduction

1.1 Motivation: A Brief History of the Navier-Stokes Equations

The Navier-Stokes equations are a set of partial differential equations that describe fluid motion. In this paper we consider the Cauchy problem for the 3D Navier-Stokes equations for incompressible flows. We begin by letting $x \in \mathbb{R}^3$ be the space variable and $t \in \mathbb{R}$, where $t \geq 0$ be the time variable. We consider the fluid velocity field of a fluid with notation $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ for the fluid velocity field, and $p(x,t)$ for the scalar pressure field. Then the *(Incompressible) Navier-Stokes equations* are given by

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \quad \textit{Change in Momentum} \tag{1.1}$$

$$\nabla \cdot u = 0, \quad \textit{Incompressibility/Divergence Free Condition} \tag{1.2}$$

where the viscosity constant is ν, and

$$u \cdot \nabla = u_1 D_1 + u_2 D_2 + u_3 D_3 \quad D_i = \frac{\partial}{\partial x_i}$$

Component by component the equations read:

$$u_{it} + \sum_{j=1}^{3} u_j D_j u_i + D_i p = \nu \Delta u_i \quad i = 1, 2, 3$$

and

$$D_1 u_1 + D_2 u_2 + D_3 u_3 = \sum_{i=1}^{3} D_i(u_i) = 0$$

Furthermore we assume that

$$u_0(x) = u(x,0) = f(x) \quad \text{with} \quad \nabla \cdot f = 0 \tag{1.3}$$

The terms of the equation are:

u_t : unsteady acceleration term

$(u \cdot \nabla)u$: the convective acceleration

∇p : the pressure gradient

$\nu \Delta u$: the viscosity term

Throughout this paper when referring to Navier-Stokes equations, we will normalize the constant ν so that $\nu = 1$. The incompressibility condition, $\nabla \cdot u = 0$, can be shown to be a limiting value of the compressible case ([30], [26], [27]). The derivation of these equations from first principles can be found in various texts on fluid mechanics and other sources. See for example [6], [29], or [31]. The history of the equations is quite extensive and is well documented ([20]). Starting with Leonard Euler in 1755 with his introduction of inviscid fluid flow, Charles Lois Maries Henri Navier in 1822 ([32]) continued refining the equations with his derivation from a molecular argument, where he introduced viscous effects. Further refinement of the equations followed from Poisson in 1829 with the equations of a compressible fluid.

Saint-Venant in 1843 ([36] and [37]) and Stokes in 1845 ([43], [45]) obtained the continuum derivation. This version is the one that is commonly introduced to students and under intense study today.

Equations that describe physical phenomena beg for solution, and the Navier-Stokes equations are no exception. There is a vast wealth of literature devoted to the solutions of Navier-Stokes (e.g. [31], [21], [22], [23], [12], [13], [14], [15]). For example, in 1934, Leray ([28]) constructed a global (in time) weak solution, and a local strong solution in $\mathbb{R}^3$. This paper, among others, reinforced the idea of specialized solutions. Among the different classes of solutions studied for the Navier-Stokes one can find classical, strong, mild, weak, very weak, uniform weak, and local Leray solutions. These different classes have themselves produced a variety of methods designed to explore the various types of solutions. Fourier analysis, statistical mechanics, distribution theory, and harmonic analysis have all played a part in attempting to analyze the equations for over two centuries. A large portion of this work focuses on the well-known fact (Appendix C, [31], and [26], for example) that at its core, the Navier-Stokes equation (1.1) is basically a non-linear heat equation. Thus, it can be written using Duhamel's principle in an integral form with heavy dependence on the initial data $u(x, 0) = f$.

Exploiting the integral form of the Navier-Stokes equations has been used to explore other aspects of the solutions, such as existence, uniqueness, and the dependence on initial data. For example, it is known that for initial data $u(x, 0) = f \in L^\infty(\mathbb{R}^n)$ the equations (1.1), (1.2), (1.3) admit a local in-time (regular) solution u with the pressure p determined by

$$p = \sum_{i,j=1}^{n} R_i R_j (u_i u_j) \tag{1.4}$$

where R_i is the Riesz transform ([4], [14], [24]). The Riesz transforms will be discussed in section 1.2. For the L^r case, where $3 \le r < \infty$, the equations (1.1), (1.2) admit a unique local in time solution u for some pressure p. As u decays at the space infinity, then (1.4) follows *a posteriori* for L^r ([21]). Kato ([22]) observed that for initial data $f \in L^\infty(\mathbb{R}^n)$, the constructed solution is bounded and may not decay at the space infinity. So even if u solves (1.1), (1.2), equation (1.4) may not follow. Kato further noted that in the most simple case, for $x \in \mathbb{R}^3$ and $t \in (0, \infty)$, one could construct a solution of the form $u(x,t) = g(t)$, $p(x,t) = -g'(t) \cdot x$. This function pair (u, p) solves (1.1) and (1.2) no matter what the function $g(t)$ is. So if u has constant initial data, the solution is not unique without *assuming* (1.4). Not only does this demonstrate a non-uniqueness to the solution, it also implies a non-decaying pressure spatially. Kato further observed that one would need to impose some control on p to obtain uniqueness other that controlling u.

In [13] it was noted that uniqueness holds if u is bounded, and p is of the form

$$p(x,t) = \pi_0 + \sum_{i,j=1}^{n} R_i R_j \pi_{ij} \tag{1.5}$$

for bounded functions π_0, π_{ij}. In particular it was noted that for $t \in (0, T)$ for a maximal time T, then uniformly

$$\pi_0, \pi_{ij} \in L^\infty \cap L^1_{loc}$$

In the same paper Kato improved upon the result by simply assuming that $p \in L^1_{loc} \cap \mathrm{BMO}$, where BMO is the space of functions of Bounded Mean Oscillation . A theorem of A. Uchiyama ([46]) indicated that if a function g was BMO , then it was of the form

$$g = \eta_0 + \sum_{i,j=1}^{n} R_i R_j \eta_{ij}$$

with some $\eta_{ij}, \eta_0 \in L^\infty(\mathbb{R}^n)$. Additionally, Sadosky ([35]), and the paper by Fefferman and Stein [9] observed that *every* $g \in \mathrm{BMO}$ could be written as

$$g = \tau_0 + \sum_{j=1}^{n} R_j \tau_j$$

where $\tau_j \in L^\infty(\mathbb{R}^n)$. In Chapter 2 and [35] it is noted that if $g \in L^\infty$, the convolution of a Calderón -Zygmund kernel $K(x)$ with g produces a function that is BMO . In Chapter 3 we discuss the fact that the (formal) pressure term (and its modification) is the solution of the Poisson pressure equation; it is the convolution of the term $D_i u_j D_j u_i$ with a Calderón -Zygmund kernel. It is of some interest to note the fact that if the velocity field has derivatives that are sufficiently smooth and small at infinity, it then turns out that the pressure is additionally a Riesz *potential* (see [40], or [10]).

Kato's paper was concerned with weak (distribution) solutions. The main result of that paper was that given condition (1.4), if (u, p) was a solution of (1.1), (1.2) with initial data in the distribution sense satisfying $u(x, 0) \in L^\infty(\mathbb{R}^3)$, and $p(x, 0) \in$ BMO , then the solution $(u, \nabla p)$ was unique, and

$$\nabla p = \sum_{i,j=1}^{n} \nabla R_i R_j (u_i u_j)$$

where $\nabla p \in \mathcal{S}'$ (see [22]). We should recall here that $\mathcal{S}$ is the *Schwartz space* and $\mathcal{S}'$ is its dual; that is, the set of all bounded linear functionals on $\mathcal{S}$ (see [39], [40], or [29]). We should further remark here that $\mathcal{S}$ and $\mathcal{S}'$ are important to Fourier and Harmonic Analysis, although for this paper our assumptions on u will preclude any involvement with these particular spaces.

The papers by Giga, et.al., specifically [13] and [14], took a different approach, and chose initial data on the space of bounded uniformly continuous functions (BUC) in $\mathbb{R}^n$, or in $L^\infty(\mathbb{R}^n)$. In the paper it was shown that if the initial value function $u(x, 0) = f(x)$ was BUC, then so was the unique solution u. In this case, however, the focus was on u as a solution to the integral (heat equation) version of the Navier-Stokes equations. In Giga's work, the set $C([0, T_0], \text{BUC})$ was defined to be the set of all bounded, uniformly continuous functions (spatially) that are defined and continuous on $[0, T_0]$, and the set C^α was representative of the set of Hölder continuous functions of order α. It was additionally shown in the same paper that

$$u \in C([\delta, T_0]; \mathrm{BUC}) \quad \text{and} \quad t^{1/2}\nabla u \in C([0, T_0]; \mathrm{BUC}) \tag{1.6}$$

and

$$\nabla u \in C^{\alpha}([\delta, T_0]; \mathrm{BUC})$$

for some α with $0 < \alpha < 1/2$ and δ such that $0 < \delta < T_0$. This showed that $t^{1/2}\nabla u$ was bounded in *some* sense. An additional result was that if $u(x, 0) = f(x)$ was a BUC function, with u satisfying the integral (heat) form of the solution to the Navier-Stokes equations, and if $\nabla u \in C^{\alpha}([\delta, T_0]; \mathrm{BUC})$, then by writing

$$p = \sum_{i,j=1}^{n} R_i R_j u_i u_j$$

we have (u, p) solving (1.1), (1.2). However, again in [13], it was noted that if one replaced the space BUC with $L^{\infty}(\mathbb{R}^n)$, the results were different. Equation (1.6) is replaced by

$$u \in C_w([\delta, T_0]; L^{\infty}) \quad \text{and} \quad t^{1/2}\nabla u \in C_w([0, T_0]; L^{\infty})$$

where C_w is the space of all L^{∞} valued weakly continuous functions defined on $[0, T_0]$.

These results, among others, are similar to the results in the work by Kreiss–Lorenz ([25]). The Kreiss–Lorenz paper is the main focus and source for this doctoral thesis. The main difference between the Kreiss–Lorenz paper and the works of Kato and Giga's is the restriction on the solutions. The KL paper concentrates on classical solutions in $C^{\infty}(\mathbb{R}^3) \cap L^{\infty}(\mathbb{R}^3)$, while Kato and Giga's ([13] and [21]) developed solutions that existed in the distribution sense (e.g. weakly continuous), with some sort of control on the pressure term p. Additionally, Kato and Giga's papers assumed their space of contention to be $\mathbb{R}^n$, for $n \geq 2$. Their results obviously can be restricted to $\mathbb{R}^3$, where the Kreiss–Lorenz results are only particular to $\mathbb{R}^3$. It must be noted that the Kreiss–Lorenz paper was able to construct similar results, but the only structure mentioned on the pressure term was that it was simply of Bounded Mean Oscillation -no breakdown of the pressure term into Riesz transforms is required or even alluded to. In Appendix B.3 and in [14] it is noted that by taking the divergence of the equation one obtains a Poisson equation of the form

$$-\Delta p = \sum_{i,j=1}^{3} D_i u_j D_j u_i$$

We may formally solve and obtain the Poisson pressure equation

$$p(x,t) = \sum_{i,j=1}^{3} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} (D_i u_j D_j u_i)(y,t)\, dy$$

The solution to the Poisson equation has an (integrable) singularity at $x = y$, and can (initially) be proven to have a solution if $(D_i u_j D_j u_i)(x,t)$ is a function of compact support. The fact of most interest here about the Poisson equation is that it is connected deeply with singular integral theory-specifically to the work of Calderón, Zygmund, Stein, Fefferman, Sadosky and others.

1.2 Singular Integral Theory

The history of singular integral theory is an interesting story in its own right, and a concise and excellent survey can be found in [41], which we will briefly reference here. The study begins with Antonio Zygmund. Zygmund's first work from the 1920's through the mid 1930's was in classical harmonic analysis. But it was during the 1930's that Zygmund considered a question. It was well known that the equation

$$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{1}{|I|} \int_I g(y)\, dy = g(x) \tag{1.7}$$

holds for almost all x, where I ranges over intervals and $g(x)$ is integrable on $\mathbb{R}$. This is due to the Lebesque differentiation theorem (see [11]). It was also well known at the time that a similar result could be applied to higher dimensions if the intervals were replaced by balls. The mathematician Stein, in his overview of the history of the roles of Calderón and Zygmund (see [41]), described the catalyst that led to further advances in the theory of singular integrals. Stein described how one counterexample could be found in the case where the sets I were rectangles with arbitrary orientation as found by Nikodym (see [41]). Additionally, he described a counterexample by Saks where the rectangles were of fixed orientation with sides parallel to the axis, if g was a general function in L^1.

It was at this point that his attention turned to a class of functions called *strong maximal functions.* Zygmund proved that for rectangles I with sides parallel to the axis, (1.7) held if g was L^p, with $p > 1$. That is:

$$\left(\int |g|^p\right)^{\frac{1}{p}} < \infty$$

He proved this by proving an inequality for a "strong" maximal function. To define such a maximal function, let $\delta > 0$ and let μ be a measure on $\mathbb{R}^n$, with f a locally integrable function. Then the mean value is given by:

$$(A_\delta g)(x) = \frac{1}{\mu(B(x,\delta))} \int_{B(x,\delta)} g(y)\, d\mu(y)$$

Then the maximal function is defined by

$$(Mf)(x) = \sup_{\delta>0} A_\delta(|g|)(x)$$

These were constructed by Hardy and Littlewood for $n = 1$, and for general n by Wiener (see [39]). It should be noted here that these maximal functions bear more than a passing resemblance to functions of Bounded Mean Oscillation (BMO), which will be discussed in the next section. Zygmund was influenced by the one dimensional Hilbert transform

$$H(f)(x) = P.V. \quad \frac{1}{\pi} \int_{-\infty}^{\infty} g(x-y)\, \frac{dy}{y}$$

where "PV" refers to the *Cauchy Principal Value.* The L^p boundedness properties were later proved by Riesz (see [41]). At this time Alberto Calderón appeared.

In the late 1940's Zygmund met Alberto Calderón. Calderón, became Zygmund's doctoral student, and later, collaborator. Calderón spurred Zygmund's study of higher dimensional singular integrals, which were higher dimensional versions of the Hilbert transform. The main candidate of study was the functional

$$T(g)(x) = \int_{\mathbb{R}^n} K(y)g(x-y)\,dy$$

where K was homogeneous of degree n, and satisfied the cancelation condition

$$\int_{\mathbb{S}^{n-1}} K(y)d\sigma(y) = 0$$

with $\sigma(y)$ as the surface measure of the unit sphere $\mathbb{S}^{n-1}$ (see [39],[40], or [35]). These are referred to as *Calderón-Zygmund*(CZ) kernels. We note here that $K(y)$ has an isolated singularity at the origin (or, for $K(x-y)$, a singularity at $x = y$). Of additional study were Calderón -Zygmund kernels that had a regularity condition, namely Dini-continuity (see Definition 2.2.9). As written this integral will fail to exist, so it is actually written as a *Cauchy Principal Value* integral:

$$\begin{aligned} T(g)(x) &\equiv \text{PV} \int_{\mathbb{R}^n} K(y)g(x-y)\,dy \\ &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} K(y)g(x-y)\,dy \\ &= \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} K(y)g(x-y)\,dy \end{aligned}$$

where g is a compactly supported function and K is otherwise nicely behaved. It was discovered that examples of the Calderón -Zygmund kernels include second derivatives of the fundamental solution operator for the Laplacian:

$$\frac{\partial^2}{\partial x_i \partial x_j}(\Delta)^{-1}$$

and the related Riesz kernel:

$$K_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$$

This is due to the well known fact that if g is of the class C^2 and has compact support

$$\frac{\partial^2}{\partial x_i \partial x_j}(g) = -R_i R_j \Delta g$$

where the R_i are the Riesz transforms discussed below (see [40]). It should be noted here that the Hilbert kernel is given by $\frac{1}{\pi x}$ with associated transform

$$H(g)(x) = P.V. \quad \frac{1}{\pi}\int_{-\infty}^{\infty} g(x-y)\,\frac{dy}{y}$$

The *Riesz kernel* is of the form

$$K_j(x) = \frac{\Omega_j(x)}{|x|^n} \quad \text{with} \quad \Omega_j(x) = \frac{x_j}{|x|}.$$

and the *Riesz transform* is given by

$$\begin{aligned} R_j(g)(x) &= \lim_{\varepsilon \to 0} c_n \int_{|y|>\varepsilon} \frac{\Omega_j(y)}{|y|^n} g(x-y)\,dy \\ &= \lim_{\varepsilon \to 0} c_n \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} g(x-y)\,dy \\ &= \text{PV} \quad c_n \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} g(x-y)\,dy \end{aligned}$$

for $j = 1, \ldots, n$, and

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}$$

This integral exists if $g \in L^p$ for $1 < p < \infty$ (see [39], [40], or [35]). This background work is illustrated in [2] and [3]. Work on singular integrals continued for over 30 years between the two men, but during this time another element of study connected to singular integral theory arose: the space of functions known as Bounded Mean Oscillation (BMO).

1.3 The Space of Functions of Bounded Mean Oscillation

The space of Bounded Mean Oscillation (BMO) is defined as follows. First let f be a locally integrable function, that is $f \in L^1_{loc}$. Define

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\ dx$$

This is the *mean value* of f over a cube (or ball) Q. Next we consider the integral

$$f_Q^\sharp = \frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx \tag{1.8}$$

The related *sharp maximal operator* is defined by

$$\Lambda^\sharp f(x) = \sup_{r>0} f^\sharp_{Q(x,r)}$$

where $Q(x,r)$ is cube of side r centered at x. Finally for a locally integrable function ($f \in L^1_{loc}$) the BMO norm is defined to be $\|f\|_{BMO} = \|\Lambda^\sharp f(x)\|_\infty$. Thus a function $f \in L^1_{loc}$ is a function of Bounded Mean Oscillation if $\|f\|_{BMO} < \infty$. This can be realized in the following way: there is an $M > 0$ such that

$$\frac{1}{|Q|} \int_Q |f(x) - f_Q|\, dx < M$$

for all cubes Q in $\mathbb{R}^n$

The space was first introduced by F. John in 1961 ([18]) in his study of mappings of a bounded set from $\mathbb{R}^n$ to $\mathbb{R}^n$ and their relationship to problems involving elastic strain. John and Nirenberg ([19]) introduced the basic notation and proved several properties. First is the fact that the space is a seminormed linear space that can be shown to be a complete normed linear space (i.e a Banach space) ([33]). It can be shown that if $\|f\|_{BMO} = 0$, then $f = C$ a.e. (almost everywhere), a constant. Additionally, it is the dual space of the Hardy space H^1. This is explored in the work by Stein ([39]), and first proved by Fefferman (see [8] and [9]). The Hardy spaces H^p are certain spaces of distributions that are related to the L^p spaces, and, in some special cases, are "better behaved". In [42], the connection between BMO and singular integrals was determined, and further reflected in the work by [35]. In particular, CZ kernels map the space $L^\infty(\mathbb{R}^n)$ to the space BMO. Thus BMO can be a "replacement" for L^∞. This reveals the main point of interest. Many operators that are unbounded or otherwise badly behaved on L^1 and L^∞ are bounded in H^1 and BMO respectively (see [39], [40], [35], or [5]). It should be noted here that both Stein and Sadosky were students of Zygmund, and Zygmund's influence upon them is very evident in their scholarly works. See, for example, [39], [40], [41], [42], or [35].

It is here we now tie together the space BMO and singular integral theory. Consider the Calderón-Zygmund operators Tf defined by

$$Tf(x) = PV \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$$

where PV refers to the *Cauchy Principal Value.* Suppose further that $K(y)$ is a Calderón-Zygmund kernel with the requisite regularity condition of Dini-continuity. Then the functional Tf maps L^∞ continuously to BMO (see Chapter 2 and [35]). It is observed in the papers [13] and [14], as well as the works by Stein ([39] and [40]), and Sadosky ([35]) that the the Riesz transforms will map an L^∞ to the space of BMO , precisely because they are such Calderón -Zygmund kernels.

In particular, consider the pressure term ∇p in the Navier-Stokes equation. By taking the divergence of the Navier-Stokes equation, we find that:

$$-\Delta p(x,t) = \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i)(x,t) = \sum_{i,j=1}^{3} D_i D_j (u_i u_j)(x,t)$$

This derivation is well known, and constructed in Appendix B.3. The solution is given by (formally)

$$p(x,t) = \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} (D_i u_j)(D_j u_i)(y,t)\,dy \tag{1.9}$$

or

$$p(x,t) = \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j (u_i u_j)(y,t)\, dy \tag{1.10}$$

where

$$C_0 = \frac{1}{4\pi}$$

However, there is a well known identity (see [40]), that if $g \in L^p$, $1 < p < \infty$, we have

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} g = -R_i R_j \Delta g$$

where the R_j are the Riesz transforms. Again, the Riesz transforms R_j map L^∞ to BMO (see [35], [39], or [40] for example). Additionally, the Riesz transform mappings from BMO to BMO are bounded. Therefore, we obtain the fact that if $u \in L^\infty$, p is in the space BMO ([14]). While this integral may exist as a BMO function, it may fail to exist in the classical sense if there is not sufficient decay on u, which is a significant problem with such an integral. The real question at this point in the discussion is the existence of the integral: Does this integral converge?

First, if $u(x,t)$ has compact support, the pressure term integral will converge. Discussions relating to this assumption of compact support can be found in [29], [7] or [17]. However, suppose that $u(x,t)$ and $Du(x,t)$ exist in the space $L^\infty(\mathbb{R}^3)$, but without compact support? This was the assumption in the Kreiss–Lorenz paper, and the consequences actually obscure the conclusions of the work. For the current discussion, we assume that (u,p) is a solution to the Navier-Stokes equations, with $u(x,0) = f$, and t existing in a time interval $0 < t < T(f)$. We further assume that $u \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$, and that $Du \in L^\infty(\mathbb{R}^3)$. The procedure used in the paper was to use the *derivatives* of the pressure to bound u and its derivatives on a small time interval. This requires a pressure term that actually exists. We will now discuss how the Kreiss–Lorenz paper handled the issue, and the subsequent consequences it produced.

The original paper's procedure was to use a C^∞ cut-off function to break the pressure term into local and global pieces. For $\delta > 0$, the C^∞ function $\phi = \phi(|x - y|/\delta)$ was set up so that $\phi(r) = 1$ if $0 \leq r \leq 1$, and $\phi = 0$ if $r \geq 2$. Then the term $u_i u_j$ in the pressure term was written as

$$u_i u_j = (\phi u_i u_j) + [(1 - \phi) u_i u_j]$$

thus splitting the integral into "local" and "global" pieces (using (1.10)):

$$\begin{aligned} p(x,t) &= \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j (u_i u_j)(y,t)\, dy \\ &= \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j (\phi u_i u_j)(y,t)\, dy \\ &+ \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j [(1-\phi)(u_i u_j)(y,t)\, dy \\ &= p_{loc}(x,t) + p_{glb}(x,t) \end{aligned}$$

It is clear that $p_{loc}(x)$ depends only on values of $u(y)$ for $|x - y| < 2\delta$, while $p_{glb}(x)$ depends on values of $u(y)$ for $|x - y| > \delta$. Thus on the boundaries, the integrants, and thus the integrals themselves, vanish.

Using this split, $p_{loc}(x)$ was explored first. The singularity at $y = x$ is integrable, and using integration by parts, the boundary integral vanishes via ϕ. Thus, on $B(x, 2\delta)$, the requisite bounds were computed. Next, $Dp_{loc}(x)$ was considered, without integration by parts. The singularity at $y = x$ is still integrable, and again the requisite bounds were obtained. The problem comes from $p_{glb}(x)$. Since $1 - \phi$ depends of values of y for $|x - y| > \delta$, we have

$$p_{glb}(x,t) = \sum_{i,j=1}^{3} C_0 \int_{|x-y|>\delta} \frac{1}{|x-y|} D_i D_j [(1-\phi)(u_i u_j)(y,t)\, dy$$

Integration by parts was performed twice. In each case, the integral at the boundary $\partial B(x,\delta)$ vanishes because at the boundary, $\phi = 1$ so that $1-\phi = 0$. Thus the integral was rewritten as

$$\sum_{i,j=1}^{3} C_0 \int_{|x-y|>\delta} K_{ij}(x-y)(1-\phi)(u_i u_j)(y,t)\, dy \tag{1.11}$$

where

$$K_{ij}(x-y) = \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5} \quad i \neq j \tag{1.12}$$

and

$$K_{jj}(x-y) = \frac{3(x_j-y_j)^2 - |x-y|^2}{|x-y|^5} \tag{1.13}$$

Herein lies the problem. At this point in the paper, a spatial derivative was applied to $p_{glb}(x)$, but the objection is that the integral will fail to exist if u does not have compact support. Observe that by transferring two derivatives to the Poisson kernel we have, for $i \neq j$

$$|K_{ij}(x-y)| = \Big|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5}\Big| \leq \frac{1}{|x-y|^3}$$

Similarly, for $i = j$, we have

$$|K_{jj}(x-y)| = \Big|\frac{3(x_j-y_j)^2 - |x-y|^2}{|x-y|^5}\Big| \leq \frac{C}{|x-y|^3}$$

This implies that for a constant u, and all $1 \leq i,j \leq 3$:

$$p_{glb}(x,t) \equiv C \int_{\delta}^{\infty} \frac{1}{r}\, dr$$

by using a transformation to polar coordinates. The latter integral integrates as $\ln r$ so as $r \to \infty$, the integral, and subsequently $p_{glb}(x)$, diverges. This makes it impossible to take the derivative of $p_{glb}(x)$, since it fails to exist to begin with.

We finally describe the relation to the space BMO. It is shown in Chapter 2 that the K_{ij} are Calderón -Zygmund kernels with the required Dini-continuity. The kernel $K_{ij}(x-y)$ is not directly integrable at $x=y$. It will be shown in Chapter 3 that we may write local pressure as

$$p_{loc}(x) = \lim_{\varepsilon\to 0}\sum_{i,j=1}^{3} C_0 \int_{2\delta>|x-y|>\varepsilon} \frac{1}{|x-y|} D_i D_j(\phi(u_i u_j)(y,t))\, dy$$

We will then integrate $p_{loc}(x)$ by parts twice. There are no boundary integrals since ϕ vanishes at the boundary. We will then show that

$$p_{loc}(x) = \lim_{\varepsilon\to 0}\sum_{i,j=1}^{3} C_0 \int_{2\delta>|x-y|>\varepsilon} K_{ij}(x-y)\phi(u_i u_j)(y,t)\, dy \tag{1.14}$$

exists. We then will perform a similar analysis on $p_{glb}(x)$. We may then combine $p_{loc}(x)$ and $p_{glb}(x)$ as a principle value integral:

$$p(x,t) = \mathrm{PV} \quad \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\, dy$$

Thus if $u \in L^\infty(\mathbb{R}^3)$, $p(x,t)$ is in the space BMO.

To summarize the preceding discussion, it is a well known fact that for $u \in L^2$, the integral describing $p(x,t)$ exists. Alternatively, given a fluid field $u \in L^\infty$ lacking sufficient decay the integral $p(x,t)$ will diverge, and so the pressure term will fail to exist. We will look closely at this problem in Chapter 3, and determine a remedy for this situation. It is here that we now turn to the paper by Kreiss and Lorenz ([25]) and define the problem at hand.

1.4 The Kreiss–Lorenz Paper

Let us return to the main problem at hand. As before, our concern is the Navier-Stokes equations:

$$u_t + (u\cdot\nabla)u + \nabla p = \nu\Delta u, \quad (1.1)$$

and

$$\nabla \cdot u = 0, \quad (1.2)$$

The papers referenced so far focus on the fluid field u. The examples of solutions in various spaces (bounded uniformly continuous, weakly continuous, Hölder continuous) all provide solutions u while also assuming or requiring some control on the pressure p to produce a solution, in particular as the space variable x approaches infinity. Clearly the structure of the pressure term demands dependence on u or the derivatives of u. However, in fluid flow, it should be clear that the fluid velocity u and the pressure term p are deeply connected. A fluid velocity field will produce pressure on a surface, while the pressure itself must interact with the velocity field. This dance back and forth between velocity and pressure must drive every system of fluid under study. So while mathematically it is productive to consider more "exotic" spaces as the ones aforementioned, we should consider simplifying the situation a bit.

First, given "real life" constraints, most fluid systems must function inside some form of containment. The flow of fluid in a pipe, from a tank, down a river, have physical constraints that move a fluid along a certain path. Therefore, there must be a bound on the fluid velocity u, as well as the spatial derivatives $D^\alpha u$. So considering $u \in L^\infty$ and $D^\alpha u \in L^\infty$ makes sense, at least on a small time interval. Second, given the nature of the formal solution to u as a solution to the non-linear heat equation, it also makes perfect sense to consider $u \in C^\infty$. Finally, the fact that the non-linear heat equation can be solved in a classical sense without resorting to "special" spaces, it is worth considering a classical-style solution.

We have not, however, considered so far a classical solution on the pressure p. That is to say, if (u, p) is a solution to the equation, and we assume u and $D^\alpha u$ L^∞ how can we deal with the pressure without *assuming* some sort of control upon it. It would be better if we actually construct a working pressure term without requiring any presumptive restrictions. In this paper the goal will be to address some of these issues, (boundedness, decay) while establishing a pathway to explore other questions (existence, uniqueness). Given the reasonable idea that $Du \in L^\infty$, the pressure term becomes a function in the space BMO by fiat. Functions of Bounded Mean Oscillation have some nice properties to exploit, and by doing so, it will allow us to construct a proper working pressure term.

It should be first noted that most papers written about the Navier-Stokes equations consider a finite *energy*. That is

$$\mathcal{E}(t) = \int |u(x,t)|^2 \, dx < \infty$$

Thus u exists in the space L^2 ([30]). In contrast, the paper from Otto Kreiss and Jens Lorenz ([25]) assumes only that $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Thus, this paper allows for an infinite energy. The Kreiss–Lorenz paper (see [25]), hereafter referred to as the KL paper, began by discussing parabolic equation systems for $u \equiv u(x,t)$:

$$u_t(x,t) = \Delta u(x,t) + D_i g(u(x,t)) \quad x \in \mathbb{R}^n, \quad t \geq 0$$

with initial condition

$$u(x,0) = f(x) \quad f \in L^\infty(\mathbb{R}^n)$$

on a maximal time interval $0 < t < T(f)$, where g was quadratic in u. It was shown that under the assumptions given on f and g, that there is a constant $c_0 > 0$ with

$$T(f) > \frac{c_0}{\|f\|_\infty^2}$$

and

$$\|u(\cdot,t)\|_\infty \leq 2\|f\|_\infty \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

Additionally, it was shown that for every $j = 1, 2, \ldots$ that there is a constant $K_j > 0$ with

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \leq K_j \|f\|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

where c_0 and K_j are independent of t and f.

The result produced is as important as the methods used in the production. The same methods are used to analyze the Navier-Stokes equations, and determine similar bounds on the velocity field u and its derivatives. First, u and derivatives of u are used to bound the pressure locally (near the singularity at x). Additionally, u and derivatives of u are used to bound the *derivatives* of the pressure both locally *and* globally. Specific attention in the KL paper is given to handling the derivatives of u in a much broader sense than is normal for paper relating to the Navier-Stokes equations.

Despite the fact that the original KL paper produced some very nice results, there were items not addressed satisfactorily in the original work . The chief problem was the analysis on the pressure term. The underlying assumption of the paper was that the velocity field $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. The original pressure term was presented as the formal integral of the Poisson:

$$p(x,t) \equiv p_{org}(x,t) = \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^3} |x-y|^{-1} D_i D_j (u_i u_j)(y,t)\, dy$$

This was decomposed into a local and global part using a C^∞ cutoff function. The purpose of this function was to provide suitable bounds on p_{loc}, and $\mathcal{D}p_{loc}$ and $\mathcal{D}p_{glb}$ in terms of u, $\mathcal{D}u$, and a number $\delta > 0$. Here the symbol $\mathcal{D}$ refers the space derivative in maximum norm (see Definition 4.3.1).

The problem with this integral is that it may fail to exist at all due to the fact that u is simply $L^\infty(\mathbb{R}^3)$. If the integral fails to exist, the subsequent calculations and bounds are essentially incorrect. At the end of the paper, however a modification

$$p(x,t) = \sum_{i,j} \frac{1}{4\pi} \lim_{R\to\infty} \int_{|y|<R} [\Phi_{ij}(x-y) - \Phi_{ij}(y)](u_i u_j)(y,t)\, dy$$

where

$$\Phi_{ij}(y) = \frac{y_i y_j}{|y|^5} \quad \text{and} \quad \Phi_{jj}(y) = \frac{3y_i^2 - |y|^2}{|y|^5}$$

was indicated. The modification is claimed to solve the Poisson pressure equation and has the benefit of being bounded so long as $|x| < R$ by an application of the Mean-Value Theorem (see [25]). The claim will be confirmed in this doctoral thesis. Thus, the pressure integral will exist, even if u is only a constant. We will note here that for a fixed point x_0, we may change the kernel to

$$\Phi_{ij}(x-y) - \Phi_{ij}(x_0 - y)$$

so that the pressure becomes

$$p(x,t) = p_{org}(x,t) + C(t) \tag{1.15}$$

that is, constructing a new integral that incorporates the original pressure term plus a time dependent constant. We may, without loss of generality, take $x_0 = 0$. If one takes a single spatial derivative in terms of x, the term $C(t)$ in (1.15) is annihilated, and (1.15) will be verified as a solution to the Poisson Pressure equation. It was shown in the original paper that the estimates:

$$\|p_{loc}\|_\infty \leq C(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)$$

$$\|\mathcal{D}p_{loc}\|_\infty \leq C(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2)$$

$$\|\mathcal{D}p_{glb}\|_\infty \leq C\delta^{-1}\|u\|_\infty^2$$

were required to obtain the final results of the paper. Here $p \equiv p_{org}$, the original pressure term. We will verify that these same estimates will follow with the addition of $C(t)$ and the rest of the calculations of the paper will follow.

The method outlined above allows us to add an arbitrary time-dependent constant to the pressure that will allow this new pressure to still solve the Poisson Pressure equation without the potential divergence problems that were not satisfactorily addressed in the original paper. The idea for this change actually is rooted in the Bounded Mean Oscillation structure of the pressure term. BMO functions modified by simply adding a constant C retain their norms; that is, given a function $p \in BMO$, the function $q = p + C$ has the same norm as p: $\|p\|_{BMO} = \|q\|_{BMO}$. This is a result of the fact that BMO functions identify if they differ by a constant. Thus, modifying the BMO pressure function by adding a judiciously chosen constant allows us to not change the pressure term in the sense of BMO , and additionally allows us to make the pressure integral finite.

The main result of the paper is as follows. Suppose that a solution (u, p) exists to the incompressible Navier-Stokes equations. We assume that $u \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$, and assume that all derivatives $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α. Further we assume that u solves (1.1) and (1.2) in some maximum time interval $0 \leq t < T(f)$. Additionally, assume that $u(x, 0) = f(x)$, and that $\nabla \cdot f = 0$. Then there is $c_0 > 0$, and for each $j = 0, 1, \ldots$ a constant K_j such that

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \leq K_j\|f\|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

and c_0 and K_j are independent of t and f. Additionally it will be shown that

$$T(f) > \frac{c_0}{\|f\|_\infty^2}$$

These are identical to the results of the parabolic problem. This will prove that all derivatives of u are bounded in maximum norm by the initial value function f, provided $f \in L^\infty(\mathbb{R}^3)$.

In this doctoral thesis we will expand and enhance the Kreiss–Lorenz paper. Since the pressure term of the Navier-Stokes equations is indeed the engine that drives the solutions, we will first review information related to singular integral theory and the space BMO , culminating in the proof of a theorem that relates the space L^∞ to the space BMO in Chapter 2. In Chapter 3, we discuss the modification of the pressure term, including how it solves the Poisson equation and the fact that it is also of the space BMO . In Chapter 4, we begin the analysis of the Kreiss–Lorenz paper.

Chapter 4 will concentrate on the bounds of u and $\mathcal{D}u$. We will rigorously prove the pertinent theorems from the paper in precise detail using the modified pressure term

$$p^*(x,t) = \frac{1}{4\pi}\sum_{i,j}\int [|x-y|^{-1} - |y|^{-1}](D_i u_j D_j u_i)(y,t)\,dy$$

and show that this modification allows us to obtain the same bounds that were constructed in the Kreiss–Lorenz paper, while addressing the issue of pressure existence.

Chapter 5 will provide the proof on the bounds on $\mathcal{D}^j u(x,t)$-the j^{th} order derivatives of u in maximum norm via an induction argument that was alluded to but not proven in the original paper. As with the original paper, we construct the results *a priori*, that is assuming the solution pair (u,p) exist. So if (u,p) solves the Navier-Stokes equations, where p is our modified pressure, we will confirm that

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \le K_j\|f\|_\infty \quad \text{for} \quad 0 < t \le \frac{c_0}{\|f\|_\infty^2}$$

for all derivatives of order j in maximum norm. We again note here that $u(x,0) = f$, and $f \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. We will conclude the paper with a sketch on how to actually *construct* a solution to the Navier-Stokes equations given our conclusions for the *a priori* case.

Chapter 2

Calderón-Zygmund Operators and BMO

2.1 Motivation

We will begin by looking at the formally derived pressure term of the Navier-Stokes equation and demonstrating its existence as a principal value integral. We continue through this chapter with a very brief survey of singular integral theory and the theory of the functions of Bounded Mean Oscillation (BMO), as presented in the works of Sadosky and Stein ([35], [39], [40]). All theorems, propositions, etc., are reproduced *directly* from Sadosky's work. Only the final key result, Theorem 2.4.1, will be proved.

It is to be *strongly* noted here that although this chapter is primarily based on Sadosky's work ([35]), notations involving operators, however, are more in the conventional spirit of Stein (see [39] or [40]). The proof presented for Theorem 2.4.1 *does* vary from Sadosky's original proof in both notation and somewhat in content.

The final key result will be proved using the earlier stated theorems. Sadosky's work is influenced, of course by the works of Stein, Calderón and Zygmund. Our motivation for study is the pressure term. For $1 \leq i, j \leq 3$ and taking the integral over $\mathbb{R}^3$ the pressure term is (formally):

$$p(x,t) = \sum_{i,j} \frac{1}{4\pi} \int |x-y|^{-1} (D_i u_j D_j u_i)(y,t)\, dy$$

This can be shown to be written in Chapter 3 alternatively as

$$p(x,t) = \mathrm{PV} \quad \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y)\, dy \tag{2.1}$$

where the $K_{ij}(x-y)$ are described by equations (1.12) and (1.13). Assuming that $u(x,t)$ is suitably smooth ($u \in C^\infty$) and has compact support, there are no problems. However, the focus of the KL paper was the velocity field u belonging to the space $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, with $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α. Clearly from the discussions in Chapter 1, this presents a problem that must be remedied. The slow decay on y on the integral version of the pressure $p(x,t)$ provides us with a considerable challenge that needs to be overcome. To deal with this challenge, it will be helpful to explore integrals that are similar in structure to $p(x,t)$. We may then pursue the idea of how to proceed with such integrals. In the next section we discuss the structure of these so-called *singular integrals*.

2.2 Singular Integral Theory-A Brief Survey

We begin this section with a definition.

Definition 2.2.1. *Let a function $h(x)$ be measurable on $\mathbb{R}^n$, and for some $x \in \mathbb{R}^n$ let h be absolutely integrable over each set $\{y : |x-y| > \varepsilon > 0\}$. Then h is integrable over $\mathbb{R}^n$ in a* principle value sense *if*

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} h(y)\, dy$$

exists and is finite. The value of this limit will be denoted by

$$PV \quad \int_{\mathbb{R}^n} h(y)\, dy$$

We note here that if

$$h(x,y) = \frac{g(y)}{|x-y|}$$

and if $g \in L^p$ where $1 \leq p < \infty$, the integral will exist as a function of x in L^p([35]). The most basic principle value integral is the Hilbert Transform:

$$\text{PV} \quad \int_{-\infty}^{\infty} \frac{g(y)}{x-y}\, dy$$

This occurs for $\mathbb{R}^1$. We may write this as

$$\lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty} \right) \frac{g(y)}{x-y}\, dy$$

We may construct analogs to the one dimensional case. If $x \in \mathbb{R}^n$, we have the following:

Definition 2.2.2. *The Riesz kernels are given by*

$$k_j(x) = \frac{x_j}{|x|^{n+1}}$$

This then suggests

Definition 2.2.3. *The* Riesz transforms *of a function h are given by*

$$R_j h(x) = PV \quad c_n \int_{\mathbb{R}^n} h(x-y) \frac{y_j}{|y|^{n+1}}\, dy = PV \quad c_n \int_{\mathbb{R}^n} h(y) \frac{x_j - y_j}{|x-y|^{n+1}}\, dy$$

for $j = 1, \ldots, n$ *and*

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}$$

If we write $K(y) = \frac{x_j - y_j}{|x-y|^{n+1}}$, then $|K(y)| \leq \frac{1}{|x-y|^n}$ as y approaches x asymptotically. This justifies us defining *singular integrals.*

Definition 2.2.4. *A* singular integral *is an integral operator of the form*

$$Th(x) = \int K(x,y)h(y)\,dy$$

where the integral is singular at $x = y$

Calderón and Zygmund developed deep theories involving singular integrals. We will explore some of them here. Consider first the Riesz kernel R_j:

$$R_j(y) = \frac{y_j}{|y|^{n+1}} = \frac{\frac{y_j}{|y|}}{|y|^n} = \frac{\Omega_j(y)}{|y|^n}$$

We note that for $\lambda > 0$

$$\Omega(\lambda y) = \frac{\lambda y_j}{|\lambda y|} = \frac{\lambda y_j}{\lambda |y|} = \frac{y_j}{|y|} = \Omega(y)$$

This suggest the homogeneity property.

Definition 2.2.5. *A function $h(x)$ is homogeneous of degree n if*

$$h(\lambda^n x) = \lambda^n h(x)$$

for all $\lambda > 0$ and for all x.

Thus the Riesz kernel is homogeneous of degree $-n$. We also recall from reference to lemma A.1.2 that

$$\int_{\mathbb{S}^{n-1}} y_j d\sigma(y) = 0$$

This suggest the concept of mean value:

Definition 2.2.6. *A function $h(x)$ has mean value zero on the unit n-sphere if*

$$\int_{S^{n-1}} h(x)\,d\sigma(x) = 0$$

This leads us to define the *Calderón -Zygmund kernels.*

Definition 2.2.7. *Consider the space* $\mathbb{R}^n$, $n > 1$. *A kernel* k *is called a Calderon-Zygmumd kernel if it is of the form:*

$$k(x) = \frac{\Omega(x)}{|x|^n}$$

with two special properties:

1. Ω *is homogeneous of degree* 0. *That is* $\Omega(\lambda x) = \Omega(x)$ *for all* $\lambda > 0$ *and for all* x.

2. Ω *has mean value zero on the unit sphere*

Thus this begets the *Calderón -Zygmund* operators

Definition 2.2.8. *Let* $h \in L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$. *For each Calderón -Zygmund Kernel the* Calderón -Zygmund *integral operator is given by*

$$Th(x) = h * k(x) = PV \quad \int_{\mathbb{R}^n} h(x-y)\frac{\Omega(y)}{|y|^n}\, dy \tag{2.2}$$

This is an example of a *convolution operator*. To make sure that the integral in equation (2.2) exists, we must have a regularity property for $\Omega(x)$. Define the *modulus of continuity* of $\Omega(x)$ on the unit sphere $\mathbb{S}^{n-1}$ by

$$\omega(\delta) = \sup_{\substack{|x-x'|<\delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')| \tag{2.3}$$

If Ω is Lipschitz continuous or C^1, then $\omega(\delta) < C\delta^\alpha$ for $\alpha > 0$. If Ω is C^1 on the unit sphere, then $\omega(\delta) < C\delta$. In any case

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty$$

This is a condition called *Dini Continuity*:

Definition 2.2.9. *A function* $\Omega(x)$ *on* S^{n-1} *is said to be* ***Dini continuous*** *if given* $\Omega(x)$ *and the modulus of continuity* $\omega(\delta)$*:*

$$\omega(\delta) = \sup_{\substack{|x-x'|<\delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')|$$

then the following condition holds

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty \tag{2.4}$$

The operator in (2.2) exists if $h \in L^p(\mathbb{R}^n)$, for $1 < p < \infty$. In particular, this exists for $h \in L^2(\mathbb{R}^n)$. Most papers involving the Navier-Stokes equations consider $u \in L^2(\mathbb{R}^n)$. Now, let us briefly return to our pressure term. Recall that our pressure is defined by

$$\begin{aligned} p(x,t) &= \lim_{\varepsilon\to 0} \sum_{i,j} \frac{1}{4\pi} \int_{|x-y|>\varepsilon} |x-y|^{-1} (D_i u_j D_j u_i)(y,t)\, dy \\ &= \lim_{\varepsilon\to 0} \sum_{i,j} \frac{1}{4\pi} \int_{|x-y|>\varepsilon} |x-y|^{-1} (D_i D_j u_i u_j)(y,t)\, dy \end{aligned}$$

If u is compactly supported, we may transfer the derivatives over to $|x-y|^{-1}$ and obtain

$$p(x,t) = \lim_{\varepsilon\to 0} \sum_{i,j} \frac{1}{4\pi} \int_{|x-y|>\varepsilon} K_{ij}(x-y)(u_i u_j)(y,t)\, dy$$

where

$$K_{ij}(x-y) = \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^5} \quad \text{and} \quad K_{jj}(x-y) = \frac{3(x_j-y_j)^2 - |x-y|^2}{|x-y|^5}$$

We may write $p(x,t)$ as

$$p(x,t) = \mathrm{PV} \quad \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\Omega_{ij}(x-y)}{|x-y|^3} (u_i u_j)(y,t)\, dy$$

where

$$\Omega_{ij}(x-y) = \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2} \quad \text{and} \quad \Omega_{jj}(x-y) = \frac{3(x_j-y_j)^2 - |x-y|^2}{|x-y|^2}$$

If $u \in L^p$ for $1 < p < \infty$, then the integral exists. If not, if u is simply $L^\infty(\mathbb{R}^n)$ for example, the integral may not exist. Therefore we must take care with this integral if u lies in the space $C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Our next section leads up to function of Bounded Mean Oscillation (BMO).

2.3 The Space BMO

Recall the Lebesgue Differentiation Theorem:

Theorem 2.3.1. *Suppose f is a locally integrable function. That is, if $f \in L^1_{loc}(\mathbb{R}^n)$*

$$\int_A |f|\, dx < \infty$$

for all measurable sets A. Then for almost every x in $\mathbb{R}^n$

$$\lim_{r\to 0} \fint_{B(x,r)} f\, dx = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x)$$

We can use this to infer the *Hardy-Littlewood maximal function*:

Definition 2.3.1. *If f is a measurable function, with $f \in L^1_{loc}(\mathbb{R}^n)$, then the* Hardy-Littlewood *maximal function is given by*

$$Mf(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} f(y)\, dy$$

where $|Q(x,r)| = r^n$ is the $n-$dimensional measure of the cube $Q(x,r)$ with sides of length r parallel to the axes.

If f is a locally integrable function, that is $f \in L^1_{\text{loc}}$, we define the *mean value* f_Q of f over the cube Q by.

Definition 2.3.2. *Let $f \in L^1_{loc}$. Then*

$$f_Q = \frac{1}{|Q|} \int_Q f(x) dx$$

is the mean value *of f over Q, where Q is taken to be a cube with sides parallel to the axes, and $|Q|$ is its Lebesgue measure.*

We now define the *mean oscillation*.

Definition 2.3.3. *Let f_Q be as in definition 2.2.1. Then the* mean oscillation *is given by*

$$f_Q^\sharp = \frac{1}{|Q|}\int_Q |f(y) - f_Q|\, dy$$

Definition 2.3.4. *The* sharp maximal operator *is given by*

$$\Lambda^\sharp f(x) = \sup_{r>0} f_{Q(x,r)}^\sharp$$

where $Q(x,r)$ is cube of side r centered at x.

Finally we have

Definition 2.3.5. *A function $f \in L^1_{loc}$ has BMO norm $\|f\|_{BMO} = \|\Lambda^\sharp f(x)\|_\infty$. A function $f \in L^1_{loc}$ is of Bounded Mean Oscillation if $\|f\|_{BMO} < \infty$. We may write this as: There is an $M > 0$ such that*

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx < M$$

for all cubes Q in $\mathbb{R}^n$

From the definition it should be clear that $L^\infty \subset BMO$. However, the natural logarithm function $\log(x)$ shows that the converse need not be true (see [39], [40], or [35]). The basic idea behind the space BMO is that certain functions in the space are "almost bounded". In some cases the functions are better behaved than they are in the space L^∞. That is to say that despite the fact the function might "blow–up", the integral of the difference between the function and its average value is indeed bounded, at least in a local sense. One interesting fact that will be exploited later is that two functions in the space are identical if they differ by a constant. That is if $p \in \text{BMO}$, then

$$\|p - q\|_{BMO} = 0 \Leftrightarrow p - q = C$$

where C is a constant. The importance of this with the modified pressure term will be discussed in Chapter 3. We now turn our attention to the main lemmas and theorems. First up is a method of decomposing $\mathbb{R}^n$ into cubes.

Lemma 2.3.1. *The Calderón -Zygmumd Lemma For $h : \mathbb{R}^n \to \mathbb{C}$ a positive integrable function and $\alpha > 0$ a fixed constant, the space $\mathbb{R}^n$ admits a decomposition $\mathbb{R}^n = P \cup Q$, $P \cap Q = \emptyset$ such that:*

1. $Q = \bigcup_{k=1}^{\infty} Q_k$*, where* Q_k *is a cube, and the interiors of the cubes are disjoint.*
2. $h(x) < \alpha$ *for* $x \in P$
3. $\alpha < \frac{1}{|Q_k|} |\int_{Q_k} h(x)\, dx < 2^n \alpha$ *for every* Q_k*,* $k = 1, 2, \ldots$

The Calderón -Zygmund lemma begets the following:

Lemma 2.3.2. (The Calderón -Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^n)$ be a positive function. Then $\mathbb{R}^n = P \cup Q$, $P \cap Q = \varnothing$, where $Q = \bigcup_{k=1}^{\infty} Q_k$, and the Q_k's are non-overlapping cubes. Then $f = g + b$ where we have $g \in L^2(\mathbb{R}^n), b(x) = 0$ almost everywhere in P, and b has mean value zero on every Q_k.*

This breakdown is needed in the proof of proposition 2.3.1. We wish to find convolution operators k with a special property. This property will give consistency in the modification of the pressure term

Definition 2.3.6. *A function K is said to satisfy the Hörmander condition with constant A if*

$$\sup_{|x|>0} \int_{|y|>2|x|} |K(x - y) - K(y)|\, dy \leq A.$$

We will denote the Fourier transform of a function h by $\hat{h}$. More information on Fourier transforms can be found in [7],[10], and [11]. The following proposition is used to establish our main results:

Proposition 2.3.1. *Let $k \in L^2(\mathbb{R}^n)$ be such that there exists an $A > 0$ for which :*

1. $|\hat{k}| \leq A$ *for all* $x \in \mathbb{R}^n$
2. k *satisfies a Hörmander condition for the same* A

Then the convolution operator T of kernel k transforms L^∞ continuously into BMO and, for all $f \in L^\infty$ there is a C depending only on A and n such that:

$$\|Tf\|_{BMO} \leq C\|f\|_\infty$$

Note that in particular if $k \in L^1 \cap L^2$, then $\hat{k}$ is continuous and bounded by the classical Riemann-Lebesque lemma. Geometrically the following lemma is of interest.

Lemma 2.3.3. *If $|x| > 2|y|$,then $|\frac{x-y}{|x-y|} - \frac{x}{|x|}| \leq 2|\frac{y}{x}|$*

This produces the following:

Lemma 2.3.4. *Let k be a kernel such that:*

1. $\int_{1\leq|x|\leq 2} |k(x)|\, dx \leq B$
2. k *follows a Hörmander condition with constant* A.

and let $k_1(x) = k(x)$ if $|x| > 1$ and zero otherwise. Then k_1 also satisfies Hörmander condition with constant $A + 2B$.

The proof of lemma 2.3.4 is produced from lemma 2.3.3. All of the previous lead up to the *Calderón -Zygmund Theorem*

Theorem 2.3.2. (The Calderón -Zygmund Theorem) *Let Ω be a function defined in $\mathbb{R}^n$ such that:*

1. *Ω is homogeneous of degree zero*

2. *Ω has mean value zero on the unit sphere.*

3. *Ω is Dini continuous (see definition 2.2.9).*

For each $\varepsilon > 0$ let T_ε be the truncated operator for every $f \in L^p$ $1 < p < \infty$ by

$$(Tf_\varepsilon)(x) = \int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy$$

Then the following hold.

1. *For all $f \in L^p(\mathbb{R}^n)$, $Tf_\varepsilon \in L^p(\mathbb{R}^n)$ and there is a constant C_p independent of ε and f such that $\|Tf_\varepsilon\|_p \leq C_p\|f\|_p$*

2. *For every $f \in L^p(\mathbb{R}^n)$ the limit of Tf_ε as $\varepsilon \to 0$ exists in L^p ($\|Tf_\varepsilon - Tf\|_p \to 0$). Furthermore $\|Tf\|_p \leq C_p\|f\|_p$*

This final theorem allows us to map L^∞ continuously to BMO . It is found in Sadosky's work [35].

2.4 The Formal Mapping of L^∞ to BMO

This is the main result we need:

Theorem 2.4.1. *Under the conditions of theorem 2.3.2, the Calderon-Zygmund operator T given by:*

$$Tf(x) = PV \int_{\mathbb{R}^n} k(x-y)f(y)\, dy$$

transforms L^∞ continuously into BMO so that

$$\|Tf\|_{BMO} \leq C\|f\|_\infty \tag{2.5}$$

Proof. Given $f \in L^\infty$ and the C-Z kernel k, we define $k_\varepsilon = k(x)$ for $|x| > \varepsilon$ and zero otherwise. We further define:

$$u_\varepsilon(x) = \int (k_\varepsilon(x-y) - k_1(-y))f(y)\,dy \tag{2.6}$$

and

$$C_\varepsilon = \int (k_\varepsilon(-y) - k_1(-y))f(y)\,dy \tag{2.7}$$

If we define $k_{\varepsilon\eta} = k_\varepsilon - k_\eta$ then we have, by (2.6) $u_{\mathcal{E}} - u_\eta = f * k_\varepsilon - f * k_\eta(x) = k_{\varepsilon\eta} * f$. Since $k_{\varepsilon\eta} \in L^1 \cap L^2$, for any $0 < \varepsilon < \eta < \infty$ and satisfies, by Lemma 2.3.4 the same hypothesis as k does, then by Proposition 2.3.1 we have that:

$$\|k_{\varepsilon\eta} * f\|_{BMO} \leq C\|f\|_\infty$$

Thus, for every cube Q:

$$\frac{1}{|Q|}\int_Q |u_\varepsilon(x) - u_\eta(x) - (u_\varepsilon)_Q + (u_\eta)_Q|dx \leq C\|f\|_\infty \tag{2.8}$$

where f_Q is the mean value of f over the cube Q. Since $u_\eta - C_\eta = \int(k_\eta(x-y) - k_\eta(-y))f(y)\,dy$ tends to 0 as $\eta \to \infty$ uniformly in Q by the truncation of k_η and the conditions k satisfies, both

$$u_\eta(x) - C_\eta \to 0 \quad \text{and} \quad (u_\eta)_Q - C_\eta \to 0 \quad \text{as} \quad \eta \to \infty \tag{2.9}$$

By (2.8)

$$\frac{1}{|Q|}\int_Q |u_\varepsilon(x) - (u_\eta(x) - C_\eta) - (u_\varepsilon)_Q + ((u_\eta)_Q - C_\eta)|dx \leq C\|f\|_\infty$$

As $\eta \to \infty$, both $(u_\eta(x) - C_\eta)$ and $(u_\eta)_Q - C_\eta)$ tend to 0. As $\varepsilon \to 0$, we find that

$$u_\varepsilon(x) \to \int_{|x-y|>0} (k(x-y) - k_1(-y))f(y)\,dy = u(x)$$

while

$$(u_\varepsilon)_Q \to (u)_Q$$

So letting $\eta \to \infty$ and then $\varepsilon \to 0$, we get

$$\frac{1}{|Q|}\int_Q |u(x) - u_Q|,\ dx \le C\|f\|_\infty$$

Finally, if

$$a = \int_{|x-y|>0} k_1(-y)\,dy$$

We may write

$$u(x) = Tf(x) + a$$

and note that $(a)_Q = a$, a constant in the space BMO . Then

$$\begin{aligned} u(x) - u_Q &= Tf(x) + a - (Tf + a)_Q \\ &= Tf(x) + a - (Tf)_Q - a \\ &= Tf(x) - (Tf)_Q \end{aligned}$$

so that

$$\frac{1}{|Q|}\int_Q |Tf(x) - (Tf)_Q|,\ dx \le C\|f\|_\infty$$

□

Some final notes. First, we may also obtain the same results with balls $B(x,r)$ as well as cubes $Q(x,r)$ (see [39], [40], and [41]). In [42] Stein proves theorem 2.4.1. Consider now

$$p(x,t) = \lim_{\varepsilon\to 0}\sum_{i,j}\frac{1}{4\pi}\int_{|x-y|>\varepsilon} K_{ij}(x-y)(u_iu_j)(y,t)\,dy$$

where $K_{ij}(x-y)$ is given by equations (1.12) and (1.13). The kernel $K_{ij}(x-y)$, like the Riesz kernel, is a Calderón -Zygmund kernel. Invariably, we have the fact that this formal Poisson pressure term is indeed a function of BMO . We should note here that if $g \in BMO$, we may write

$$g = \eta_0 + \sum_{j=1}^{n} R_j \eta_j$$

where $\eta_i \in L^\infty(\mathbb{R}^n)$.

2.5 The Riesz Kernels and the Space BMO

In Theorem 2.4.1 we proved the fact that if $k(x)$ was a Calderón -Zygmund (CZ) kernel, then

$$Kf(x) = \text{PV} \quad \int_{\mathbb{R}^n} k(x-y) f(y)\, dy$$

transformed L^∞ continuously into BMO . Recalling that

$$k(x) = \frac{\Omega(x)}{|x|^n}$$

we would like to explore the types of candidates for $\Omega(x)$ that would allow $k(x)$ to be a CZ kernel.

Lemma 2.5.1. *If*

$$\Omega(x) = \frac{x_j}{|x|}$$

(the Riesz kernel) then $k(x)$ is a CZ kernel that is Dini-continuous.

This is discussed in Appendix A.2. Additionally, the expressions

$$K_{ij} = \frac{x_i x_j}{|x|^2} \quad \text{and} \quad K_{jj} = \frac{x_j^2 - |x|^2}{|x|^2}$$

also satisfies these properties (see Theorem A.2.2). The three properties are important, because they are needed in showing that CZ kernels satisfy the Hörmander condition (see Definition 2.3.6). As a matter of fact, this can be stated in a theorem. (see Majda [31] and Sadosky [35]).

Theorem 2.5.1. *Let k(x) be a Calderón Zygmund kernel. That is*

$$k(x) = \frac{\Omega(x)}{|x|^n}$$

where $\Omega(x)$ *has mean value of zero on the unit sphere, and is homogeneous of degree* 0. *Further, suppose that* $\Omega(x)$ *is Dini-continuous. Then* $k(x)$ *satisfies the Hörmander condition:*

$$\sup_{|x|>0} \int_{|y|>2|x|} |K(x-y) - K(y)|\, dy \leq A.$$

The proof of this can be found in Appendix A.2, Theorem A.2.3. Since the Riesz Kernel, and the product of Riesz kernels satisfy the properties of homogeneity, mean value 0 and Dini-continuity, they satisfy the Hörmander condition, and thus all of the above theorems. Additionally, the expression K_{ij} for all i, j also satisfies these three properties. Then both the Riesz kernels and the K_{ij} will satisfy the conditions of Theorem 2.4.1 and in either case, functions will be mapped from L^∞ continuously to BMO . We will show in the next chapter that the pressure term $p(x,t)$ exists as a function in the space Bounded Mean Oscillation .

Chapter 3

The Pressure Term

3.1 The Pressure and the Space BMO

The usual assumption on the fluid velocity u in the Navier-Stokes equations is that $u \in L^2(\mathbb{R}^3)$. That is to say that

$$\Big(\int_{\mathbb{R}^3} |u(x,t)|^2\, dx\Big)^{\frac{1}{2}} < \infty$$

This allows for a finite energy:

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} |u(x,t)|^2\, dx < \infty$$

However, in consideration of functions in the space BMO , we here will concentrate on functions that are $L^\infty(\mathbb{R}^3)$. That is to say for the remainder of this work that we will set our fluid field $u(x,t)$ such that for all t in a maximal time interval $0 < t < T(f)$:

$$u(x,t) \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \quad \text{and} \quad D^\alpha u(x,t) \in L^\infty(\mathbb{R}^3)$$

for all orders α. This allows for an infinite energy. The last chapter established a pathway to the set of functions of Bounded Mean Oscillation . We now establish the fact that the pressure term is indeed a member of this set of functions. In order to do this, we must establish the conditions required by the theorem 2.4.1. Consider the (formal) pressure term of the Navier-Stokes equations

$$p(x,t) \equiv p_{org}(x,t) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} |x-y|^{-1} D_i D_j (u_i u_j)(y,t)\, dy \tag{3.1}$$

where $C_0 = \frac{1}{4\pi}$. As written, the concern is two fold. First, the behavior of the integral near the singularity ($x = y$), and second the slow decay of the integrant for large values of y. Although this integral may not exist in the classical sense, it can be shown to exist in the space of functions of Bounded Mean Oscillation .

We will now show that the pressure term can be written as a principal value integral. The purpose here is to reconstruct the pressure term in a form that will exploit the results in Chapter 2. By no means are we to assume the *classical* existence of the integral, especially if $u \in L^\infty(\mathbb{R}^3)$.

Lemma 3.1.1. *Consider the (formal) pressure term from the Navier-Stokes equations:*

$$p(x,t) \equiv p_{org}(x,t) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} |x-y|^{-1} D_i D_j (u_i u_j)(y,t)\, dy$$

The pressure term can be rewritten as

$$p(x,t) = PV \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\, dy$$

where

$$K_{ij}(x-y) = \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}$$

and

$$K_{jj}(x-y) = \frac{3(x_j - y_j)^2 - |x-y|^2}{|x-y|^5}$$

Proof. Fix t in a maximal time interval $[0, T)$, and fix $\rho > 0$. We define a C^∞ cutoff function ϕ where $\phi(r) = 1$ for $0 \leq r \leq 1$, and $\phi(r) = 0$ for $r \geq 2$. We will take ϕ to be

$$\phi \equiv \phi\Big(\frac{|x-y|}{\rho}\Big)$$

for fixed $\rho > 0$. We write $u_i u_j = (\phi u_i u_j) + [(1-\phi)u_i u_j]$ and break up the integral.

$$\begin{aligned}
p(x,t) &= \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j (u_i u_j)(y,t)\, dy \\
&= \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j (\phi u_i u_j)(y,t)\, dy \\
&+ \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} \frac{1}{|x-y|} D_i D_j [(1-\phi)(u_i u_j)(y,t)\, dy \\
&= p_{loc}(x,t) + p_{glb}(x,t)
\end{aligned}$$

From here we will suppress the t in out notation. It should be noted that the "local" part near the singularity depends on values of $u(y)$ where $0 \leq |y| \leq 2\rho$, while the global part depends values of $u(y)$ where $|y| \geq \rho$. Now, consider $p_{loc}(x)$. Since ϕ vanishes at the boundary, if we integrate by parts the integral at the boundary will vanish. However, the problem is that if one takes two derivatives of $1/|x-y|$ we obtain K_{ij}, where the singularity at $y = x$ is not directly integrable. We will show that the integral locally exists as a limit for $\varepsilon \to 0$

Fix $\varepsilon > 0$ and consider the integral for our ϕ

$$I_{1(i,j)}(\varepsilon, x) = C_0 \int_{\varepsilon<|x-y|<2\rho} \frac{1}{|x-y|} D_i D_j (\phi u_i u_j)(y,t)\, dy$$

We may integrate $I_{ij}(\varepsilon, x)$ by parts twice, using the standard integration by parts formula (see [7]). Again, it is noted that on the boundary, $\phi = 0$, so the boundary integrals vanish. We obtain

$$I_{1(i,j)}(\varepsilon, x) = C_0 \int_{\varepsilon<|x-y|<2\rho} D_i D_j \Big(\frac{1}{|x-y|}\Big)(\phi u_i u_j)(y,t)\, dy$$

or

$$I_{1,(i,j)}(\varepsilon, x) = C_0 \int_{\varepsilon<|x-y|<2\rho} K_{ij}(x-y)(\phi u_i u_j)(y,t)\, dy$$

We note that $\phi = 1$ on $\varepsilon < |x-y| < \rho$, so we break up the integral into two pieces:

$$\begin{aligned} I_{1(i,j)}(\varepsilon, x) &= C_0 \int_{\varepsilon<|x-y|<2\rho} K_{ij}(x-y)(\phi u_i u_j)(y,t)\, dy \\ &= C_0 \int_{\varepsilon<|x-y|<\rho} K_{ij}(x-y)(u_i u_j)(y,t)\, dy \\ &+ C_0 \int_{\rho<|x-y|<2\rho} K_{ij}(x-y)(\phi u_i u_j)(y,t)\, dy \\ &= T_{1(i,j)}(\varepsilon, x) + T_{2(i,j)}(\varepsilon, x) \end{aligned}$$

Now $T_{2(i,j)}(\varepsilon, x)$ exists regardless of ε. Our main concern is T_1.

Concerning T_1, the term K_{ij} defined by (3.7), (3.8), and (3.9), is a Calderón -Zygmund Kernel (see Definition 2.2.7). In particular, it has mean value zero on the three dimensional unit sphere $\mathbb{S}^2$. This is due to the fact that from Lemma A.1.2

$$\int_{\mathbb{S}^2} (y_i)(y_j)\, d\sigma(y) = 0$$

for $i \neq j$ and

$$\int_{\mathbb{S}^2} \frac{3(y_i)^2 - |y|^2}{|y|^2}\, d\sigma(y) = 0$$

for $i = j$. If $\mathbb{S}^2_{x,a} = \{y : |x-y| = a\}$ is the surface of the sphere of radius a centered at x, we then find that

$$\int_{\mathbb{S}^2_{x,a}} (x_i - y_i)(x_j - y_j)\, d\sigma(y) = 0 \tag{3.2}$$

for $i \neq j$ and

$$\int_{\mathbb{S}^2_{x,a}} \frac{3(x_i - y_i)^2 - |x-y|^2}{|x-y|^2}\, d\sigma(y) = 0 \tag{3.3}$$

for $i = j$, where σ is the surface measure of the sphere in $\mathbb{R}^3$:

$$\sigma(\mathbb{S}^2) = a^2\omega_3 = a^2\frac{2(\pi)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}$$

For (3.2) and (3.3), this is just the unit sphere in $\mathbb{R}^3$ shifted to the center at x instead of 0, with a scaling factor of a^2.

For any i, j where $1 \leq i, j \leq 3$, we note that in view of our assumptions $(u_iu_j)(x,t)$ and its' derivatives exist for all $x \in \mathbb{R}^3$. We have

$$\rho^2\omega_3 \int_{\partial B(x,\rho)} \frac{(x_i - y_i)(x_j - y_j)(u_iu_j)(x)}{|x-y|^5}\, d\sigma(y) = 0 \tag{3.4}$$

Similarly, for $\varepsilon > 0$, we have

$$\varepsilon^2\omega_3 \int_{\partial B(x,\varepsilon)} \frac{(x_i - y_i)(x_j - y_j)(u_iu_j)(x)}{|x-y|^5}\, d\sigma(y) = 0 \tag{3.5}$$

Let A be the set $B(x,\rho)\backslash B(x,\varepsilon) = \{y : \varepsilon < |x-y| < \rho\}$, and let ∂A be the boundary. Subtracting (3.4) and (3.5) we find that

$$\omega_3(\rho^2 - \varepsilon^2) \int_{\partial A} \frac{(x_i - y_i)(x_j - y_j)(u_iu_j)(x)}{|x-y|^5}\, d\sigma(y) = 0$$

We note that by polar coordinates

$$\int_A \frac{(x_i - y_i)(x_j - y_j)(u_iu_j)(x)}{|x-y|^5}\, dy = C(\rho,\varepsilon) \int_{\partial A} \frac{(x_i - y_i)(x_j - y_j)(u_iu_j)(x)}{|x-y|^5}\, d\sigma(y) = 0$$

where $C(\rho, \varepsilon) = \omega_3(\rho^2 - \varepsilon^2)$. We first write

$$\begin{aligned}
T_{1(i,j)}(\varepsilon, x) &= C_0 \int_{\varepsilon<|x-y|<\rho} K_{ij}(x-y)(u_iu_j)(y,t)\, dy \\
&= C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}(u_iu_j)(y)\, dy \\
&= C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}(u_iu_j)(y)\, dy - 0 \\
&= C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}(u_iu_j)(y)\, dy \\
&- C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}(u_iu_j)(x)\, dy \\
&= C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5}[(u_iu_j)(y) - (u_iu_j)(x)]\, dy
\end{aligned}$$

Now we will use the fact that $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and the transformation to polar coordinates to produce our results.

$$\begin{aligned}
|T_{1(i,j)}(\varepsilon, x)| &= \left| C_0 \int_A \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} [(u_i u_j)(y) - (u_i u_j)(x)]\, dy \right| \\
&\leq C_0 \int_A \left| \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} \right| \cdot |(u_i u_j)(y) - (u_i u_j)(x)|\, dy \\
&\leq C_0 \int_A \frac{1}{|x-y|^3} \cdot |(u_i u_j)(y) - (u_i u_j)(x)|\, dy
\end{aligned}$$

We may now suitably bound the integral if we can determine a bound for $(u_i u_j)(y) - (u_i u_j)(x)$.

Considering the expression $|(u_i u_j)(y) - (u_i u_j)(x)|$. we may write

$$\begin{aligned}
|(u_i u_j)(y) - (u_i u_j)(x)| &= |u_i(y)u_j(y) - u_i(x)u_j(x)| \\
&= |u_i(y)u_j(y) - u_i(x)u_j(y) + u_i(x)u_j(y) - u_i(x)u_j(x)| \\
&\leq |u_j(y)| \cdot |u_i(y) - u_i(x)| + |u_i(x)| \cdot |u_j(y) - u_j(x)| \\
&\leq C\|u\|_\infty |u(y) - u(x)|
\end{aligned}$$

Define

$$\psi(t) = u((1-t)x + ty)$$

Then $\psi(1) = u(y)$, $\psi(0) = u(x)$, and $\psi'(t) = -\nabla u((1-t)x + ty) \cdot (x - y)$. We note that

$$\begin{aligned}
|u(y) - u(x)| &= |\psi(1) - \psi(0)| \\
&= \left| \int_0^1 \psi'(t)\, dt \right| \\
&\leq \max_{0 \leq t \leq 1} |\psi'(t)| \\
&\leq |x - y| \cdot \|\nabla u\|_\infty
\end{aligned}$$

by the Cauchy-Schwarz inequality and the Mean Value Theorem. We now have

$$\begin{aligned}|(u_iu_j)(y)-(u_iu_j)(x)| &\leq C\|u\|_\infty|u(y)-u(x)| \\ &\leq C\|u\|_\infty|x-y|\cdot\|\nabla u\|_\infty \\ &= C\|u\|_\infty\|\nabla u\|_\infty|x-y|\end{aligned}$$

In view of our assumptions on u and Du, both $\|u\|_\infty$ and $\|\nabla u\|_\infty$ are finite. We may now properly bound $|T_{1(i,j)}(\varepsilon,x)|$.

Returning to $T_{1(i,j)}(\varepsilon,x)$, and converting to polar coordinates, we now compute

$$\begin{aligned}|T_{1(i,j)}(\varepsilon,x)| &\leq C_0\int_A \frac{1}{|x-y|^3}\cdot|(u_iu_j)(y)-(u_iu_j)(x)|\,dy \\ &\leq C_0C_1\int_A \frac{1}{|x-y|^3}\|u\|_\infty\|\nabla u\|_\infty|x-y|\,dy \\ &= C_0C_1\|u\|_\infty\|\nabla u\|_\infty\int_A \frac{1}{|x-y|^3}\cdot|x-y|\,dy \\ &\leq C_2\int_A \frac{1}{|x-y|^2}\,dy \\ &= C_2\omega_2\int_\varepsilon^\rho \frac{1}{r^2}r^2\,dr \\ &= C(\rho-\varepsilon)\end{aligned}$$

Then

$$\begin{aligned}\lim_{\varepsilon\to 0}|T_{1(i,j)}(\varepsilon,x)| &\leq \lim_{\varepsilon\to 0}C(\rho-\varepsilon) \\ &= C(\rho-0) \\ &= C\rho \\ &< \infty\end{aligned}$$

again for fixed x and $\rho>0$. Finally summing over all $1\leq i,j\leq 3$:

$$\lim_{\varepsilon\to 0}\sum_{i,j}|T_{1(i,j)}(\varepsilon,x)| \leq C\rho$$

for fixed x and ρ.

If $i = j$, then we use

$$\int_{\mathbb{S}^2_{x,a}} \frac{3(x_j - y_j)^2 - |x-y|^2}{|x-y|^2}\, d\sigma(y) = 0$$

to obtain

$$C(\rho, \varepsilon) \int_A \frac{3(x_j - y_j)^2 - |x-y|^2}{|x-y|^5} [(u_j u_j)(y) - (u_j u_j)(x)]\, dy$$

We observe that

$$\Big| \frac{3(x_j - y_j)^2 - |x-y|^2}{|x-y|^5} \Big| \le \frac{C|x-y|^2}{|x-y|^5} \le \frac{C}{|x-y|^3}$$

and also

$$|(u_j u_j)(y) - (u_j u_j)(x)| = |u_j^2(y) - u_j^2(x)| \le C\|u\|_\infty |u(y) - u(x)|$$

so that by a similar integration the requisite bound is obtained in the case $i = j$. In any case

$$\lim_{\varepsilon \to 0} \sum_{i,j} |T_{1(i,j)}(\varepsilon, x)| \le C\rho \tag{3.6}$$

for all $1 < i, j < 3$, for fixed x, and ρ.

A quick computation on $T_{2(i,j)}(\varepsilon, x)$ reveals

$$|T_{2(i,j)}(\varepsilon, x)| \le C \ln 2$$

where C depends on ϕ and $\|u\|_\infty$. This computation is independent of ε. Given equation (3.6), we have

$$\lim_{\varepsilon \to 0} |I_{1(i,j)}(\varepsilon, x)| \le \lim_{\varepsilon \to 0} |T_{1(i,j)}(\varepsilon, x) + T_{2(i,j)}(\varepsilon, x)| < \infty$$

by the triangle inequality. The limit as $\varepsilon \to 0$ exists and is finite. Summing over all $1 \le i, j \le 3$:

$$\lim_{\varepsilon \to 0} \sum_{i,j} |I_{1(i,j)}(\varepsilon, x)| < \infty$$

Let

$$I_{1(i,j)}(x) = C_0 \int_{B(x,2\delta)} K_{ij}(x-y)(\phi u_i u_j)(y,t)\, dy$$

Then

$$p_{loc}(x) = \sum_{i,j=1}^{3} I_{1(i,j)}(x)$$

As $\varepsilon \to 0$, $I_{1(i,j)}(\varepsilon, x) \to I_{1(i,j)}(x)$, as a principal value integral. We sum over all $1 \leq i, j \leq 3$ to produce

$$\begin{aligned} p_{loc}(x) &= \sum_{i,j} I_{1(i,j)}(x) \\ &= \lim_{\varepsilon \to 0} \sum_{i,j} I_{1(i,j)}(\varepsilon, x) \\ &= \mathrm{PV} \quad \sum_{i,j} I_{1(i,j)}(x) \\ &= \mathrm{PV} \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)\phi(u_i u_j)(y,t)\, dy \end{aligned}$$

For $p_{glb}(x)$, the argument is much simpler. Given ϕ and for fixed $\varepsilon > 0$ let

$$I_{2(i,j)}(x,\varepsilon) = C_0 \int_{|x-y|>\rho>\varepsilon} \frac{1}{|x-y|} D_i D_j [(1-\phi)(u_i u_j)](y,t)\, dy$$

and

$$I_{2(i,j)}(x) = C_0 \int_{|x-y|>\rho>0} K_{ij}(x-y)[(1-\phi)(u_i u_j)](y,t)\, dy$$

Letting $\varepsilon \to 0$, $I_{2(i,j)}(x,\varepsilon)$ becomes $I_{2(i,j)}(x)$, and summing over all $1 \leq i, j \leq 3$, $I_{2(i,j)}(x)$ is $p_{glb}(x)$. We integrate $I_{2(i,j)}(x,\varepsilon)$ by parts, again noting at the boundary that $\phi = 1$, so that $1 - \phi = 0$, resulting in the integral at the boundary vanishing. We then have

$$I_{2(i,j)}(x,\varepsilon) = C_0 \int_{|x-y|>\rho>\varepsilon} K_{i,j}(x-y)[(1-\phi)(u_i u_j)](y,t)\, dy$$

As $\varepsilon \to 0$ and summing over all $1 \le i, j \le 3$ we obtain:

$$\lim_{\varepsilon \to 0} \sum_{i,j} I_{2(i,j)}(\varepsilon, x) = \text{PV} \quad \sum_{i,j} I_{2(i,j)}(x) = \text{PV} \quad p_{glb}(x)$$

So we may write $p_{glb}(x)$ as a principal value integral

$$p_{glb}(x) = \text{PV} \quad p_{glb}(x)$$

We may now recombine $p_{loc}(x)$ and $p_{glb}(x)$ to produce

$$\begin{aligned} p(x,t) &= p_{loc}(x,t) + p_{glb}(x,t) \\ &= \text{PV} \quad \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(\phi u_i u_j)(y,t)\, dy \\ &= \text{PV} \quad \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)[(1-\phi) u_i u_j](y,t)\, dy \\ &= \text{PV} \quad \sum_{i,j=1}^{3} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y) u_i u_j(y,t)\, dy \end{aligned}$$

This is the required result. □

We note that in the above proof, the trick was to consider a region very close to the singularity and demonstrate that the singularity was integrable in a *principal value sense*-that is, the singularity was integrable by essentially encapsulating it in a ball of radius ε, and showing that the integral "locally" was finite as $\varepsilon \to 0$.

The principal value expression of the pressure is a *little* better than the original in terms of its behavior as y approaches infinity. *almost* integrable. However, due to the slow decay on y, this integral may fail to exist in the classical sense if $u \in L^\infty(\mathbb{R}^3)$. The pressure term *can* be shown to exist in the space of functions of Bounded Mean Oscillation .

Consider now

$$p(x,t) = \mathrm{PV} \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\,dy$$

The term $K_{ij}(x-y)$, the kernel, is of the form

$$K_{ij}(x) = \frac{\Omega_{ij}(x)}{|x|^3} \tag{3.7}$$

By observation, we obtain

$$\Omega_{ij}(x)\frac{x_i x_j}{|x|^2}, \tag{3.8}$$

for $i \neq j$,

$$\Omega_{jj}(x) = \frac{3x_j^2 - |x|^2}{|x|^2} \tag{3.9}$$

for $i = j$. The conclusions that for all $1 \leq i, j \leq 3$ the term $\Omega_{ij}(x)$ is Dini continuous, has mean value zero around the unit sphere, and is homogeneous of degree 0 are proved in Theorem A.2.2 (see definitions 2.2.5, 2.2.6 and 2.2.9). Thus we can now prove the following:

Proposition 3.1.1. *Consider the Poisson pressure solution from the Navier-Stokes equations*

$$p(x,t) = PV \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\,dy$$

*where $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α. The (formal) pressure term in the Navier-Stokes equations is a function of bounded mean oscillation (**BMO**).*

Proof. Let

$$g(y,t) = (u_i u_j)(y,t)$$

where $1 \leq i, j \leq 3$. We note that

$$\begin{aligned} |g(y,t)| &= |(u_i u_j)(y,t)| \\ &\leq |u_i(y,t)| \cdot |u_j(y,t)| \\ &\leq \|u\|_\infty^2 \end{aligned}$$

Since $u \in L^\infty(\mathbb{R}^3)$, we have $\|g\|_\infty \leq \|u\|_\infty^2$ is finite and $g \in L^\infty(\mathbb{R}^3)$. Recall that the (formal) pressure term in the Navier-Stokes equation:

$$\begin{aligned} p(x,t) &= \sum_{i,j} C_0 \int_{\mathbb{R}^3} |x-y|^{-1} D_i D_j (u_i u_j)(y,t)\, dy \\ &= \text{PV} \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y) g(y,t)\, dy \end{aligned}$$

where

$$K_{ij}(x) = \frac{\Omega_{ij}}{|x|^3}$$

and

$$\Omega_{ij}(x) = \frac{x_i x_j}{|x|^2}$$

for $i \neq j$,

$$\Omega_{jj}(x) = \frac{3x_j^2 - |x|^2}{|x|^2}$$

for $i = j$. It was established in Theorem A.2.2 that the $\Omega_{ij}(x)$ were homogeneous of degree zero for all i, j, $\Omega_{ij}(x)$ had mean value zero on the unit sphere, and that $\Omega_{ij}(x)$ are Dini continuous. Thus the functions K_{ij} are clearly Calderon-Zygmund kernels from Definition 2.2.7. Since for all i, j the function $g(y,t) = (u_i u_j)(y,t)$ is L^∞, the conditions of Theorem 2.4.1 are satisfied, and we have that for all i, j the integrals

$$\text{PV} \quad C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\, dy$$

lie in the space of functions of bounded mean oscillation (**BMO**). Clearly then the sum over all i, j of these integrals:

$$p(x,t) = \text{PV} \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\, dy$$

must also lie in the space as the sum is finite. Thus the pressure term from the Navier-Stokes equation is in the space BMO . □

It is clear that the pressure term is bounded in the sense BMO . The next section will establish a refinement of the pressure term that will allow us to determine the viability of L^∞ functions as solutions to the Navier-Stokes equations.

3.2 A Slight Modification

It is here we will turn our attention to the pressure term as the central object of study in the Navier-Stokes equations. We have discussed in Section 1.1 the paths taken by Giga ([13], [14], [15] as well as Kato ([21], [22]). Again, their paths were of more of a distributional (weak) concentration. Others, like Majda (see [31]), turned to the vorticity (Leray) formulation, where the pressure is entirely removed from the Navier-Stokes equation. However, the pressure is an important part of fluid mechanics. While removing it might be supported by the mathematical necessity of solving the equation, it still is present. It isn't going anywhere.

To that end, perhaps a study of it's close relative, the *compressible* Navier-Stokes may shed some light. In 1988, Kreiss, Lorenz, and Naughton ([27]) explored the compressible Navier-Stokes equations in a periodic case. For a small Mach number, they proved that the solution of the compressible equations consists of the solutions to the incompressible case plus a function that is "highly oscillatory in time" and can be described by wave equations (in local time). It was noted that this wave equation part could be suppressed by an initialization ([27], [26]). It was shown that the incompressible Navier-Stokes equations were a limit of the compressible case as the Mach number tends to zero.

There is no "free constant" in the compressible case. In the incompressible limit, the idealized constant becomes arbitrary which is not meaningful for physical reasons. However, since only space derivatives appear in the pressure term of the incompressible Navier-Stokes equations, then adding a judiciously chosen time-dependent constant will not matter. In this work, we construct a pressure for the incompressible Navier-Stokes equations that does not contain a free constant. In future work, we will address how the constructed pressure is related to the pressure that one obtains from the compressible equations as the Mach number tends to zero.

A further justification for the addition of a time-dependent constant to the formal solution of the Poisson pressure equation incorporates elements of singular integral theory, and the theory of the space of functions of BMO . Applying the modification of a constant will allow us to properly address the results of the Kreiss–Lorenz paper. Again, to reaffirm, all Proposition 3.1.1 does is to state that the singular integral that is the pressure p exists as a function Bounded Mean Oscillation . The integral may fail to exist, but through a limiting process we can control the growth over any cube or ball so that the integral

$$\sup_Q \frac{1}{|Q|} \int_Q |p(y) - p_Q| \, dy < \infty$$

over all cubes (or balls) Q. As was mentioned in Chapter 2, functions identified in the space BMO differ by a constant. So, since $p \in$ BMO , we can add any time dependent constant $C(t)$ to p without changing the element in the space. Since the pressure term lies in BMO , $p + C(t)$ will also lie in the space, and

$$\|p + C(t)\|_{BMO} = \|p\|_{BMO}$$

Also, from Theorem 2.4.1 as

$$p(x,t) = \text{PV} \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} K_{ij}(x-y)(u_i u_j)(y,t)\,dy$$

we have

$$\|p\|_{BMO} \leq C\|u\|_\infty^2$$

which is finite in view of our assumption that $u \in L^\infty(\mathbb{R}^3)$. Then we may conclude that

$$\|p + C(t)\|_{BMO} = \|p\|_{BMO} \leq C\|u\|_\infty^2$$

For physical reasons this is not meaningful, but mathematically this will allow us to modify the pressure solution without changing its underlying structure.

In Chapter 2 we observed that one of the conditions that was instrumental in proving that a function lies in the space BMO was the *Hörmander condition* (see Definition 2.3.6)

$$\sup_{|x|>0} \int_{|y|>2|x|} |K(x-y) - K(y)|\,dy \leq A.$$

This gives us a bit of hope in trying to find a slight modification of the (formal) Poisson pressure term that will work even if u is L^∞, but not necessarily L^2. In addition, the proof of Theorem 2.4.1 gave us an idea on how to deal with such (possibly) unbounded integrals. First recall that the (formal) pressure term of the Navier-Stokes equations is given by

$$p(x,t) = \sum_{i,j} C_0 \int K_{ij}(x-y)(u_i u_j)(y,t)\,dy \tag{3.10}$$

We will now suppress the time t in our notation. The term $K_{ij}(x-y)$, the kernel, is of the form

$$K_{ij}(x) = \frac{\Omega_{ij}}{|x|^3},$$

where the term $\Omega_{ij}(x)$ is given by

$$\Omega_{ij}(x) = \frac{x_i x_j}{|x|^2}$$

for $i \neq j$,

$$\Omega_{jj}(x) = \frac{3x_j^2 - |x|^2}{|x|^2}$$

for $i = j$, where

$$C_0 = \frac{1}{4\pi}$$

We will now define the modification.

Definition 3.2.1. *Let (u, p) be a solution to the Navier-Stokes equations, and suppose that $u \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$, with $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α, and where $t \in [0, T)$ for a finite $T \in \mathbb{R}$. The **modified Poisson pressure** is given by*

$$p^*(x,t) = \sum_{i,j} C_0 \int [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\, dy \tag{3.11}$$

where $C_0 = \frac{1}{4\pi}$ and the kernel K_{ij} is given by equations (3.7), (3.8), and (3.9)

We will now show that the integral defined in (3.11) exists. The singularities at $y = 0$ and $y = x$ will be shown to not be a problem by using the fact that u is a C^∞ function. We will be able to show that the integral exists as a principal value integral. The following lemma will be needed (see for example [29] or [10]):

Lemma 3.2.1. *Suppose that $K_{i,j}(y)$ is the kernel described by equations (3.7), (3.8), and (3.9). Then for $|y| > 2|x|$*

$$|K_{ij}(y-x) - K_{ij}(y)| \leq \frac{C|x|}{|y|^4}$$

Proof. We note here that we may write:

$$K_{ij}(x-y) = \frac{\Omega_{ij}}{|x-y|^3} = \frac{\Omega_{ij}}{|y-x|^3}$$

Let

$$\psi(t) = K_{ij}(y - tx), \quad 0 \le t \le 1$$

Then

$$\begin{aligned} |K_{ij}(y-x) - K_{ij}(y)| &= |\psi(1) - \psi(0)| = \Big| \int_0^1 \psi'(t)\, dt \Big| \\ &\le \max_{0 \le t \le 1} |\psi'(t)| \end{aligned}$$

Since

$$\psi'(t) = -x \cdot \nabla K_{ij}(y - tx)$$

by the Cauchy-Schwarz inequality

$$|\psi'(t)| \le |x||\nabla K_{ij}(y - tx)| \le \frac{C|x|}{|y - tx|^4}$$

Also, as $0 \le t \le 1$, we have:

$$|y - tx| \ge |y| - t|x| \ge |y| - |x| \ge \frac{1}{2}|y|$$

Thus

$$\frac{1}{|y - tx|} \le \frac{2}{|y|}$$

and

$$\frac{1}{|y - tx|^4} \le \frac{16}{|y|^4}$$

whence

$$|K_{ij}(y-x) - K_{ij}(y)| \le \max_{0 \le t \le 1} |\phi'(t)| \le \frac{C|x|}{|y - tx|^4} \le \frac{C|x|}{|y|^4}$$

Thus we obtain:

$$|K_{ij}(y-x) - K_{ij}(y)| \le \frac{C|x|}{|y|^4}$$

for $|y| > 2|x|$ $\square$

We now proceed with the theorem.

Theorem 3.2.1. *Suppose that* (u, p) *is a solution to the Navier-Stokes equations, and suppose that* $u \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$*, with* $D^\alpha u \in L^\infty(\mathbb{R}^3)$ *for all orders* α*, and where* $t \in [0, T)$ *for a finite* $T \in \mathbb{R}$*. If*

$$p^*(x,t) = PV \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\, dy$$

is the modified Poisson pressure (Definition 3.2.1), where $K_{ij}(y)$ *is defined by* (3.7), (3.8)*, and* (3.9)*. then* $p^*(x,t)$ *exists; that is* $p^* < \infty$*.*

Proof. First, if $x = 0$, the $p^*(x,t) = 0$, and there is nothing to prove. So fix $x \neq 0$, and let $t \in [0, T)$, with $T \in \mathbb{R}$, and $T > 0$. Let $1 \leq i, j \leq 3$, and fix $R > 0$ large enough so that $R > 2|x|$. It was noted for a fixed $\rho > 0$ that

$$\lim_{\varepsilon \to 0} \sum_{i,j} C_0 \int_{\varepsilon < |x-y| < \rho} K_{ij}(x-y)(u_i u_j)(y,t)\, dy < \infty \tag{3.12}$$

in the proof of Lemma 3.1.1. Using the same argument of Lemma 3.1.1, we can also show that

$$\lim_{\varepsilon \to 0} \sum_{i,j} C_0 \int_{\varepsilon < |y| < \rho} K_{ij}(y)(u_i u_j)(y,t)\, dy < \infty \tag{3.13}$$

as well. It is essentially same argument with $x = 0$. This means that the singularities both at 0 and x are integrable.

Next, consider y far from 0 or x, say $|y| > 2|x|$. Suppressing the t in our notation, we now reconsider I_{glb} where

$$I_{glb}(i,j) = C_0 \int_{R > |y| > 2|x|} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\, dy$$

and recall Lemma 3.2.1. With a change to polar coordinates we compute:

$$\begin{aligned}
|I_{glb}(i,j)| &= \Big|C_0 \int_{R>|y|>2|x|} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y)\, dy\Big| \\
&\leq C_0 \|u\|_\infty^2 \sum_{i,j} \int_{R>|y|>2|x|} |K_{ij}(x-y) - K_{ij}(y)|\, dy \\
&= C_0 C_1 \int_{R>|y|>2|x|} \frac{C|x|}{|y|^4}\, dy \\
&\leq C_0 C_1 C|x| \int_{R>|y|>2|x|} \frac{1}{|y|^4}\, dy \\
&\leq C_2 |x| \int_{2|x|}^{R} \frac{1}{r^2}\, dr \\
&= C_2 |x| \Big[\frac{1}{2|x|} - \frac{1}{R}\Big] \\
&= C_2 \Big(\frac{1}{2} - \frac{|x|}{R}\Big)
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{R\to\infty} |I_{glb}(i,j)| &\leq \lim_{R\to\infty} C_2 \Big(\frac{1}{2} - \frac{|x|}{R}\Big) \\
&\leq C_3
\end{aligned}$$

so that

$$I_{glb} = \lim_{R\to\infty} \sum_{i,j} I_{glb}(i,j) < \infty \tag{3.14}$$

Thus I_{glb} is finite for large values of $R > 2|x|$.

We now consider the overall integral. Define a C^∞ cutoff function ϕ with $\phi(r) = 1$ for $0 \leq r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$. Write ϕ as

$$\phi \equiv \phi\Big(\frac{|y|}{|x|}\Big).$$

Further, we start off by restricting ourselves to the region $A = \{y : \varepsilon < |y| < R\}$. We consider the following:

$$\begin{aligned}
I_{\varepsilon,R}(x) &= \sum_{i,j} C_0 \int_A [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\,dy \\
&= \sum_{i,j} C_0 \int_A [K_{ij}(x-y) - K_{ij}(y)]\phi(u_i u_j)(y,t)\,dy \\
&+ \sum_{i,j} C_0 \int_A [K_{ij}(x-y) - K_{ij}(y)](1-\phi)(u_i u_j)(y,t)\,dy \\
&= I_{loc(\varepsilon,R)}(x) + I_{glb(\varepsilon,R)}(x)
\end{aligned}$$

since $I_{loc}(\varepsilon, R)$ depends on values of $u(y)$ for $r < 2$, and $I_{glb}(\varepsilon, R)$ depends on values of $u(y)$ for $r > 1$. For I_{loc}, the region under consideration is $\{y : \varepsilon < |y| < 2|x|\}$.

$$\begin{aligned}
I_{loc(\varepsilon,R)}(x) &= \sum_{i,j} C_0 \int_{\varepsilon<|y|<2|x|} [K_{ij}(x-y) - K_{ij}(y)]\phi(u_i u_j)(y,t)\,dy \\
&= \sum_{i,j} C_0 \int_{\varepsilon<|y|<2|x|} K_{ij}(x-y)\phi(u_i u_j)(y,t)\,dy \\
&+ \sum_{i,j} C_0 \int_{\varepsilon<|y|<2|x|} K_{ij}(y)\phi(u_i u_j)(y,t)\,dy \\
&= \sum_{i,j} T_1(i,j) + T_2(i,j)
\end{aligned}$$

First consider $T_2(i,j)$. As $\varepsilon \to 0$, $T_2(i,j)$ is just equation (3.13) with $\rho = 2|x|$ since the singularity is integrable. This yields

$$\lim_{\varepsilon\to 0} \sum_{i,j} T_2(i,j) < \infty$$

For $T_1(i,j)$, consider

$$\lim_{\varepsilon\to 0} C_0 \int_{\varepsilon<|x-y|<2|x|} K_{ij}(x-y)\phi(u_i u_j)(y,t)\,dy < \infty$$

from the proof of Lemma 3.1.1 with $\rho = 2|x| < R$. Since $|x-y| < |x|$, $|y| - |x| < |x-y| + |x| - |x| = |x-y| < |x|$, and since by definition $0 \leq \phi \leq 1$:

$$\begin{aligned} T_1(i,j) &= C_0 \int_{\varepsilon<|y|<2|x|} K_{ij}(x-y)\phi(u_iu_j)(y,t)\,dy \\ &= C_0 \int_{\varepsilon<|y|-|x|<|x|} K_{ij}(x-y)\phi(u_iu_j)(y,t)\,dy \\ &\leq C_0 \int_{\varepsilon<|x-y|<|x|} K_{ij}(x-y)\phi(u_iu_j)(y,t)\,dy \\ &\leq C_0 \int_{\varepsilon<|x-y|<2|x|} K_{ij}(x-y)\phi(u_iu_j)(y,t)\,dy \\ &\leq C_0 \int_{\varepsilon<|x-y|<2|x|} K_{ij}(x-y)(u_iu_j)(y,t)\,dy \end{aligned}$$

Then, using equation (3.12), as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} \sum_{i,j} T_1(i,j) < \infty$$

Thus, as $\varepsilon \to 0$, $I_{loc(\varepsilon,R)}(x) < \infty$ for fixed x. Then, as $R \to \infty$:

$$I_{loc} = \sum_{i,j} C_0 \int_{0<|y|<2|x|} [K_{ij}(x-y) - K_{ij}(y)]\phi(u_iu_j)(y,t)\,dy < \infty$$

Next, $I_{glb}(\varepsilon, R)$ can by written as

$$\begin{aligned} I_{glb(\varepsilon,R)}(x) &= \sum_{i,j} C_0 \int_{|y|>|x|} [K_{ij}(x-y) - K_{ij}(y)](1-\phi)(u_iu_j)(y,t)\,dy \\ &= \sum_{i,j} C_0 \int_{B(0,2|x|)\setminus B(0,|x|)} [K_{ij}(x-y) - K_{ij}(y)](1-\phi)(u_iu_j)(y,t)\,dy \\ &+ \sum_{i,j} C_0 \int_{2|x|<|y|<R} [K_{ij}(x-y) - K_{ij}(y)](u_iu_j)(y,t)\,dy \\ &= J_1(\varepsilon,R) + J_2(\varepsilon,R) \end{aligned}$$

As $\varepsilon \to 0$, both integrals exist on sets outside the singularities. As $R \to \infty$ J_1 is bounded on the annulus $\{y : |x| < |y| < 2|x|\}$. Finally, $J_2(\varepsilon, R)$ is just the result

(3.14) as R approaches infinity. So as $R \to \infty$, the sum $I_{loc(\varepsilon,R)}(x)$ is finite by fiat, and $I_{glb(\varepsilon,R)}(x)$ is finite via (3.14). Thus as $\varepsilon \to 0$, $R \to \infty$, and for fixed x:

$$I_{\varepsilon,R}(x) < \infty$$

This is precisely the pressure term, and we finally have

$$p^*(x,t) = \text{PV} \quad \sum_{i,j} C_0 \int [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\, dy < \infty$$

□

We note that

Corollary 3.2.1. *Theorem 3.2.1 is valid if one replaces*

$$K_{ij}(x-y) - K_{ij}(y)$$

with

$$K_{ij}(x-y) - K_{ij}(x_0 - y)$$

for a fixed $x_0 \in \mathbb{R}^3$

Proof. The proof is the same as Theorem 3.2.1, but instead of using $B = B(0,\rho) \setminus B(0,\varepsilon)$ we replace B with $B = B(x_0,\rho) \setminus B(x_0,\varepsilon)$, for a point x_0 in $\mathbb{R}^3$. Without loss of generality we may shift x_0 to 0, and the theorem still holds. □

Now that we have a candidate for a pressure term that actually exists, it remains to prove that it actually solves the Poisson pressure equation. We now turn to that task.

3.3 The Relation of p^* to the Poisson Pressure Equation

We realize from the definition 3.2.1 that essentially $p^*(x,t)$ can be simply written as

$$p^*(x,t) = p(x,t) - C(t) \tag{3.15}$$

in the sense of functions of BMO , since

$$\begin{aligned} p^*(x,t) &= \sum_{i,j} C_0 \int [K_{ij}(x-y) - K(y)](u_i u_j)(y,t)\,dy \\ &= \sum_{i,j} C_0 \int K_{ij}(x-y)(u_i u_j)(y,t)\,dy - \sum_{i,j} C_0 \int K_{ij}(y)(u_i u_j)(y,t)\,dy \\ &= p(x,t) - C(t) \end{aligned}$$

that is, our modification is simply the original (formal) pressure modified by the addition of a time dependent constant. Additionally note that, as always, the above exists as a *principal value* integral. However, this "split" may not exist in the classical sense. We now prove

Theorem 3.3.1. *Let*

$$p^*(x,t) = PV \quad \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\,dy$$

Then p^* is a solution to the Poisson pressure equation

$$-\Delta p(x,t) = \sum_{i,j} (D_i u_j)(D_j u_i)(x,t)$$

Proof. We begin by applying $D_{k,x}(p^*(x,t))$ under the integral sign, and writing $C_0 = \frac{1}{4\pi}$. Once again, we will use a C^∞ cutoff function $\phi(r)$ with $\phi(r) = 1$ for $0 \le r \le 1$ and 0 for $r > 2$. We shall take $\phi \equiv \phi(|x-y|)$. Suppressing the t in our notation, we write $(u_i u_j)(y,t) = (u_i u_j)(y)$. Using ϕ, we may write $u_i u_j = \phi(u_i u_j)(y) + (1-\phi)(u_i u_j)(y) =$

$g_1(x,y) + g_2(x,y)$. We must be careful here; we can't simply split the integrals and take derivatives separately, since separately the integrals may not exist. We do all of our work under the integral sign.

To this end, we write

$$M(x,y) = [K_{ij}(x-y) - K_{ij}(y)](u_iu_j)(y)$$

and apply $D_{k,x}$ to this expression.

$$\begin{aligned}
D_{k,x}(M(x,y)) &= D_{k,x}([K_{ij}(x-y) - K_{ij}(y)](u_iu_j)(y)) \\
&= D_{k,x}(K_{ij}(x-y)(u_iu_j)(y)) - D_{k,x}(K_{ij}(y)(u_iu_j)(y)) \\
&= D_{k,x}(K_{ij}(x-y)(u_iu_j)(y)) \\
&= D_{k,x}(K_{ij}(x-y)(g_1(x,y) + g_2(x,y))) \\
&= D_{k,x}(K_{ij}(x-y)(g_1(x,y)) + D_{k,x}(K_{ij}(x-y)(g_2(x,y))) \\
&= D_{k,x}(K_{ij}(x-y)(\phi(u_iu_j)(y)) + D_{k,x}(K_{ij}(x-y)[(1-\phi)(u_iu_j)(y)])
\end{aligned}$$

Then we produce

$$\begin{aligned}
D_{k,x}(p^*(x)) &= PV \quad \sum_{i,j} D_{k,x}\Big(C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](u_iu_j)(y)\,dy\Big) \\
&= PV \quad \sum_{i,j} D_{k,x}\Big(C_0 \int_{\mathbb{R}^3} M(x,y)\,dy\Big) \\
&= PV \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} D_{k,x}(M(x,y)) \\
&= \lim_{\varepsilon\to 0} \sum_{i,j} C_0 \int_{|x-y|>\varepsilon} D_{k,x}[K_{ij}(x-y)(\phi(u_iu_j)(y))]\,dy \\
&+ \lim_{\varepsilon\to 0} \sum_{i,j} C_0 \int_{|x-y|>\varepsilon} D_{k,x}K_{ij}(x-y)[(1-\phi)(u_iu_j)(y)]\,dy \\
&= \lim_{\varepsilon\to 0}(I_1(x,\varepsilon) + I_2(x,\varepsilon))
\end{aligned}$$

We will now break up I_1 and I_2 into suitable pieces to apply another spatial derivative.

We consider $I_1(x,\varepsilon)$ first. It depends only on those values of $u(y)$ where $|x-y|<2$. Since $\phi u_i u_j$ is a function of compact support, we may employ the same integration by parts procedure found in Evans ([7]). We observe that the derivative for I_1 exists on $\varepsilon < |x-y| < 2$. We now compute:

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1(x,\varepsilon) &= \lim_{\varepsilon\to 0}\sum_{i,j} C_0 \int_{|x-y|>\varepsilon} D_{k,x}[K_{ij}(x-y)(\phi(u_i u_j)(y))]\,dy \\
&= \lim_{\varepsilon\to 0}\sum_{i,j} D_{k,x}\Big(C_0 \int_{\varepsilon<|x-y|<2} [K_{ij}(x-y)(\phi(u_i u_j)(y))]\,dy\Big) \\
&= \lim_{\varepsilon\to 0}\sum_{i,j} D_{k,x}\Big(C_0 \int_{\varepsilon<|x-y|<2} \frac{1}{|x-y|}(D_i D_j \phi u_i u_j(y))\,dy\Big) \\
&= \lim_{\varepsilon\to 0}\sum_{i,j} C_0 \int_{\varepsilon<|x-y|<2} D_{k,x}[|x-y|^{-1}(D_i D_j \phi u_i u_j(y))]\,dy \\
&= \lim_{\varepsilon\to 0} C_0 \int_{\varepsilon<|x-y|<2} D_{k,x}[|x-y|^{-1}\sum_{i,j}(D_i D_j \phi u_i u_j)(y)]\,dy \\
&= \lim_{\varepsilon\to 0} C_0 \int_{\varepsilon<|x-y|<2} D_{k,x}[|x-y|^{-1}G(y)]\,dy \\
&= C_0 \int_{0<|x-y|<2} D_{k,x}[|x-y|^{-1}G(y)]\,dy
\end{aligned}$$

where $G(y)$ is given by

$$G(y) = \sum_{i,j}(D_i D_j \phi u_i u_j)(y)$$

and $G(y)$ is a function of compact support. As before, the singularity is integrable at $x=y$, and we may now write

$$\begin{aligned}
I_1(x) &= C_0 \int_{0<|x-y|<2} D_{k,x}[|x-y|^{-1}G(y)]\,dy \\
&= C_0 \int_{0<|x-y|<1} D_{k,x}[|x-y|^{-1}G(y)]\,dy \\
&+ C_0 \int_{1<|x-y|<2} D_{k,x}[|x-y|^{-1}G(y)]\,dy \\
&= J_1(x) + J_2(x)
\end{aligned}$$

As is well known (for example [7] or [17]) for $G(y)$ of compact support

$$-\Delta_x(J_1(x)) = G(x) \tag{3.16}$$

and, for sets not containing the singularity we observe that

$$\Delta_x(J_2(x)) = 0 \tag{3.17}$$

We can do the same thing with $I_2(x, \varepsilon)$. We split the integral into two pieces one with $1-\phi$, until $|x-y| = 2$, and then $1-\phi = 1$. Using this information and ϕ, we may actually break up $D_{k,x}(p^*(x))$ into three integrals: One on the region $B(x, 1)$, one on the annulus $\{1 < |x-y| < 2\}$, and $|x-y| > 2$. Now $\phi = 1$ until $|x-y| = 1$; then it decreases toward 0. $1-\phi$, on the other hand, is 0 until $|x-y| = 1$; then it increases to 1. We reintegrate $J_2(x)$ by parts to obtain

$$J_2(x) = \sum_{i,j} C_0 \int_{1<|x-y|<2} D_{k,x}[K_{ij}(x-y)(\phi(u_i u_j)(y))]$$

again using ϕ.

Splitting the integrals I_1 and I_2 into their respective "main pieces" plus pieces on the annulus. The derivatives of the principal value integrals can now be written:

$$\begin{aligned}
D_{k,x}(p^*(x)) &= J_1(x) + J_2(x) + I_2(x) \\
&= C_0 \int_{0<|x-y|<1} D_{k,x}[|x-y|^{-1} G(y)]\, dy \\
&+ \sum_{i,j} C_0 \int_{1<|x-y|<2} D_{k,x}[K_{ij}(x-y)(\phi(u_i u_j)(y))] \\
&+ \sum_{i,j} C_0 \int_{1<|x-y|<2} D_{k,x}[K_{ij}(x-y)(1-\phi)(u_i u_j)(y)]\, dy \\
&+ \sum_{i,j} C_0 \int_{|x-y|>2} D_{k,x}[K_{ij}(x-y)(u_i u_j)(y))] \\
&= D_{k,x} p_{loc}(x) + D_{k,x}(p_{annulus}(x)) + D_{k,x}(p_{glb}(x))
\end{aligned}$$

Applying another space derivative $D_{k,x}$ to each piece summing over all $1 \leq k \leq 3$ we may write

$$\Delta_x(p^*(x)) = \Delta_x(p_{loc}(x)) + \Delta_x(p_{annulus}(x)) + \Delta_x(p_{glb}(x))$$

First, as cited above ([7] or [17]),

$$\Delta_x(p_{loc}) = \Delta_x(J_1(x)) = -\sum_{i,j}(D_i u_j)(D_j u_i)(x,t)$$

Additionally, (from [7] or [17]), it is also known that for values of y far from x

$$\Delta_x \frac{1}{|x-y|} = 0$$

This is also true for the derivatives of the kernel $1/|x-y|$ as well so that

$$\Delta_x(K_{ij}(x-y)) = 0$$

for $|x-y| > 1$. Thus, on the annulus the same result applies. We now have:

$$\begin{aligned}
\Delta_x(p^*(x)) &= \Delta_x(p_{loc}(x) + p_{sd}(x) + p_{glb}(x))) \\
&= \Delta_x(p^*_{loc}(x)) + \Delta(p_{sd}(x)) + \Delta_x(p^*_{glb}(x)) \\
&= \sum_{i,j}(D_i D_j u_i u_j)(x,t) + 0 + 0 \\
&= \sum_{i,j}(D_i D_j u_i u_j)(x,t) \\
&= -\sum_{i,j}(D_i u_j D_j u_i)(x,t)
\end{aligned}$$

Thus $p^*(x,t)$ is a solution to the Poisson Pressure equation. □

We now state a prescient fact:

Corollary 3.3.1. *$p^*(x,t)$ is in the space BMO*

Proof. From Proposition 3.1.1 $p(x,t)$ is in BMO . From [39] or [40] if C is a constant, then C is in the space BMO . The integral

$$PV \quad \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(y)(u_i u_j)(y))] \, dy$$

exists in the space BMO as a constant. The sum or difference of two BMO functions is BMO. Thus $p^*(x,t)$ is in BMO. □

In closing off this chapter it should be noted that the observation that the (formal) pressure term $p(x,t)$ lies in the space BMO has been made elsewhere; see for example, [13], or [14]. The main aim here was to facilitate certain techniques similar to those used to prove 2.4.1 to "even out" the pressure term and allow it to exist for functions that are both L^∞ and C^∞. In the next chapter we turn to using the pressure modification to establish the bounds of the original Kreiss–Lorenz paper.

Chapter 4

The Kreiss–Lorenz Paper-Establishing Methodology

4.1 The Kreiss–Lorenz Paper: An Overview

The original Kreiss–Lorenz paper began with auxiliary results for the heat equation, which will be reviewed in section 4.2. From here, a procedure was established in the following section to establish bounds in maximum norm for parabolic systems

$$u_t = \Delta u + D_i g(u)$$

with initial condition

$$u(x,0) = f(x) \quad f \in L^\infty(\mathbb{R}^n)$$

on a time interval $0 \le t < T(f)$. It is well-known that the solution is C^∞ in a maximal interval $0 \le t < T(f)$ where $0 < T(f) \le \infty$. The main assumptions here were that u was a solution of the inhomogeneous heat equation and that g was quadratic in u. The results established were similar to those later in the paper for the

Navier-Stokes equations. For the initial results in the parabolic case, it was shown that under the assumptions given on f and g, that there is a constant $c_0 > 0$ with

$$T(f) > \frac{c_0}{\|f\|_\infty^2}$$

and

$$\|u(x,t)\| \leq 2\|f\|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{\|f\|_\infty^2}$$

Additionally, it was shown that for every $j = 1, 2, \ldots$ that there is a constant $K_j > 0$ with

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \leq K_j \|f\|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{\|f\|_\infty^2}$$

where c_0 and K_j are independent of t and f. The paper then turned its attention to the Navier-Stokes equations (1.1) and (1.2). Bounds were established for the pressure locally, and both locally and globally for the measure of all space derivatives of the pressure in maximum norm:

$$\|p_{loc}\|_\infty \leq C(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)$$

$$\|\mathcal{D}p_{loc}\|_\infty \leq C(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2)$$

$$\|\mathcal{D}p_{glb}\|_\infty \leq C\delta^{-1}\|u\|_\infty^2$$

An implied induction argument indicated that there was a constant c_0 independent of t and f such that

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \leq K_j \|f\|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

It should be noted here that the theorems proved in the paper involving the pressure term should have, in actuality, involved (3.11). Most papers involving the Navier-Stokes equations involve a finite energy; that is $u \in L^2$, which allows for the existence of the *original* pressure term. However, in the Kreiss–Lorenz paper the velocity field is assumed to be such that $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, but *not necessarily* in L^2, which allows for an infinite energy. The end result is that while the pressure term exists locally, the global existence is in doubt. This was not adequately addressed in the original paper. While one could use the BMO norm to bound the pressure field p, the most one can hope for using this norm is a bound only on every ball or cube in $\mathbb{R}^3$; using this norm does nothing to address the behavior over the whole space $\mathbb{R}^3$, where the pressure could diverge for large y in the integral. This is rectified by the modification discussed in Chapter 3.

What must be implied from the original paper is that there must be sufficient decay on u to allow the pressure integral to exist globally at all. The goal of this paper is clear: we must allow for an infinite energy, *and* guarantee the existence of the integral globally. This was actually accomplished in Chapter 3. The modification of the pressure exists over all of $\mathbb{R}^3$ as a principal value integral and solves the Poisson pressure equation. In the following few sections we will revisit the Kreiss–Lorenz paper. We will once again prove in detail the pertinent results from the paper. In addition, we will prove statements not proven in the original- for example the induction proof of the fact that

$$t^{j/2}\|\mathcal{D}^j u(x,t)\|_\infty \leq K_j\|f\|_\infty \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{\|f\|_\infty^2}$$

for all $j = 0, 1, 2, \ldots$.

Some of the proofs in the original Kreiss–Lorenz work were not proved explicitly, except for the main result. The bounds on p_{loc}, $\mathcal{D}p_{loc}$, and $\mathcal{D}p_{gl}$ were obtained by bounds on the related integrals and implied bounds on the combination of derivatives of u and the C^∞ cutoff function ϕ. In this paper we will use our *modified Poisson pressure* to prove the required bounds. Appendix D contains the somewhat tedious calculations on the derivatives of u and ϕ. The subsequent sections here will derive the same results as the original paper, but will be presented in more detail. Finally, the main result, that the derivatives of all orders on u are bounded in maximum norm by the initial value function $u(x,0) = f(x)$. This means that the solution (u,p) is controlled by the initial value function f.

4.2 The Kreiss–Lorenz Paper: Auxiliary Results for the Heat Equation

In this section we will review section 2 of the Kreiss–Lorenz paper. Section 2 discussed auxiliary results of the heat equation. We recall that (see equation (C.1.2) or [7])

$$e^{\Delta t} f = e^{\Delta t} * f = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

We will take $n = 3$. If $f \in L^\infty(\mathbb{R}^3)$, then the solution of

$$u_t = \Delta u, \quad u(x,0) = f \tag{4.1}$$

is denoted by

$$u(\cdot, t) = u(t) = e^{\Delta t} f$$

It is well-known that (see [10])

$$\|e^{\Delta t} f\|_\infty \le \|f\|_\infty, \quad t \ge 0 \tag{4.2}$$

and

$$\|\mathcal{D}^j e^{\Delta t} f\|_\infty \le C_j t^{-j/2} \|f\|_\infty \tag{4.3}$$

In the following C, C_j, etc. are positive constants independent of t and f. Suppose that $F \in L^\infty(\mathbb{R}^n \times [0,T])$, and consider the solution of

$$u_t = \Delta u + F(x,t), \quad u(x,0) = 0 \tag{4.4}$$

given by

$$u(t) = \int_0^t e^{\Delta(t-s)} F(s)\, ds$$

We then find

$$\|u(t)\|_\infty \leq 2t^{1/2} \max_{0\leq s\leq t}\{s^{1/2}\|F(s)\|_\infty\} \tag{4.5}$$

We will need estimates on the solution to the equation

$$u_t = \Delta u + D_i F(x,t), \quad u(x,0) = 0$$

As D_i commutes with the heat semi group, we use (4.3) to obtain

$$\|u(t)\|_\infty \leq C \max_{0\leq s\leq t}\{s^{1/2}\|F(s)\|_\infty\} \tag{4.6}$$

Equation (4.3), (4.5), and (4.6) will be used in determining bounds of the heat equation version of the Navier-Stokes equation.

4.3 The Kreiss–Lorenz Paper: Estimates for the Navier-Stokes Equations

These results are found in Section 4 of Kreiss–Lorenz ([25]). In the original paper some of the calculations were indicated; here they will be computed in rigorous detail. As is well-known, the (incompressible) Navier-Stokes equations are given by equations ((1.1),(1.2))

$$u_t + (u \cdot \nabla)u + \nabla p = \nu\Delta u \quad \nabla \cdot u = 0$$

with initial condition

$$u(x,0) = f(x) \quad \nabla \cdot f = 0$$

In the paper it is assumed that $f \in L^\infty(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$. The goal was to prove *a priori* estimates of the derivatives of u in terms of the maximum norm of the initial value function $u(x,0) = f(x)$, assuming that the solution existed and was C^∞ for a maximal time interval $0 \leq t < T(f)$. Again, as in the KL paper, we will assume the existence of a solution *a priori*, and assume that for all time t in a maximal time interval $0 \leq t < T(f)$ that $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and that $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α. Recalling that

$$\|f\|_\infty = \sup_x \|f\| \quad \text{with} \quad \|f\|^2 = \sum_i f_i^2(x)$$

we will now define the measurement of all space derivatives of order j in maximum norm.

Definition 4.3.1. *Let $\|f\|_\infty$ be the usual L^∞ norm, and let*

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \textit{for} \quad \alpha = (\alpha_1, \dots, \alpha_n)$$

be a multi-index, and $|\alpha| = \sum \alpha_i$. For any $j = 0, 1, 2, \dots$, set

$$\|\mathcal{D}^j u(t)\|_\infty = \|\mathcal{D}^j u(\cdot, t)\|_\infty = \max_{|\alpha|=j} \|D^\alpha u(\cdot, t)\|_\infty \tag{4.7}$$

Then $\|\mathcal{D}^j u(t)\|_\infty$ measures all space derivatives of order j in maximum norm.

Now, as $f \in L^\infty$, we have the existence of a constant M such that $\|f\|_\infty = M < \infty$. f is bounded over all space variables. We will show that this will guarantee the boundedness of u on a maximal time interval $0 \leq t < T(f)$.

We begin as follows. Suppressing the variables x and t in our notation, define $Q = Q(x,t)$ to be

$$Q = -\nabla p - u \cdot \nabla u = -\nabla p - \sum_j D_j(u_j u)$$

Recall from appendix B.3 that the pressure is determined by the **Poisson Pressure Equation**: equations (B.3.1) and (B.3.3):

$$\begin{aligned} -\Delta p(x,t) &= \sum_{i,j=1}^{3} D_i D_j (u_i u_j) \\ &= \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) \end{aligned}$$

Formally (see Appendix B.2), the solution is given by

$$p(x,t) = \mathrm{PV} \quad \sum_{i,j} \frac{1}{4\pi} \int_{\mathbb{R}^n} |x-y|^{-1} D_i D_j (u_i u_j)(y,t)\, dy$$

We, however, will use here our modification (3.11):

$$p^*(x,t) = \mathrm{PV} \quad \sum_{i,j} C_0 \int [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y,t)\, dy \tag{4.8}$$

with $C_0 = \frac{1}{4\pi}$ and where K_{ij} is the kernel defined by equations (3.7), (3.8), and (3.9).

As we determined through Corollary 3.3.1, the pressure term belongs to the space of functions Bounded Mean Oscillation . The pressure p is not generally in L^∞, but this will not be a problem for us as we will use the *derivatives* of u to derive estimates of ∇p, and thus the heat equation version of the Navier-Stokes equation. And, since *this* pressure term exists on the whole space, there will be no problems. As it turns out, the modification to the original pressure contributes nominally to the *local* pressure term. Since

$$D_{k,x}\left(\frac{1}{|y|}\right) = 0$$

we will see that the modification does not contribute to the *derivatives* of the pressure either locally or globally.

Since δ in this incarnation is arbitrary, it may be any number we wish. It will be convenient to choose $\delta = \sqrt{t}$. Appendix D contains the detailed calculations on the bounds of $\phi, \phi', \phi'', \phi'''$ as well as the bounds on

$$|D_i\phi u_i u_j|, \quad |D_i D_j \phi u_i u_j| \quad |D_i D_j D_{k,x} \phi u_i u_j|$$

These will be used to establish the bounds on p_{loc} as well as the *derivatives* of p_{loc} and p_{glb} in terms of the norms of u and $\mathcal{D}u$. The estimates in the following theorem are valid for all t where $0 < t \leq T(f)$.

Theorem 4.3.1. *Let u be a solution to the Navier-Stokes equations*

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \Delta u \quad \nabla \cdot u = 0$$

with initial condition

$$u(x,0) = f(x) \quad \nabla \cdot f = 0$$

where the pressure $p(x,t) \equiv p(x)$ is given by

$$p(x) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y)\, dy$$

There is a constant $C > 0$, independent on t, δ, and f, so that the following estimates hold:

$$\|p_{loc}\|_\infty \leq C(\|u\|_\infty^2 + \delta \|u\|_\infty \|\mathcal{D}u\|_\infty) \tag{4.9}$$

$$\|\mathcal{D}p_{loc}\|_\infty \leq C(\delta^{-1}\|u\|_\infty^2 + \delta \|\mathcal{D}u\|_\infty^2) \tag{4.10}$$

$$\|\mathcal{D}p_{glb}\|_\infty \leq C\delta^{-1}\|u\|_\infty^2 \tag{4.11}$$

Proof. We begin by fixing x, and fix R so large that $R > 2|x|$, and $R > 2\delta$ for fixed $\delta > 0$. Define $\psi(|y|)$ such that $\psi(y) = 1$ for $0 \leq |y| \leq R$, and $\psi(y) = 0$ for

$|y| \geq R+1$. Suppressing the t in our notation we write

$$
\begin{aligned}
p(x) &= \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)(y)\, dy \\
&= \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)]\psi(y)(u_i u_j)(y)\, dy \\
&+ \sum_{i,j} C_0 \int_{\mathbb{R}^3} [K_{ij}(x-y) - K_{ij}(y)](1-\psi(y))(u_i u_j)(y)\, dy \\
&= I_1(x) + I_2(x)
\end{aligned}
$$

Now $I_2(x)$ is not important for our consideration. We note that $\psi = 0$ if $|y| > R+1$. We integrate by parts twice, noting the $\psi = 0$ on the boundary. We break up the integral into two pieces: one on $B(0,R)$, and one one the annulus $B(0,R{+}1)\backslash B(0,R)$. We reintegrate the second integral by parts to obtain the original form-again the boundary inegrals are 0. We may then write

$$
\begin{aligned}
I_1(x) &= \sum_{i,j} C_0 \int_{B(0,R+1)} [K_{ij}(x-y) - K_{ij}(y)]\psi(y)(u_i u_j)(y)\, dy \\
&= \sum_{i,j} C_0 \int_{B(0,R)} [|x-y|^{-1} - |y|^{-1}] D_i D_j (u_i u_j)(y)\, dy \\
&+ \sum_{i,j} C_0 \int_{B(0,R+1)\backslash B(0,R)} [K_{ij}(x-y) - K_{ij}(y)]\psi(y)(u_i u_j)(y)\, dy \\
&= J_1(x) + J_2(x)
\end{aligned}
$$

This yields

$$
p(x) = J_1(x) + J_2(x) + I_2(x)
$$

We now further refine the break up of J_1:

$$\begin{aligned} J_1(x) &= \sum_{i,j} C_0 \int_{B(0,R)} [|x-y|^{-1} - |y|^{-1}] D_i D_j (u_i u_j)(y)\, dy \\ &= \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (u_i u_j)(y)\, dy \\ &- \sum_{i,j} C_0 \int_{B(0,R)} |y|^{-1} D_i D_j (u_i u_j)(y)\, dy \\ &= L_1(x) - L_2(x) \end{aligned}$$

Our job now is to isolate integrals around each of the singularities separately. Then, through a limiting process, we will rewrite the pressure term into a "local" part around the singularities, and a "global" part without them. We start with $L_1(x)$. We will suppress the argument of ϕ and the argument y of u in our notation for what follows. Choose another C^∞ cut-off function $\phi(r)$ with

$$\phi(r) = 1 \quad \text{for} \quad 0 \le r \le 1, \quad \phi(r) = 0 \quad r \ge 2$$

We let

$$\phi_1 \equiv \phi(\delta^{-1}|x-y|)$$

and write $u_i u_j = \phi_1 u_i u_j + (1-\phi_1) u_i u_j$. We now break up $L_1(x)$:

$$\begin{aligned} L_1(x) &= \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (u_i u_j)\, dy \\ &= \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (\phi_1 u_i u_j)\, dy \\ &+ \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (1-\phi_1)(u_i u_j)\, dy \\ &= M_1(x) + M_2(x) \end{aligned}$$

Given ϕ_1, it dies off a ball of radius 2δ. We now may rewrite $M_1(x)$.

$$\begin{aligned} M_1(x) &= \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (\phi_1 u_i u_j)\, dy \\ &= \sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-1} D_i D_j (\phi_1 u_i u_j)\, dy \end{aligned}$$

Similarly, for $L_2(x)$, we use $\phi_2 \equiv \phi(\delta^{-1}|y|)$ and obtain

$$\begin{aligned} L_2(x) &= \sum_{i,j} C_0 \int_{B(0,R)} |y|^{-1} D_i D_j (u_i u_j)\, dy \\ &= \sum_{i,j} C_0 \int_{B(0,2\delta)} |y|^{-1} D_i D_j (\phi_2 u_i u_j)\, dy \\ &+ \sum_{i,j} C_0 \int_{B(0,R)} |y|^{-1} D_i D_j (1-\phi_2)(u_i u_j)\, dy \\ &= M_3(x) + M_4(x) \end{aligned}$$

The "local" part is $M_1(x) + M_3(x)$, while the "global" is $M_2(x) + M_4(x)$. We may reintegrate $M_2(x)$ and $M_4(x)$ by parts to reacquire the original form of the pressure, because $B(0,R)$ is a finite set, and the functions $1-\phi_1$ and $1-\phi_1$ die on their respective boundaries. Let

$$G(x) = \lim_{\varepsilon \to 0} \sum_{i,j} C_0 \int_{B(\varepsilon,R)} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j)\, dy$$

We now compute:

$$\begin{aligned}
M_2(x) + M_4(x) &= \sum_{i,j} C_0 \int_{B(0,R)} |x-y|^{-1} D_i D_j (1-\phi_1)(u_i u_j)\, dy \\
&+ \sum_{i,j} C_0 \int_{B(0,R)} |y|^{-1} D_i D_j (1-\phi_2)(u_i u_j)\, dy \\
&= \sum_{i,j} C_0 \int_{B(0,R)} K_{ij}(x-y)(1-\phi_1)(u_i u_j)\, dy \\
&+ \sum_{i,j} C_0 \int_{B(0,R)} K_{ij}(y)(1-\phi_2)(u_i u_j)\, dy \\
&= \sum_{i,j} C_0 \int_{B(0,R)} [K_{ij}(x-y) - K_i j(y)](u_i u_j)\, dy \\
&- \sum_{i,j} C_0 \int_{B(0,R)} K_{ij}(x-y)\phi_1(u_i u_j)\, dy \\
&- \sum_{i,j} C_0 \int_{B(0,R)} K_{ij}(x-y)\phi_2(u_i u_j)\, dy \\
&= \lim_{\varepsilon \to 0} \sum_{i,j} C_0 \int_{B(\varepsilon,R)} [K_{ij}(x-y) - K_{ij}(y)](u_i u_j) \\
&+ (-M_1(x) - M_3(x)) \\
&= G(x) - M_1(x) - M_3(x)
\end{aligned}$$

The singularities at $y = x$ and $y = 0$ are integrable, and this exists as a principle value integral. Letting $R \to \infty$ we obtain

$$\lim_{R \to \infty} G(x) = p(x)$$

Now, as $R \to \infty$, both $J_2(x) \to 0$ and $I_2(x) \to 0$, and we may rewrite $p(x)$ in term of the "local" pieces about the singularities and the "global" pieces away from them:

$$\begin{aligned}
p(x) &= M_1(x) + M_3(x) + [p(x) - (M_1(x) + M_3(x))] \\
&= p_{loc}(x) + p_{glb}(x)
\end{aligned}$$

For the purposes of the theorem we are only concerned with $p_{loc}(x)$, and the derivatives of $p(x)$ both locally and globally.

We first turn or attention to the bounds of $p_{loc}(x)$. For M_1, we transfer one derivative to $|x-y|^{-1}$ by the usual procedure of writing the principal value integral as a limit, integrating by parts *once*, and then letting $\varepsilon \to 0$ to produce:

$$|M_1(x)| \leq \sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-2} |D_j(\phi_1 u_i u_j)|\, dy$$

We note from the calculations in Appendix D, in particular results of Corollary D.2.1 (equation (D.2.3)), that

$$|D_j(\phi_1 u_i u_j)| \leq C_1(\delta^{-1}\|u\|_\infty^2 + \|u\|_\infty \|\mathcal{D}u\|_\infty)$$

We find that [10] or [11] gives us the polar coordinate method for computing the integral

$$\int_{B(x,2\delta)} \frac{1}{|x-y|^2}\, dy$$

We compute

$$\begin{aligned}
\int_{B(x,2\delta)} \frac{1}{|x-y|^2}\, dy &= \int_{|x-y|\leq 2\delta} \frac{1}{|x-y|^2}\, dy \\
&= \omega_3 \int_0^{2\delta} \frac{1}{r^2} r^2\, dr \\
&= \omega_3 \int_0^{2\delta} dr \\
&\leq C_2 \delta
\end{aligned}$$

This constant depends on the surface measure of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, ω_3 (see Appendix A.1). We now have

$$\begin{aligned}
\|M_1\|_\infty &\leq \sum_{i,j} C_0 \int_{B(x,2\delta)} \frac{1}{|x-y|^2} |D_j(\phi_1 u_i u_j)|\, dy \\
&\leq C_3(\|u\|_\infty^2 + \delta \|u\|_\infty \|\mathcal{D}u\|_\infty)
\end{aligned}$$

As for $M_3(x)$, the computations are almost identical. Essentially, ϕ_2 is ϕ_1 at $x = 0$. Then

$$\begin{aligned}\|M_3\|_\infty &\leq \sum_{i,j} C_0 \int_{B(0,2\delta)} \frac{1}{|x-y|^2} |D_j(\phi_2 u_i u_j)|\, dy \\ &\leq C_4(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)\end{aligned}$$

Then we finally compute

$$\begin{aligned}\|p_{loc}\|_\infty &= \|M_1 + M_3\|_\infty \\ &\leq \|M_1\|_\infty + \|M_3\|_\infty \\ &\leq C_3(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty) \\ &+ C_4(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty) \\ &\leq C_A(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)\end{aligned}$$

This limiting process has finally produced

$$\|p_{loc}\|_\infty \leq C_A(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)$$

We have obtained a bound for $p_{loc}(x)$, involving only the singularities x and 0.

Recall in Chapter 3 we took a spatial derivative of our pressure term, when we proved it to be a solution of the Poisson pressure equation. We perform the exact same procedure here to estimate the derivatives for $p_{loc}(x)$ and $p_{glb}(x)$. Again we apply the derivative $D_{k,x}$ under the integral of equation (4.8) *without* ϕ, and then write $u_i u_j = \phi u_i u_j + (1-\phi)u_i u_j$, where $\phi = \phi(|x-y|/\delta)$ in this case. We integrate the integral involving $B(x, 2\delta)$ by parts, observing that $\phi = 0$ on the boundary. We obtain

$$\begin{aligned} D_{k,x}(p(x)) &= \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{k,x}[|x-y|^{-1} D_i D_j \phi(u_i u_j)(y)]\, dy \\ &+ \sum_{i,j} C_0 \int_{\mathbb{R}^3} D_{k,x}[K_{ij}(x-y)(1-\phi)(u_i u_j)(y)]\, dy) \\ &= D_{k,x}(p_{loc}(x)) + D_{k,x}(p_{glb}(x)) \end{aligned}$$

Thus we have

$$D_{k,x}(p_{loc}(x)) = \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{k,x}[|x-y|^{-1} D_i D_j \phi(u_i u_j)(y)]\, dy \tag{4.12}$$

and

$$D_{k,x}(p_{glb}(x)) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} D_{k,x}[K_{ij}(x-y)(1-\phi)(u_i u_j)(y)]\, dy \tag{4.13}$$

We now begin to estimate the derivatives of the pressure. We work with the local pressure first (equation (4.12)).

$$\begin{aligned} D_{k,x}(p_{loc}(x)) &= \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{k,x}(|x-y|^{-1} D_i D_j(\phi u_i u_j))\, dy \\ &= \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{k,x}(|x-y|^{-1}) D_i D_j(\phi u_i u_j)\, dy \\ &+ \sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-1} D_{k,x}(D_i D_j(\phi u_i u_j))\, dy \\ &= I_1 + I_2 \end{aligned}$$

where

$$I_1 = \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{k,x}(|x-y|^{-1}) D_i D_j(\phi u_i u_j)\, dy$$

and

$$I_2 = \sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-1} D_{k,x}(D_i D_j(\phi u_i u_j))\, dy$$

Let us first consider I_1. We note from Corollary D.2.2 in Appendix D.2 that there is a constant C_1, independent of δ, t, and f.

$$|D_iD_j(\phi_1 u_i u_j)| \leq C_1[\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-2}\|u\|_\infty^2] = C_1 H(u)$$

Also note that since

$$D_{k,x}(|x-y|^{-1}) = \frac{x_k - y_k}{|x-y|^3}$$

we have

$$|D_{k,x}(|x-y|^{-1})| = \left|\frac{x_k - y_k}{|x-y|^3}\right| \leq |x-y|^{-2}$$

Thus

$$\begin{aligned}
|I_1| &= \Big|\sum_{i,j}\int_{B(x,2\delta)} D_{k,x}(|x-y|^{-1})D_iD_j(\phi u_i u_j)\,dy\Big| \\
&\leq \sum_{i,j} C_0 \int_{B(x,2\delta)} |D_{k,x}(|x-y|^{-1})||D_iD_j(\phi u_i u_j)|\,dy \\
&\leq \sum_{i,j} C_0 \int_{B(x,2\delta)} |D_{k,x}(|x-y|^{-1})|H(u)\,dy \\
&\leq C_0 C_1 H(u)\int_{B(x,2\delta)} |D_{k,x}(|x-y|^{-1})\,dy \\
&\leq C_0 C_1 H(u)\int_{B(x,2\delta)} |x-y|^{-2}|\,dy \\
&= C_0 C H(u)\omega_3 \int_{|x-y|\leq 2\delta} \frac{1}{r^2} r^2\,dr \\
&= C_0 C H(u)\omega_3(2\delta) \\
&= C_0 C \omega_3(2\delta) C_1[\|Du\|_\infty^2 + \delta^{-1}\|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-2}\|u\|_\infty^2] \\
&\leq C_4(\delta\|\mathcal{D}u\|_\infty^2 + \|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-1}\|u\|_\infty^2)
\end{aligned}$$

It is a well-known identity that there is a constant C such that for $\varepsilon > 0$:

$$|AB| \leq C\Big(\varepsilon^2 A^2 + \frac{1}{\varepsilon^2}B^2\Big) \tag{4.14}$$

If we let $A = \|\mathcal{D}u\|_\infty$, $B = \|u\|_\infty$, and $\varepsilon = \sqrt{\delta}$, we apply (4.14), and find that

$$\|u\|_\infty \|\mathcal{D}u\|_\infty \leq C_5(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)$$

so that

$$\begin{aligned}
|I_1| &\leq C_4(\delta\|\mathcal{D}u\|_\infty^2 + \|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-1}\|u\|_\infty^2 \\
&\leq C_4(\delta\|\mathcal{D}u\|_\infty^2 + C_5(\delta\|Du\|_\infty^2 + \delta^{-1}\|u\|_\infty^2) + \delta^{-1}\|u\|_\infty^2 \\
&\leq C_6(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)
\end{aligned}$$

We finally conclude

$$|I_1| \leq C_6(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)$$

Next we turn to I_2:

$$I_2 = \sum_{i,j}\int_{B(x,2\delta)} |x-y|^{-1}(D_iD_j(D_{k,x}(\phi u_i u_j)))\,dy$$

First we observe that in view of Corollary D.2.3 we have

$$|D_iD_j(D_{k,x}(\phi_2 u_i u_j)| \leq C_1^*[\delta^{-1}\|\mathcal{D}u\|_\infty^2 + \delta^{-2}\|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-3}\|u\|_\infty^2] = G(u)$$

We note here as in the previous case that C_1^* is independent of t, δ, and f. The derivation, like the case for C_1, can be found in Appendix D. Thus we have

$$\begin{aligned}
|I_2| &= \Big|\sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-1} D_{k,x}(D_iD_j(D_{k,x}\phi u_i u_j))\,dy\Big| \\
&\leq C_0\sum_{i,j}\int_{B(x,2\delta)} |x-y|^{-1}|D_{k,x}(D_iD_j(D_{k,x}\phi u_i u_j))|\,dy \\
&\leq C_0 G(u)\int_{B(x,2\delta)} |x-y|^{-1}\,dy \\
&= C_0 G(u)\omega_3 \int_0^{2\delta} \frac{1}{r} r^2\,dr \\
&= C_0 G(u)\omega_3 2\delta^2 \\
&= C_0\omega_3 2\delta^2 C_1^*[\delta^{-1}\|\mathcal{D}u\|_\infty^2 + \delta^{-2}\|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-3}\|u\|_\infty^2] \\
&= C_2^*(\delta\|\mathcal{D}u\|_\infty^2 + \|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-1}\|u\|_\infty^2)
\end{aligned}$$

We note here that C_2^* is a combination of the surface measure of the unit sphere in $\mathbb{R}^3$, C_0, and C_1^*. C_1^* depends only on the maximum norms of ϕ, ϕ', etc. We note here that C_2 does not depend on t, δ, and f. Again, letting $A = \|\mathcal{D}u\|_\infty$, $B = \|u\|_\infty$, and $\varepsilon = \sqrt{\delta}$, and applying (4.14), we find that

$$\|u\|_\infty \|\mathcal{D}u\|_\infty \leq C_3(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)$$

and once again

$$\begin{aligned}
|I_2| &\leq C_2(\delta\|\mathcal{D}u\|_\infty^2 + \|u\|_\infty\|\mathcal{D}u\|_\infty + \delta^{-1}\|u\|_\infty^2) \\
&\leq C_2(\delta\|\mathcal{D}u\|_\infty^2 + C_3(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2) + \delta^{-1}\|u\|_\infty^2) \\
&\leq C_2^* C_3(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2) \\
&= C_4^*(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)
\end{aligned}$$

Again we conclude

$$|I_2| \leq C_4^*(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)$$

We find now that

$$\begin{aligned}
|D_{k,x}(p_{loc1}(x))| &= |I_1 + I_2| \\
&\leq |I_1| + |I_2| \\
&\leq C_6(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2) + C_4^*(\delta\|Du\|_\infty^2 + \delta^{-1}\|u\|_\infty^2) \\
&\leq C_B(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)
\end{aligned}$$

Thus

$$|Dp_{loc}| \leq C_B(\delta\|\mathcal{D}u\|_\infty^2 + \delta^{-1}\|u\|_\infty^2)$$

This finally indicates that for the maximum norm of the derivatives

$$\|\mathcal{D}p_{loc}\|_\infty \leq C_B(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2)$$

We note that in performing the above calculations integration by parts was **not** used. Additionally, the cases for $(D_{ii}D_{k,x})(\phi)$ $(D_{kk}D_{k,x})(\phi)$ are similar and the requisite bounds follow.

Finally, we derive the estimate on $\|\mathcal{D}p_{glb}\|_\infty$. By equation (4.13) we have

$$p_{glb}(x) = \sum_{i,j} C_0 \int_{\mathbb{R}^3} D_{k,x}[K_{ij}(x-y)(1-\phi)(u_i(y)u_j(y))]\,dy$$

By definition of ϕ, we find that on $B(x,\delta)$, $1-\phi = 0$.

$$p_{glb}(x) = \sum_{i,j} C_0 \int_{|x-y|>\delta} D_{k,x}[K_{ij}(x-y)(1-\phi)(u_i(y)u_j(y))]\,dy$$

We apply $D_{k,x} = \partial/\partial x_k$ under the integral sign, and note that as $\phi(r) = 0$ for $r \geq 2$, $\phi'(r) = 0$ $r \geq 2$. Noting that $K_{ij}(x-y) = D_iD_j(|x-y|^{-1}$, for all i,j, we now compute:

$$\begin{aligned}
D_{k,x}(p_{glb}(x)) &= \sum_{i,j} C_0 \int_{|x-y|\geq\delta} D_{k,x}((D_iD_j(|x-y|^{-1})(1-\phi)(u_iu_j))\,dy \\
&= \sum_{i,j} C_0 \int_{|x-y|\geq\delta} (D_iD_jD_{k,x}(|x-y|^{-1}))(1-\phi)(u_iu_j)\,dy \\
&+ \sum_{i,j} C_0 \int_{|x-y|\geq\delta} (D_iD_j(|x-y|^{-1}))(D_{k,x}(1-\phi))(u_iu_j)\,dy \\
&= \sum_{i,j} C_0 \int_{|x-y|\geq\delta} (D_iD_jD_{k,x}(|x-y|^{-1}))(1-\phi)(u_iu_j)\,dy \\
&+ \sum_{i,j} C_0 \int_{2\delta\geq|x-y|>\delta} (D_iD_j(|x-y|^{-1}))(D_{k,x}(-\phi))(u_iu_j)\,dy \\
&= I_1 + I_2
\end{aligned}$$

where

$$I_1 = \sum_{i,j} C_0 \int_{|x-y|\geq\delta} (D_iD_jD_{k,x}(|x-y|^{-1}))(1-\phi)(u_iu_j)\,dy$$

and

$$I_2 = \sum_{i,j} C_0 \int_{2\delta\geq|x-y|>\delta} (D_iD_j(|x-y|^{-1}))(D_{k,x}(-\phi))(u_iu_j)\,dy$$

We now compute the bounds on I_1 and I_2. First, for I_1, we note that

$$(D_i D_j D_{k,x}(|x-y|^{-1})) = \frac{-5(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|x-y|^7}$$

which yields

$$|(D_i D_j D_{k,x}(|x-y|^{-1}))| \leq \frac{C_1}{|x-y|^4}$$

Also note that $|1-\phi| \leq 1 + \|\phi\|_\infty \leq C_2$ We now compute

$$\begin{aligned}
|I_1| &= \Big|\sum_{i,j} C_0 \int_{|x-y|\geq\delta} (D_i D_j D_{k,x}(|x-y|^{-1}))(1-\phi)(u_i u_j)\,dy\Big| \\
&\leq C_0 \sum_{i,j} \int_{|x-y|\geq\delta} |(D_i D_j D_{k,x}(|x-y|^{-1}))||(1-\phi)||(u_i u_j)|\,dy \\
&\leq C_0 C_2 C_1 \|u\|_\infty^2 \int_{|x-y|\geq\delta} \frac{1}{|x-y|^4}\,dy \\
&= \omega_2 C_0 C_1 C_2 \|u\|_\infty^2 \lim_{R\to\infty} \int_\delta^R \frac{1}{r^4} r^2\,dr \\
&= \omega_2 C_0 C_1 C_2 \|u\|_\infty^2 \lim_{R\to\infty} \Big[-\frac{1}{R} + \frac{1}{\delta}\Big] \\
&= C_3 \delta^{-1} \|u\|_\infty^2
\end{aligned}$$

Next we turn to I_2. We recall that

$$|D_{k,x}(\phi)| = \Big|\frac{x_k-y_k}{\delta|x-y|}\Big| \leq \frac{C}{\delta}$$

from Lemma D.1.1. Also we have

$$|(D_i D_j(|x-y|^{-1}))| = \Big|\frac{3(x_i-y_i)(x_j-y_j)}{|x-y|^5}\Big| \leq \frac{C_1^*}{|x-y|^3}$$

We now compute

$$\begin{aligned}
|I_2| &= \Big|\sum_{i,j}\frac{1}{4\pi}\int_{2\delta\geq|x-y|>\delta}(D_iD_j(|x-y|^{-1}))(D_{k,x}(-\phi))(u_iu_j)\,dy \\
&\leq C_0\sum_{i,j}\int_{2\delta\geq|x-y|>\delta}|(D_iD_j(|x-y|^{-1}))||(D_{k,x}(-\phi))||(u_iu_j)|\,dy \\
&\leq C_0C_1^*\delta^{-1}\|u\|_\infty^2\int_{2\delta\geq|x-y|>\delta}\frac{1}{|x-y|^3}\,dy \\
&= C_0C_1^*\omega_2\delta^{-1}\|u\|_\infty^2\int_\delta^{2\delta}\frac{1}{r^3}r^2\,dr \\
&= C_0C_1^*\omega_2\delta^{-1}\|u\|_\infty^2(\ln(2\delta)-\ln(\delta)) \\
&= C_0C_1^*\omega_2\delta^{-1}\|u\|_\infty^2\ln(2) \\
&= C_4\delta^{-1}\|u\|_\infty^2
\end{aligned}$$

This now implies

$$\begin{aligned}
|D_{k,x}p_{glb}(x)| &= |I_1+I_2| \\
&\leq |I_1|+|I_2| \\
&\leq C_3\delta^{-1}\|u\|_\infty^2+C_4\delta^{-1}\|u\|_\infty^2 \\
&\leq C_{gl}\delta^{-1}\|u\|_\infty^2
\end{aligned}$$

By taking the maximum norm of $D_{k,x}p_{glb}(x)$ we find that

$$\|\mathcal{D}p_{glb}\|_\infty\leq C_{gl}\delta^{-1}\|u\|_\infty^2$$

where C_{gl} is independent of t, δ, and f. Defining C by

$$C=\max\{C_A,C_B,C_{gl}\}$$

we obtain the bounds (4.9), (4.10), (4.11). □

The important fact to note here is that gradient of the pressure ∇p can be estimated in terms of that maximum norm of the derivatives. That is

$$\begin{aligned}
|\nabla p| &= |\nabla(p_{loc} + p_{glb})| \\
&\le |\nabla p_{loc}| + |\nabla p_{glb}| \\
&\le \|\mathcal{D} p_{loc}\|_\infty + \|\mathcal{D} p_{glb}\|_\infty \\
&\le C(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2) + C\delta^{-1}\|u\|_\infty^2 \\
&\le C(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2)
\end{aligned}$$

The gradient of the pressure term of the Navier-Stokes equation can be bounded in terms of the zero and first order *derivatives in maximum norm* of the velocity field. We will now show in turn that these terms will be bounded in terms of the L^∞ norm of the initial value function f. Recall that

$$u_t = \Delta u + Q, \quad Q = -\nabla p - u \cdot \nabla u, \quad u(x,0) = f$$

Writing $Q = Q_{loc} + Q_{gl}$, where

$$Q_{loc} = -\nabla p_{loc} - \sum_j D_j(u_j u) = D_j(-p_{loc} - \sum_j u_j u) \tag{4.15}$$

and

$$Q_{glb} = -\nabla p_{glb} \tag{4.16}$$

we will prove the following lemma:

Lemma 4.3.1. *Set*

$$V(t) = \|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty, \quad 0 < t < T(f) \tag{4.17}$$

There is a constant $C > 0$, independent of t and f, such that

$$V(t) \le C\|f\|_\infty + Ct^{1/2} \max_{0\le s\le t} V^2(s), \quad 0 < t < T(f) \tag{4.18}$$

Proof. Using Lemma 4.3.1, we note that there is a constant C independent of t, δ, and f such that

$$\|p_{loc}\|_\infty \le C(\|u\|_\infty^2 + \delta\|u\|_\infty\|\mathcal{D}u\|_\infty)$$

$$\|\mathcal{D}p_{loc}\|_\infty \le C(\delta^{-1}\|u\|_\infty^2 + \delta\|\mathcal{D}u\|_\infty^2)$$

$$\|\mathcal{D}p_{glb}\|_\infty \le C\delta^{-1}\|u\|_\infty^2$$

Now, since δ was arbitrary, let $\delta = t^{1/2}$ in the above expressions and obtain

$$\|p_{loc}\|_\infty + |u_j u|_\infty \le C(\|u\|_\infty^2 + t^{1/2}\|u\|_\infty\|\mathcal{D}u\|_\infty) \tag{4.19}$$

$$\|Q_{loc}\|_\infty \le C(t^{-1/2}\|u\|_\infty^2 + t^{1/2}\|\mathcal{D}u\|_\infty^2) \tag{4.20}$$

$$\|Q_{gl}\|_\infty \le Ct^{-1/2}\|u\|_\infty^2 \tag{4.21}$$

Suppressing the x variable in our notation, recall that the solution u to the Navier-Stokes equations can be written as

$$u(t) = e^{\Delta t}f + \int_0^t e^{\Delta(t-s)}Q(s)\,ds = u_h(t) + u_{nh}(t)$$

First note that from (4.2)

$$\|u_h\|_\infty = \|e^{\Delta t}f\|_\infty \le \|f\|_\infty$$

Next, we consider the non-homogeneous term $u_{nh}(t)$, and write

$$\begin{aligned} u_t &= \Delta u + Q_{loc} + Q_{gl} \\ &= \Delta u + D_k(-p_{loc} - \sum_j u_j u) - \nabla p_{glb} \\ &= \Delta u + D_k F_1 + F_2 \end{aligned}$$

with $F_1 = -p_{loc} - \sum_j u_j u$, and $F_2 = \nabla p_{glb}$. The solution to

$$u_t = \Delta u + D_j F_1$$

is determined by the fact that Q_{loc} is obtained by applying one space derivative to p_{loc} and $u_j u$. Then by equations (4.6), and (4.19), and referring to this solution as u_1

$$\begin{aligned}
\|u_1\|_\infty &\leq C \max_{0\leq s\leq t}\{s^{1/2}\| - p_{loc} - \sum_j u_j u\|_\infty\} \\
&\leq C \max_{0\leq s\leq t}\{s^{1/2}C(\|u\|_\infty^2 + s^{1/2}\|u\|_\infty\|\mathcal{D}u\|_\infty)\} \\
&\leq C \max_{0\leq s\leq t}\{s^{1/2}\|u(s)\|_\infty^2 + s\|u(s)\|_\infty\|\mathcal{D}u(s)\|_\infty\}
\end{aligned}$$

The solution to

$$u_t = \Delta u + F_2$$

is given by equations (4.5) and (4.21). Denoting this by u_2:

$$\begin{aligned}
\|u_2\|_\infty &\leq 2t^{1/2} \max_{0\leq s\leq t}\{s^{1/2}\|Q_{gl}(s)\|_\infty\} \\
&\leq 2t^{1/2} \max_{0\leq s\leq t}\{s^{1/2}Cs^{-1/2}\|u(s)\|_\infty^2\} \\
&\leq Ct^{1/2} \max_{0\leq s\leq t} \|u(s)\|_\infty^2
\end{aligned}$$

Thus the solution to

$$u_t = \Delta u + Q_{loc} + Q_{gl}$$

is bounded by

$$\begin{aligned}
\|u_{nh}(t)\|_\infty &= \|u_1 + u_2(t)\|_\infty \\
&\leq \|u_1(t)\|_\infty + \|u_2\|_\infty \\
&\leq C \max_{0\leq s\leq t}\{s^{1/2}\|u(s)\|_\infty^2 + s\|u(s)\|_\infty\|\mathcal{D}u(s)\|_\infty\} + Ct^{1/2} \max_{0\leq s\leq t} \|u(s)\|_\infty^2
\end{aligned}$$

We now compute:

$$\begin{aligned}
\|u\|_\infty &\leq \|u_h + u_{nh}\|_\infty \\
&\leq \|u_h\|_\infty + \|u_{nh}\|_\infty \\
&\leq \|f\|_\infty + C \max_{0\leq s\leq t}\{s^{1/2}\|u(s)\|_\infty^2 + s\|u(s)\|_\infty\|\mathcal{D}u(s)\|_\infty\} \\
&+ Ct^{1/2} \max_{0\leq s\leq t} \|u(s)\|_\infty^2 \\
&\leq \|f\|_\infty + Ct^{1/2} \max_{0\leq s\leq t}(\|u(s)\|_\infty^2 + s^{1/2}\|u(s)\|_\infty\|\mathcal{D}u(s)\|_\infty) \\
&\leq \|f\|_\infty + Ct^{1/2} \max_{0\leq s\leq t}(\|u(s)\|_\infty^2 + s^{1/2}\|u(s)\|_\infty\|\mathcal{D}u(s)\|_\infty + s\|\mathcal{D}u(s)\|_\infty^2) \\
&= \|f\|_\infty + Ct^{1/2} \max_{0\leq s\leq t} V^2(s)
\end{aligned}$$

Next, consider $v(t) = D_k u(t)$. We have

$$v_t = \Delta v + D_k Q$$

and, from equations (4.20) and (4.21) we have

$$\|Q\|_\infty \leq \| \leq C(t^{-1/2}\|u\|_\infty^2 + t^{1/2}\|\mathcal{D}u\|_\infty^2) \tag{4.22}$$

Now, from equation (4.3) with $j = 1$ we have

$$\|\mathcal{D}e^{\Delta t} f\|_\infty \leq Ct^{-1/2}\|f\|_\infty$$

for the homogeneous part. For the non-homogeneous part equation (4.6) yields

$$\begin{aligned}
\|v\|_\infty &\leq C \max_{0\leq s\leq t}\{s^{1/2}\|Q(s)\|_\infty\} \\
&= C \max_{0\leq s\leq t}\{s^{1/2}(C(s^{-1/2}\|u(s)\|_\infty^2 + s^{1/2}\|\mathcal{D}u(s)\|_\infty^2))\} \\
&\leq C \max_{0\leq s\leq t}(\|u(s)\|_\infty^2 + s\|\mathcal{D}u(s)\|_\infty^2) \\
&\leq C \max_{0\leq s\leq t}(\|u(s)\|_\infty^2 + s\|\mathcal{D}u(s)\|_\infty^2) \\
&\leq C \max_{0\leq s\leq t} V^2(s)
\end{aligned}$$

Then for $v = D_k u$

$$
\begin{aligned}
t^{1/2}\|v(t)\|_\infty &\leq t^{1/2}(\|\mathcal{D}e^{\Delta t} f\|_\infty + \|v\|_\infty) \\
&\leq t^{1/2}(Ct^{-1/2}\|f\|_\infty + C \max_{0\leq s\leq t} V^2(s) \\
&= C\|f\|_\infty + Ct^{1/2} \max_{0\leq s\leq t} V^2(s)
\end{aligned}
$$

Thus the lemma is proved for u and the derivatives $D_k u$. □

Lemma 4.3.1 allows us to estimate $\|u\|_\infty$ and $\|\mathcal{D}u\|_\infty$ in terms of $\|f\|_\infty$ for a small time interval.

Lemma 4.3.2. *Let $C > 0$ denote the constant in estimate* (4.18) *from Lemma 4.3.1. Let*

$$c_0 = \frac{1}{16C^4}$$

Then

$$T(f) > \frac{c_0}{\|f\|_\infty^2}$$

and

$$\|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty < 2C\|f\|_\infty \quad \textit{for} \quad 0 \leq t < \frac{c_0}{\|f\|_\infty^2} \tag{4.23}$$

Proof. We use the definition of $V(t)$ in equation (4.17). Assume that equation (4.23) does not hold. and that

$$0 \leq t \leq \frac{c_0}{\|f\|_\infty^2}$$

Then there are times t where

$$\|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty \geq 2C\|f\|_\infty$$

Let t_0 be the smallest time for which $V(t_0) = 2C\|f\|_\infty$. Then

$$\begin{aligned} 2C\|f\|_\infty &= V(t_0) \\ &\leq C\|f\|_\infty + Ct_0^{1/2} \max_{0\leq s\leq t_0} V^2(s) \\ &\leq C\|f\|_\infty + Ct_0^{1/2} 4C^2\|f\|_\infty^2 \end{aligned}$$

Then

$$C\|f\|_\infty \leq Ct_0^{1/2} 4C^2\|f\|_\infty^2$$

and

$$1 \leq 4C^2 t_0^{1/2}\|f\|_\infty$$

We now have

$$t_0 \geq \frac{c_0}{\|f\|_\infty^2}$$

This contradiction of the time interval proves (4.23). Next, since

$$\lim \sup_{t\to T(f)} \|u(t)\|_\infty = \infty$$

if $T(f)$ is finite, then as $t \to T(f)$ we have, since

$$\|u\|_\infty \leq \|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty < 2C\|f\|_\infty$$

which yields $2C\|f\|_\infty = \infty$, a contradiction as $f \in L^\infty$ so we must have

$$T(f) > \frac{c_0}{\|f\|_\infty^2}$$

The theorem is now proved. □

We will point out here that the calculations for Theorem 4.3.1 were carried out in the original KL paper without much detail. In this paper the appendix establishes the important bounds on the derivatives of ϕ and the variations on the derivatives $D_{ij}(\phi u_i u_j)$, etc. This is important as the whole crux of the paper depends on the bounds on the local pressure and the *derivatives* of the local and global pressure in terms of u and $\|\mathcal{D}u\|_\infty$ (equations (4.9), (4.9), and(4.9)). It is crucial that the independence of the constants from t and f is established here.

Lemma 4.3.2 produces equation (4.23). Again, this was calculated in the original paper, but the main point of this is that

$$\|u\|_\infty \leq C\|f\|_\infty$$

and

$$t^{1/2}\|\mathcal{D}u\|_\infty \leq C\|f\|_\infty$$

Explicitly, the velocity field u is bounded above by $\|f\|_\infty$. This means that u depends on the initial value function f. Additionally, this is actually the beginning of the induction proof alluded to but not formally proven in the original paper. In the final chapter we conclude the paper with estimates on the derivatives of u for all orders j. We will show that if u is a solution to the Navier-Stokes equation that all derivatives of u are bounded in maximum norm by $\|f\|_\infty$. That is

$$t^{j/2}\|\mathcal{D}^j u\|_\infty \leq K_j\|f\|_\infty$$

for all orders j in maximum norm.

Chapter 5

The Kreiss–Lorenz Paper: Estimates for the Derivatives

5.1 Computations for Derivative Estimates

We now prove the final theorem of this paper. We show that if u is a solution to the Navier-Stokes equation that there are constants K_j

$$t^{j/2}\|\mathcal{D}^j u(t)\|_\infty \leq K_j \|f\|_\infty$$

We again assume throughout this chapter, as in Chapter 4, that $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and that $D^\alpha u \in L^\infty(\mathbb{R}^3)$ for all orders α. Additionally, we assume that t exists on a maximal time interval $0 \leq t \leq T(f)$. The following theorems involving the pressure are proved in Appendix E (Theorem E.1.1).

Theorem 5.1.1. *Consider the Navier-Stokes equation*

$$v_t = \Delta v + D^j Q, \quad v = D^j u$$

u a solution, and where

$$Q = -\nabla p - u \cdot \nabla u$$

Let $j \geq 1$ and assume that for $0 \leq k \leq j-1$ there are constants K_k independent of t and f such that

$$t^{k/2}\|\mathcal{D}^k u(t)\|_\infty \leq K_k\|f\|_\infty \quad \textit{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2} \tag{5.1}$$

Then there exists a constant C independent of t and f such that

$$\|D^{j-1}(D_{q,x}(p_{loc}(x))\|_\infty \leq C(\|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2) \tag{5.2}$$

Our next theorem concerns $\|D_{q,x}p_{glb}\|_\infty$. It is proved in Appendix E (Theorem E.2.1).

Theorem 5.1.2. *Consider the Navier-Stokes equation*

$$v_t = \Delta v + D^j Q, \quad v = D^j u$$

u a solution, and where

$$Q = -\nabla p - u \cdot \nabla u$$

Let $j \geq 1$ and assume that for $0 \leq k \leq j-1$ there are constants K_k independent of t and f such that

$$t^{k/2}\|\mathcal{D}^k u(t)\|_\infty \leq K_k\|f\|_\infty \quad \textit{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

Then there exists a constant C independent of t and f such that

$$\|D^{j-1}(D_{q,x}(p_{glb}(x))\|_\infty \leq C\|f\|_\infty^2 t^{-j/2} \tag{5.3}$$

We now will prove the main proposition.

Proposition 5.1.1. *Consider the Cauchy problem for the Navier-Stokes equations (1.1) and (1.2), where $f \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and $\nabla \cdot f = 0$. Then there is a constant $c_0 > 0$, and for every $j = 0, 1, 2, \ldots$ there is a constant K_j such that for an interval*

$$0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

we have

$$t^{j/2}\|\mathcal{D}^j u(t)\|_\infty \leq K_j \|f\|_\infty$$

The constants c_0 and K_j are independent of t and f.

Proof. We consider an induction argument. Let t be in the interval

$$0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

where c_0 is the constant from Lemma 4.3.1. Letting $j = 0$, Lemma 4.3.2 yields that on the considered interval

$$\|u(t)\|_\infty \leq \|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty < 2C\|f\|_\infty$$

We let $K_0 = 2C$, and obtain $\|u(t)\|_\infty \leq K_0\|f\|_\infty$. Next, consider $j = 1$. Again by Lemma 4.3.2 we have

$$t^{1/2}\|\mathcal{D}u\|_\infty \leq \|u\|_\infty + t^{1/2}\|\mathcal{D}u\|_\infty < 2C\|f\|_\infty$$

Again, letting $K_1 = 2C$, we obtain

$$t^{1/2}\|\mathcal{D}u\|_\infty \leq K_1\|f\|_\infty$$

We note that the constants $K_0 = K_1 = 2C$ are independent of t and f via Lemma 4.3.1. Let $j \geq 1$ and assume that for $0 \leq k \leq j - 1$

$$t^{k/2}\|\mathcal{D}^k u(t)\|_\infty \leq K_k\|f\|_\infty \quad \text{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2} \tag{5.4}$$

Applying D^j to $u_t = \Delta u + Q(s)$, and letting $v = D^j u$ we obtain

$$v_t = \Delta v + D^j Q$$

and the solution

$$v(t) = D^j e^{\Delta t} f + \int_0^t e^{\Delta(t-s)} D^j Q(s)\, ds$$

We must now estimate

$$\|v(t)\|_\infty = \|\mathcal{D}^j u(t)\|_\infty$$

We note that

$$\begin{aligned}\|v(t)\|_\infty &= \left\|D^j e^{\Delta t} f + \int_0^t e^{\Delta(t-s)} D^j Q(s)\, ds\right\|_\infty \\ &\leq \|D^j e^{\Delta t} f\|_\infty + \left\|\int_0^t e^{\Delta(t-s)} D^j Q(s)\, ds\right\|_\infty \\ &= \|T_1\|_\infty + \|T_2\|_\infty\end{aligned}$$

First note that taking the maximum estimate of all D^j of order j and using (4.3) we have

$$\|T_1\|_\infty \leq \|\mathcal{D}^j e^{\Delta t} f\|_\infty \leq L_j t^{-j/2} \|f\|_\infty$$

where L_j is a constant not depending on t or f. We now consider T_2:

$$\begin{aligned}T_2 &= \int_0^t e^{\Delta(t-s)} D^j Q(s)\, ds \\ &= \int_0^{t/2} e^{\Delta(t-s)} D^j Q(s)\, ds + \int_{t/2}^t e^{\Delta(t-s)} D^j Q(s)\, ds \\ &= J_1 + J_2\end{aligned}$$

We first consider J_1. Applying integration by parts j times, on the interval $[0, t/2]$, the integral exists and we compute

$$\begin{aligned}
\|J_1\|_\infty &= \Big\| \int_0^{t/2} e^{\Delta(t-s)} D^j Q(s)\, ds \Big\|_\infty \\
&= \Big\| \int_0^{t/2} D^j e^{\Delta(t-s)} Q(s)\, ds \Big\|_\infty \\
&\le \|Q\|_\infty \Big\| \int_0^{t/2} D^j e^{\Delta(t-s)}\, ds \Big\|_\infty \\
&= \|Q\|_\infty C_1 \int_0^{t/2} (t-s)^{-j/2} \\
&\le \|Q\|_\infty C_2 (t^{-j/2+1} - (t/2)^{-j/2+1}) \\
&\le \|Q\|_\infty C_3 t^{-j/2+1}
\end{aligned}$$

Recall that from equation (4.22), and the induction assumption

$$\begin{aligned}
\|Q\|_\infty &\le C(t^{-1/2}\|u\|_\infty^2 + t^{1/2}\|\mathcal{D}u\|_\infty^2) \\
&\le C(t^{-1/2}K_0^2\|f\|_\infty^2 + t^{1/2}t^{-1}K_1^2\|f\|_\infty^2) \\
&\le C_4 t^{-1/2}\|f\|_\infty^2
\end{aligned}$$

Then

$$\begin{aligned}
\|J_1\|_\infty &\le \|Q\|_\infty C_3 t^{-j/2+1} \\
&\le C_4 t^{-1/2}\|f\|_\infty^2 C_3 t^{-j/2+1} \\
&\le C_5 \|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

Next when estimating J_2 we can only transfer from $D^j Q$ to the heat semigroup one derivative. Moving more derivatives will cause the integral to be non- integrable. We have by integration by parts

$$\begin{aligned}
J_2 &= \int_{t/2}^{t} e^{\Delta(t-s)} D^j Q(s)\, ds \\
&= \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} Q(s)\, ds \\
&= \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} (Q_{loc}(s) + Q_{gl}(s))\, ds \\
&= \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} Q_{loc}(s)\, ds + \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} Q_{gl}(s)\, ds \\
&= J_3 + J_4
\end{aligned}$$

Now, recall that $Q_{loc} = D_i(-p_{loc} - u_i u)$ (4.15) so that by replacing this expression in J_3 we have

$$\begin{aligned}
J_3 &= \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} (D_i(-p_{loc} - u_i u))(s)\, ds \\
&= \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} D_i(-p_{loc}(s))\, ds - \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} (D_i(u_i u)(s))\, ds \\
&= S_1 + S_2
\end{aligned}$$

First we bound S_1. Recalling from Theorem 5.1.1 that there is a constant C_1 independent of t and f such that

$$\|D^{j-1}(D_{q,x} p_{loc})\|_\infty \leq C_1(\|f\|_\infty \|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2} \|f\|_\infty^2 + t^{-j/2} \|f\|_\infty^2)$$

we estimate $\|S_1\|_\infty$:

$$
\begin{aligned}
\|S_1\|_\infty &= \left\| \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} D_i(-p_{loc}(s))\, ds \right\|_\infty \\
&\leq \int_{t/2}^{t} De^{\Delta(t-s)} \|D^{j-1}(D_{q,x} p_{loc})\|_\infty\, ds \\
&\leq \int_{t/2}^{t} (t-s)^{-1/2} \|D^{j-1}(D_{q,x}) p_{loc}\|_\infty\, ds \\
&\leq \int_{t/2}^{t} (t-s)^{-1/2} C_1 (\|f\|_\infty \|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + s^{-j/2}\|f\|_\infty^2)\, ds \\
&\leq \int_{t/2}^{t} (t-s)^{-1/2} C_1 \|f\|_\infty \|\mathcal{D}^j u\|_\infty\, ds + C_1 \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2} \|f\|_\infty^2)\, ds \\
&\leq S_3 + S_4
\end{aligned}
$$

We consider S_3. Theorem D.3.2 yields the result that

$$
\int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2}\, ds = t^{(1-j)/2} B_{1/2}(1/2, 1 + j/2) = B t^{(1-j)/2}
$$

where $B = B_{1/2}(1/2, 1 + j/2)$ is the *incomplete beta function*. Then S_3 can be bounded by

$$
\begin{aligned}
S_3 &= \int_{t/2}^{t} (t-s)^{-1/2} C_1 \|f\|_\infty \|\mathcal{D}^j u\|_\infty\, ds \\
&= C_1 \int_{t/2}^{t} (t-s)^{-1/2} \|\mathcal{D}^j u\|_\infty\, ds \\
&= C_1 \|f\|_\infty \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2} s^{j/2} \|\mathcal{D}^j u\|_\infty\, ds \\
&\leq C_1 \|f\|_\infty \max_{0 \leq s \leq t} \{s^{j/2} \|\mathcal{D}^j u(s)\|_\infty\} \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2}\, ds \\
&\leq C_1 B \|f\|_\infty t^{(1-j)/2} \max_{0 \leq s \leq t} \{s^{j/2} \|\mathcal{D}^j u(s)\|_\infty\} \\
&\leq C_2 \|f\|_\infty t^{(1-j)/2} \max_{0 \leq s \leq t} \{s^{j/2} \|\mathcal{D}^j u(s)\|_\infty\}
\end{aligned}
$$

Next for S_4:

$$\begin{aligned} S_4 &= C_1 \int_{t/2}^{t} (t-s)^{-1/2} t^{-j/2} \|f\|_\infty^2)\, ds \\ &= C_1 \|f\|_\infty^2 \int_{t/2}^{t} (t-s)^{-1/2} t^{-j/2} \|f\|_\infty^2)\, ds \\ &= C_1 \|f\|_\infty^2 B t^{(1-j)/2} \\ &\leq C_3 \|f\|_\infty^2 t^{(1-j)/2} \end{aligned}$$

So we have

$$\begin{aligned} \|S_1\|_\infty &\leq S_3 + S_4 \\ &\leq C_2 \|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t} \{ s^{j/2} \|\mathcal{D}^j u(s)\|_\infty \} + C_3 \|f\|_\infty^2 t^{(1-j)/2} \end{aligned}$$

Now for S_2. We first observe that

$$D_i(u_i u)(s) = (uDu_i)(s) + (u_i Du)(s)$$

For each of these pieces, we use Lemma D.3.1 (equation (D.3.1)) with $l = j$. We note by the induction argument that, for $0 \leq m \leq j-1$

$$\|\mathcal{D}^m u\|_\infty \leq t^{-m/2} K_m \|f\|_\infty$$

Writing $M_1(j) = M_{1j}$:

$$\begin{aligned} \|D^{j-1}(D_i(u_i u)(s))\|_\infty &\leq M_{1j}(\|u\|_\infty \|\mathcal{D}^j u\|_\infty + t^{-j/2} \|f\|_\infty^2) \\ &\leq M_{1j} C_1 \|f\|_\infty \|\mathcal{D}^j u\|_\infty + C_2 t^{-j/2} \|f\|_\infty^2 \\ &\leq M_{2j} \|f\|_\infty \|\mathcal{D}^j u\|_\infty + C_2 t^{-j/2} \|f\|_\infty^2 \end{aligned}$$

It is important to note here that M_{2j} is a constant that depends on

$$\max_m K_m \quad \text{and} \quad \max \binom{n}{r} \quad 0 \leq m, r, n \leq j-1$$

and not on t or f. Then we compute:

$$\begin{aligned}
\|S_2\|_\infty &= \left\| \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1}(D_i(u_i u)(s))\, ds \right\|_\infty \\
&\leq \int_{t/2}^{t} De^{\Delta(t-s)} \|D^{j-1}(D_i(u_i u)(s))\|_\infty\, ds \\
&\leq \int_{t/2}^{t} (t-s)^{-1/2}(M_{2j}\|f\|_\infty \|\mathcal{D}^j u\|_\infty + C_2 t^{-j/2}\|f\|_\infty^2)\, ds \\
&\leq M_{2j}\|f\|_\infty \int_{t/2}^{t} (t-s)^{-1/2}\|\mathcal{D}^j u\|_\infty\, ds + \int_{t/2}^{t} (t-s)^{-1/2} C_2 t^{-j/2}\|f\|_\infty^2\, ds \\
&\leq M_{3j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_4\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

through the same computation as for S_1. Then we bound J_3

$$\begin{aligned}
\|J_3\|_\infty &= \|S_1 + S_2\|_\infty \\
&\leq \|S_1\|_\infty + \|S_2\|_\infty \\
&\leq M_{4j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_3\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

Now for J_4. Recalling that, from Theorem 5.1.2 the existence of a constant C_6 independent of t and f such that

$$\|D^{j-1}(D_{q,x}(p_{glb}(x))\|_\infty \leq C\|f\|_\infty^2 t^{-j/2}$$

$$\begin{aligned}
\|J_4\|_\infty &= \left\| \int_{t/2}^{t} De^{\Delta(t-s)} D^{j-1} Q_{gl}(s)\, ds \right\|_\infty \\
&\leq C_6\|f\|_\infty^2 \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2}\, ds \\
&\leq C_6\|f\|_\infty^2 B t^{(1-j)/2} \\
&= C_7\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

Then we have

$$\begin{aligned}
\|J_2\|_\infty &\leq \|J_3\|_\infty + \|J_4\|_\infty \\
&\leq M_{4j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} \\
&+ C_3\|f\|_\infty^2 t^{(1-j)/2} + C_7\|f\|_\infty^2 t^{(1-j)/2} \\
&\leq M_{5j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_8\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

Thus T_2 is bounded by

$$\begin{aligned}
\|T_2\|_\infty &\leq \|J_1\|_\infty + \|J_2\|_\infty \\
&\leq C_5\|f\|_\infty^2 t^{(1-j)/2} \\
&+ M_{5j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_8\|f\|_\infty^2 t^{(1-j)/2} \\
&\leq M_{6j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_9\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

We finally bound $\|v(t)\|_\infty$:

$$\begin{aligned}
\|v(t)\|_\infty &\leq \|T_1\|_\infty + \|T_2\|_\infty \\
&\leq L_j t^{-j/2}\|f\|_\infty \\
&+ M_{6j}\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C_9\|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

We now have an expression for $t^{j/2}\|v(t)\|_\infty$:

$$\begin{aligned}
t^{j/2}\|v(t)\|_\infty &\leq t^{j/2}(L_j t^{-j/2}\|f\|_\infty \\
&+ M_j\|f\|_\infty t^{(1-j)/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} \\
&+ C\|f\|_\infty^2 t^{(1-j)/2}) \\
&\leq L_j\|f\|_\infty \\
&+ M_j\|f\|_\infty t^{1/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\} + C\|f\|_\infty^2 t^{1/2} \\
&\leq C_j\|f\|_\infty + C\|f\|_\infty^2 t^{1/2} + C\|f\|_\infty t^{1/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\}
\end{aligned}$$

Now, as $v = D^j u$, we maximize the resulting estimates of $t^{j/2}\|D^j u\|_\infty$ over all derivatives D^j of order j and derive

$$t^{j/2}\|\mathcal{D}^j u\|_\infty \leq C_j\|f\|_\infty + C_j\|f\|_\infty^2 t^{1/2} + C_j\|f\|_\infty t^{1/2} \max_{0\leq s\leq t}\{s^{j/2}\|\mathcal{D}^j u(s)\|_\infty\}$$

Now, define

$$\psi(t) = t^{j/2}\|\mathcal{D}^j u\|_\infty$$

We have the estimate

$$\psi(t) \leq C_j\|f\|_\infty + C_j\|f\|_\infty^2 t^{1/2} + C_j\|f\|_\infty t^{1/2} \max_{0\leq s\leq t}\psi(s)$$

Recall the assumption that

$$0 < t \leq \frac{c_0}{\|f\|_\infty^2} \quad c_0 = \frac{1}{16C^4}$$

where C is the constant from Lemma 4.3.1, equation (4.18). Then

$$t^{1/2}\|f\|_\infty \leq \sqrt{c_0}$$

and the term $C\|f\|_\infty^2 t^{1/2}$ is bounded by

$$C\|f\|_\infty^2 t^{1/2} = (C\|f\|_\infty)(\|f\|_\infty t^{1/2}) \leq C\|f\|_\infty\sqrt{c_0} = C\sqrt{c_0}\|f\|_\infty$$

so that

$$\begin{aligned}\psi(t) &\leq C_j\|f\|_\infty + C\sqrt{c_0}\|f\|_\infty + C\|f\|_\infty t^{1/2}\max_{0\leq s\leq t}\psi(s)\\ &\leq (C_j + C\sqrt{c_0})\|f\|_\infty + C\|f\|_\infty t^{1/2}\max_{0\leq s\leq t}\psi(s)\\ &\leq C_j\|f\|_\infty + C_j\|f\|_\infty t^{1/2}\max_{0\leq s\leq t}\psi(s)\end{aligned}$$

Thus we have

$$\psi(t) \leq C_j\|f\|_\infty + C_j\|f\|_\infty t^{1/2}\max_{0\leq s\leq t}\psi(s) \quad \text{for} \quad 0 \leq t \leq \frac{c_0}{\|f\|_\infty^2} \tag{5.5}$$

We note that the constant C_j is a maximum of all constants appearing in the above and is independent of t and f. Fix this constant so that (5.5) holds. Let

$$c_j = \min\left\{c_0, \frac{1}{4C_j^2}\right\}$$

We first claim that

$$\psi(t) < 2C_j\|f\|_\infty \quad \text{for} \quad 0 \le t \le \frac{c_j}{\|f\|_\infty^2}$$

Assume not. Then let $0 < t_0 < c_j/\|f\|_\infty^2$ denote the smallest time with $\psi(t_0) = 2C_j\|f\|_\infty$. Then from (5.5)

$$\begin{aligned} 2C_j\|f\|_\infty &= \psi(t_0) \le C_j\|f\|_\infty + C_j\|f\|_\infty t_0^{1/2} \max_{0\le s\le t_0} \psi(s) \\ &\le C_j\|f\|_\infty + C_j\|f\|_\infty t_0^{1/2} \cdot 2C_j\|f\|_\infty \\ &= C_j\|f\|_\infty + 2t_0^{1/2} C_j^2 \|f\|_\infty^2 \end{aligned}$$

or

$$2C_j\|f\|_\infty \le C_j\|f\|_\infty + 2C_j^2\|f\|_\infty^2$$

Then

$$C_j\|f\|_\infty \le 2t_0^{1/2} C_j^2 \|f\|_\infty^2$$

and

$$1 \le 2C_j\|f\|_\infty t_0^{1/2}$$

This forces

$$t_0 \ge \frac{1}{4\|f\|_\infty^2 C_j^2} \ge \frac{c_j}{\|f\|_\infty^2}$$

a contradiction. So we must have

$$t^{j/2}\|\mathcal{D}^j u\|_\infty \le 2C_j\|f\|_\infty \tag{5.6}$$

Then the statement is true for j with $K_j = 2C_j$. Suppose now that

$$T_j = \frac{c_j}{\|f\|_\infty^2} \leq t \leq \frac{c_0}{\|f\|_\infty^2} = T_0 \tag{5.7}$$

Then we restart the argument at $t - T_j$. As $T_j \leq t \leq T_0$, $0 \leq t - T_j \leq T_0 - T_j \leq T_0$ From Lemma 4.3.2 we have

$$\|u(t - T_j)\|_\infty \leq 2\|f\|_\infty$$

and we obtain

$$t^{j/2}\|\mathcal{D}^j u\|_\infty \leq 2C_j\|f\|_\infty \quad \text{for} \quad 0 \leq t \leq T_j$$

For $t = T_j$ we obtain

$$T^{j/2}\|\mathcal{D}^j u\|_\infty \leq 4C_j\|f\|_\infty \quad \text{for} \quad 0 \leq t \leq T_j \tag{5.8}$$

Finally, for any t with (5.6) we have

$$T_0 = \frac{c_0}{c_j}T_j$$

and if $t \leq T_0$

$$t^{j/2} \leq T_0^{j/2} = \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}$$

Now we have

$$\begin{aligned} t^{j/2}\|\mathcal{D}^j u\|_\infty &\leq T_0^{j/2}\|\mathcal{D}^j u\|_\infty \\ &\leq \left(\frac{c_0}{c_j}\right)^{j/2} T_j^{j/2}\|\mathcal{D}^j u\|_\infty \\ &\leq 4C_j\left(\frac{c_0}{c_j}\right)^{j/2}\|f\|_\infty \\ &= K_j\|f\|_\infty \end{aligned}$$

Thus we have a K_j for which $t^{j/2}\|\mathcal{D}^j u\|_\infty \leq K_j\|f\|_\infty$, and this completes the proof of the theorem. □

5.2 Applications of the Results

After the lengthy proof of Proposition 5.1.1 we can finally conclude that for all derivatives of order j in maximum norm

$$t^{j/2}\|\mathcal{D}^j u\|_\infty \leq C_j \|f\|_\infty$$

where the constants C_j are independent of t and f. In particular, if $f \in L^\infty(\mathbb{R}^3)$, we have that

$$\|u\|_\infty < \infty$$

This indicates that the maximum norm of the initial value function $f = u(x, 0)$ controls all of the derivatives of u in maximum norm. We now outline an application for Proposition 5.1.1 for future work: the construction of a solution. The *a priori* estimates of Proposition 5.1.1 will guarantee the construction, and will help us construct an algorithm that will produce a solution (u, p) of the Navier-Stokes equations (1.1) and (1.2).

We begin as follows. Suppose that at $t = 0$, we have $u(x, 0) = f(x)$, where $f \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and $\nabla \cdot f = 0$. Define $u^0 = f$. We will construct a sequence of solutions to both u and p. As the modified pressure term exists as an integral for $u \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we define

$$p_{ij}^{n+1} = \int [K_{ij}(x - y) - K_{ij}(y)](u_j^n u_i^n)(y, t)\, dy \tag{5.9}$$

where the $K_{ij}(y)$ are defined by equations (3.7), (3.8), and (3.9). We next define

$$p^{n+1}(x) = \sum_{i,j} p_{ij}^{n+1}(x) \tag{5.10}$$

At $n = 0$, p^1 exists by virtue of f and Theorem 3.2.1. Now consider the Navier-Stokes equation and define a sequence

$$u_t^{n+1} = \Delta u^{n+1} - (u^n \cdot \nabla u^n + \nabla p^{n+1})$$

or, by writing $Q^n(x,t) = -(u^n \cdot \nabla u^n + \nabla p^{n+1})$

$$u_t^{n+1} = \Delta u^{n+1} + Q^n$$

Then, as is shown in [7], this is solved by

$$u^{n+1} = e^{\Delta t} f + \int_0^t e^{\Delta(t-s)} Q^n(s)\, ds \tag{5.11}$$

where

$$e^{\Delta t} h = \int_{\mathbb{R}^n} \Psi(y,t) h(y)\, dy = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{|y|^2}{4t}} h(y)\, dy$$

Thus the procedure is as follows. Let $u^0 = f$. Solve for p^1 using $u^0 = f$ and equations (5.9) and (5.10). By Theorem 3.3.1 p^1 is a solution to the Poisson pressure equation, and by Corollary 3.3.1, p^1 will be BMO . Next, use p^1, and $u^0 = f$ to solve for u^1 using equation (5.11). This u_1 will be $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by virtue of Proposition 5.1.1. Then, placing u_1 back into the pressure equation (5.9) and solving, we find a pressure p^2 which is BMO . Using p^2 and u^1 we now solve for u^2 again in equation (5.11). Continuing this process we construct a sequence p^n that are all solutions to the Poisson pressure equation. The u^n exist in $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Additionally, by the material in Chapter 4 the u^n are uniformly bounded by $\|f\|_\infty$, independent of n. This follows for the derivatives $\|\mathcal{D}^j u^n(x,t)\|_\infty$ as well. Thus, each u^n will be in $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and, by construction, each p^n will exist and be a function of $Bounded Mean Oscillation$.

We may then show that this sequence converges to a limit in $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, by way of the Arzela-Ascoli theorem. The convergence of u_n to u will guarantee the convergence of the terms of the pressure sequence to a limit p in BMO . The convergence of all derivatives of u can also be shown in view of Proposition 5.1.1. Finally, we will note that the limit of the velocity field will be unique in $C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, while that of the pressure term will be unique up to a (time dependent) constant in the space BMO .

Appendix A

Background: Spaces, Derivatives, Integrals

A.1 Some Applications of Integration with Polar Coordinates

We will now explore a few applications related to polar coordinates, which will be of necessity in the paper. First we will justify a general expression for the **surface measure** $\sigma(\mathbb{S}^{n-1})$. We begin with a lemma (see [10] or [11]):

Lemma A.1.1. $\int_{\mathbb{R}^n} e^{-a|x|^2}\,dx = \left(\frac{\pi}{a}\right)^{\frac{n}{2}}$

The gamma function (see [10]) is defined as

$$\Gamma(p) = \int_0^\infty s^{p-1}e^{-s}ds$$

for $\mathfrak{Re} \quad p > 0$. Then

$$(\pi)^{(\frac{n}{2})} = \frac{\sigma(\mathbb{S}^{n-1})}{2}\Gamma\left(\frac{n}{2}\right)$$

It can be shown that (see [10] or [11]):

$$\sigma(\mathbb{S}^{n-1}) = \omega_n = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{A.1.1}$$

If $n = 2$, $\sigma(\mathbb{S}^1) = 2\pi$, so we find that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Similarly, using $n = 3$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} = \Gamma\left(1 + \frac{1}{2}\right) = (1 - \frac{1}{2})\Gamma\left(\frac{1}{2}\right)$$

Generally

$$\Gamma\left(k + \frac{1}{2}\right) = (k - \frac{1}{2})(k - \frac{3}{2}) \cdots \frac{1}{2}\sqrt{\pi}$$

We write

$$\sigma(\mathbb{S}^{n-1}) = \omega_n = \frac{2(\pi)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{A.1.2}$$

We also note that the measure of the **volume** of the unit ball is given by

$$\alpha(n) = V(B_1(0)) = |B_1(0)| = \frac{\omega_n}{n}$$

We may then write $n\alpha(n) = \omega(n)$. Note that if $n = 2$, $\alpha(2) = \pi$, while if $n = 3$, $\alpha(3) = \frac{4\pi}{3}$. Another use of the measure is the use in Cartesian coordinates. Let $n = 3$. In Cartesian coordinates we can write the formula for the radius of the unit sphere $\mathbb{S}^2$ in terms of the point $x = (x_1, x_2, x_3)$:

$$\sum_{i=1}^{3} x_i^2 = 1$$

We can the show that

Lemma A.1.2.

$$\int_{\mathbb{S}^2} x_i x_j \, d\sigma(x) = \begin{cases} 0, & i \neq j; \\ \frac{4\pi}{3}, & i = j. \end{cases}$$

A.2 Other Integral Relationships

One important concept is the **average value** of a function. We define the average value of a locally integrable function f on the ball $B(x,r)$ to be:

$$\fint_{B(x,r)} f\,dx = \frac{1}{|B(x,r)|}\int_{B(x,r)} f\,dy$$

where $|B(x,r)|$ is measure or volume of the open ball $B(x,r)$. Now we define the space L^1 to be the space of all integrable, functions such that

$$\int_{\mathbb{R}^n} |f| < \infty$$

A function f is in the space L^1_{loc} if

$$\int_K |f| < \infty$$

for all compact sets $K \subset \mathbb{R}^n$. Using this we can prove a version (from [11]) of the **Lebesgue Differentiation Theorem**:

Theorem A.2.1. ***Lebesgue Differentiation Theorem***

$$\lim_{r\to 0}\fint_{B(x,r)} f\,dx = \lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} f\,dx = f(x)$$

a.e.; that is outside a set of measure zero. We have

$$|B(x,r)| = \alpha(n)r^n$$

and

$$|\partial B(x,r)| = \omega_n r^{n-1}$$

Then

$$\lim_{r\to 0}\fint_{B(x,r)} f\,dx = \lim_{r\to 0}\frac{1}{\alpha(n)r^n}\int_{B(x,r)} f\,dx = f(x)$$

almost everywhere (a.e.). Recall the definition of Dini continuity (Definition 2.2.9):

Definition A.2.1. *A function $\Omega(x)$ on S^{n-1} is said to be* ***Dini continuous*** *if for*

$$\omega(\delta) = \sup_{\substack{|x-x\prime|<\delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')|$$

then

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty$$

Recall from Definition 2.2.7 that Calderón -Zygmund Kernels have mean value of 0 on $\mathbb{S}^2$ and are homogeneous of degree 0. Some CZ kernels also have the property of Dini continuity. We note that the Riesz kernels

$$\frac{x_j}{|x|}$$

are trivially CZ kernels.

Theorem A.2.2. *Consider the function:*

$$K_{ij}(x) = \frac{\Omega_{ij}(x)}{|x|^3}$$

where

$$\Omega_{ij}(x) = \frac{x_i x_j}{|x|^2} \quad \textit{and} \quad \Omega_{jj} = \frac{3x_j^2 - |x|^2}{|x|^2}$$

Then for all i, j, the function $\Omega_{ij}(x)$ has the following properties:

1. *$\Omega_{ij}(x)$ is homogeneous of degree 0*

2. *$\Omega_{ij}(x)$ has mean value zero around the unit 3-sphere $\mathbb{S}^2$*

3. *$\Omega_{ij}(x)$ is Dini continuous.*

Properties 1 and 2 yield that K_{ij} is a Calderón Zygmund kernel.

Proof. The term $K_{ij}(x)$, the kernel, must be of the form

$$K_{ij}(x) = \frac{\Omega_{ij}}{|x|^3}, \quad K_{jj}(x) = \frac{\Omega_{jj}}{|x|^3} \tag{A.2.1}$$

where the term $\Omega_{ij}(x)$ is given by

$$\Omega_{ij}(x) = \frac{x_i x_j}{|x|^2} \tag{A.2.2}$$

for $i \neq j$,

$$\Omega_{jj}(x) = \frac{3x_j^2 - |x|^2}{|x|^2} \tag{A.2.3}$$

for $i = j$, where C_0 is a constant that can be taken to be

$$C_0 = \frac{1}{4\pi}$$

We must have $\Omega_{ij}(x)$ be Dini continuous, have mean value zero around the unit sphere, and be homogeneous of degree 0. See definitions 2.2.5, 2.2.6 and 2.2.9. We will take each in turn. Trivially, $\Omega_{ij}(x)$ is homogeneous of degree zero. To show that Ω has mean value zero around the unit sphere, we first consider $\Omega_{ij}(x)$ for $i \neq j$:

$$\begin{aligned}
\int_{\mathbb{S}^2} \Omega_{ij}(x)\, dS &= \int_{\mathbb{S}^2} C_0 \frac{x_i x_j}{|x|^2}\, dS \\
&= \int_{\mathbb{S}^2} C_0 x_i x_j \, d\sigma(x) \\
&= 0
\end{aligned}$$

from lemma A.1.2. Next, for $i = j$

$$\begin{aligned}
\int_{\mathbb{S}^2} \Omega_{jj}(x)\, dS &= \int_{\mathbb{S}^2} \frac{3x_j^2 - |x|^2}{|x|^2}\, dS \\
&= \left[3 \int_{\mathbb{S}^2} x_j^2 \, d\sigma(x) - \int_{\mathbb{S}^2} d\sigma(x)\right] \\
&= \left[3\frac{4\pi}{3} - 4\pi\right] \\
&= (4\pi - 4\pi) \\
&= 0
\end{aligned}$$

again from Lemma A.1.2. Finally, we establish Dini continuity. We first consider the case $i = j$ and write $\Omega_{jj}(x) = \Omega(x)$ We have:

$$\begin{aligned}
|\Omega(x) - \Omega(z)| &= \left|\frac{3x_j^2 - |x|^2}{|x|^2} - \frac{3z_j^2 - |z|^2}{|z|^2}\right| \\
&= C_1\left|\frac{x_j^2}{|x|^2} - \frac{z_j^2}{|z|^2}\right| \\
&= C_1\left|\frac{x_j}{|x|} - \frac{z_j}{|z|}\right|\left|\frac{x_j}{|x|} + \frac{z_j}{|z|}\right| \\
&\leq C_2\left|\frac{x_j}{|x|} - \frac{z_j}{|z|}\right|
\end{aligned}$$

Then we have

$$\begin{aligned}
\omega(\delta) &= \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} |\Omega(x) - \Omega(z)| \\
&= \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} C_2\left|\frac{x_j}{|x|} - \frac{z_j}{|z|}\right| \\
&\leq \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} |x_j - z_j| \\
&\leq C_3|x - z| \\
&< C_3\delta
\end{aligned}$$

Thus

$$\omega(\delta) = \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} |\Omega(x) - \Omega(z)| < C_3\delta$$

We then have:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta \leq \int_0^1 \frac{C_3\delta}{\delta}\, d\delta \leq C_3$$

The next case is for $i \neq j$ and again write $\Omega_{ij}(x) = \Omega(x)$

$$\begin{aligned} |\Omega(x) - \Omega(z)| &= \left| \frac{x_i x_j}{|x|^2} - \frac{z_i z_j}{|z|^2} \right| \\ &= C_1 \left| \frac{x_i x_j}{|x|^2} - \frac{x_i z_j}{|x||z|} + \frac{x_i z_j}{|x||z|} - \frac{z_i z_j}{|z|^2} \right| \\ &= C_1 \left| \frac{x_i}{|x|} \left[\frac{x_j}{|x|} - \frac{z_j}{|z|} \right] + \frac{z_j}{|z|} \left[\frac{x_i}{|x|} - \frac{z_i}{|z|} \right] \right| \end{aligned}$$

If we let, for each x and z

$$\left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| = \max_i \left| \frac{x_i}{|x|} - \frac{z_i}{|z|} \right|$$

we obtain, via the triangle inequality:

$$\begin{aligned} |\Omega(x) - \Omega(z)| &\leq \left| \frac{x_i}{|x|} \right| \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| + \left| \frac{z_j}{|z|} \right| \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| \\ &\leq C_1 \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| + C_2 \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| \\ &\leq C_3 \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| \end{aligned}$$

We then have, when $|x| = |z| = 1$:

$$\begin{aligned} \omega(\delta) &= \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} |\Omega(x) - \Omega(z)| \\ &\leq \sup_{\substack{|x-z|<\delta \\ |x|=|z|=1}} C_3 \left| \frac{x_l}{|x|} - \frac{z_l}{|z|} \right| \\ &\leq C_3 |x_l - z_l| \\ &\leq C_3 |x - z| \\ &\leq C_3 \delta \end{aligned}$$

By definition we have $\omega(\delta) \leq C_5\delta$ and we obtain:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta \leq \int_0^1 \frac{C_5\delta}{\delta}\, d\delta \leq C_5$$

Thus we have $\Omega_{ij}(x)$ is Dini continuous for all i, j. The three properties are proved. By definition 2.2.7 K_{ij} is a Calderón Zygmund kernel. □

The properties are important to establish a Hörmander condition.

Theorem A.2.3. *Let k(x) be a Calderón Zygmund kernel. That is*

$$k(x) = \frac{\Omega(x)}{|x|^n}$$

where $\Omega(x)$ has mean value of zero on the unit sphere, and is homogeneous of degree 0. Further, suppose that $\Omega(x)$ is Dini-continuous. Then $k(x)$ satisfies the Hörmander condition:

$$\sup_{|x|>0} \int_{|y|>2|x|} |k(x-y) - k(y)|\, dy \leq A.$$

Proof. If

$$k(x) = \frac{\Omega(x)}{|x|^n}$$

then we consider

$$\int_{|y|>2|x|} \Big| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(y)}{|y|^n} \Big|\, dy$$

With a little algebra we may write

$$\begin{aligned} I &= \int_{|y|>2|x|} \Big|\frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(y)}{|y|^n}\Big|\, dy \\ &\leq \int_{|y|>2|x|} |\Omega(y)| \Big[\frac{1}{|x-y|^n} - \frac{1}{|y|^n}\Big]\, dy \\ &+ \int_{|y|>2|x|} \frac{|\Omega(x-y) - \Omega(y)|}{|x-y|^n}\, dy \\ &= I_1 + I_2 \end{aligned}$$

First I_1. Since Ω is a CZ kernel, it is bounded for $|y| \geq 2|x|$. Thus $|\Omega(x)| \leq B$. For the other part of the integrand we note that

$$\frac{1}{|x-y|^n} - \frac{1}{|y|^n} = \frac{|y|^n - |x-y|^n}{|x-y|^n |y|^n}$$

Now, noting that since $|y| > 2|x|$, $|x-y| \geq \frac{|y|}{2}$. Additionally, $|y| - |x-y| \leq |x|$. Thus

$$\begin{aligned}
\frac{|y|^n - |x-y|^n}{|x-y|^n |y|^n} &= \frac{[|y| - |x-y|] \sum_{j=0}^{n-1} |y|^{n-1-j} |x-y|^j}{|x-y|^n |y|^n} \\
&= [|y| - |x-y|] \sum_{j=0}^{n-1} |y|^{-1-j} |x-y|^{j-n} \\
&\leq |x| \sum_{j=0}^{n-1} |y|^{-1-j} \left(\frac{|y|}{2} \right)^{j-n} \\
&= |x||y|^{-n-1} \sum_{j=0}^{n-1} 2^{n-j} \\
&= |x||y|^{-n-1} [2(2^{n-1} - 1)] \\
&= C|x||y|^{-n-1}
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 &= \int_{|y|>2|x|} |\Omega(y)| \Big[\frac{1}{|x-y|^n} - \frac{1}{|y|^n} \Big] \, dy \\
&\leq \int_{|y|>2|x|} BC|x||y|^{-n-1} \, dy \\
&= BC|x| \int_{|y|>2|x|} |y|^{-n-1} \, dy \\
&= BC|x|\omega_n \int_{2|x|}^{\infty} \frac{1}{r^2} \, dr \\
&= C_1 |x| \frac{1}{2|x|} \\
&= C_2
\end{aligned}$$

So I_1 is bounded. As for I_2, we note that from the above

$$\frac{1}{|x-y|^n} \leq \frac{2^n}{|y|^n}$$

Using the homogeneity property, since

$$\Omega(x) = \Omega\Big(\frac{x}{|x|}\Big)$$

and since (see [35])

$$\Big|\frac{x-y}{|x-y|} - \frac{y}{|y|}\Big| \leq 2\frac{|x|}{|y|}$$

using the Dini-continuity property we have, by definition

$$|\Omega(x-y) - \Omega(y)| \leq \omega\Big(2\frac{|x|}{|y|}\Big)$$

Then

$$\begin{aligned}
I_2 &= \int_{|y|>2|x|} \frac{|\Omega(x-y) - \Omega(y)|}{|x-y|^n}\, dy \\
&\leq 2^n \int_{|y|>2|x|} \frac{\omega\Big(2\frac{|x|}{|y|}\Big)}{|y|^n}\, dy \\
&= C\omega_n \int_{2|x|}^{\infty} \frac{\omega\Big(2\frac{|x|}{r}\Big)}{r}\, dr \\
&= C\omega_n \int_{2|x|}^{\infty} \omega\Big(2\frac{|x|}{r}\Big)\frac{dr}{r} \\
&= C_1 \int_0^1 \frac{\omega(\delta)}{\delta} d\delta \\
&< \infty
\end{aligned}$$

since Ω is Dini-continuous. So $I = I_1 + I_2$ is finite. □

Appendix B

The Poisson Equation

In this section we will discuss the construction of the solution of the Laplace and Poisson Equations. We will be using material from primarily Evans ([7]), John ([17]), and [38].

B.1 The Laplace Equation and the Solution

The Laplace equation is given by

$$\Delta u - \sum_{i=1}^{n} u_{x_i x_i} = 0$$

where the u_{x_i} are partial derivatives respect to variable x_i, and $u = u(x_1, \ldots x_n)$. If $x \in \mathbb{R}^n$, that is $x = (x_1, x_2, \ldots, x_n)$, then the Euclidean distance given by

$$|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

It is well known that

Definition B.1.1. *The fundamental solution of the Laplace equation for $n \geq 3$ is given by*

$$\Phi(x) = \frac{2-n}{\omega_n}|x|^{2-n}$$

For $n = 3$, this is

$$\Phi(x) = -\frac{1}{4\pi}\frac{1}{|x-y|}$$

Of special interest in the paper is the radial part of the kernel, $|x-y|$, and its derivatives. Let

$$r(x) = |x| = \sqrt{\sum_{i=1}^{3} x_i^2}$$

and define

$$r_i = \frac{\partial r}{\partial x_i}$$

We now compute the derivatives and their bounds:

$$r_i(x) = -\frac{x_i}{|x|} \quad |r_i(x)| \leq C \tag{B.1.1}$$

$$r_{ij}(x) = -\frac{x_i x_j}{|x|^3} \quad |r_{ij}(x)| \leq \frac{C}{|x|} \tag{B.1.2}$$

$$r_{jj}(x) = \frac{|x|^2 - x_j^2}{|x|^3} \quad |r_{jj}(x)| \leq \frac{C}{|x|} \tag{B.1.3}$$

Additionally, we have have

$$r_{ijk}(x) = -\frac{3x_i x_j x_k}{|x|^5} \quad |r_{ijk}(x)| \leq \frac{C}{|x|^2} \tag{B.1.4}$$

$$r_{jjk}(x) = \frac{x_k(3x_j^2 - |x|^2)}{|x|^5} \quad |r_{jjk}(x)| \leq \frac{C}{|x|^2} \tag{B.1.5}$$

$$r_{kkk}(x) = \frac{3x_k(x_k^2 - |x|^2)}{|x|^5} \quad |r_{kkk}(x)| \leq \frac{C}{|x|^2} \tag{B.1.6}$$

These equations are used throughout the paper.

B.2 The Poisson Equation and the Solution

We now turn our attention to the solution of the Poisson equation. The background material can be found in either Evans ([7]) or John ([17]). First, let us write the inhomogeneous Laplace equation, otherwise known as the *Poisson* equation:

$$-\Delta u = g$$

We will now first consider a solution for suitable g. Let $g \in C_0^2$, that is g is twice differentiable on a set of compact support. The solution can be found in [7] or [17].

Theorem B.2.1. *The solution of the Poisson equation*

$$-\Delta u = g$$

in $\mathbb{R}^n$ *is given by*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)g(y)\,dy$$

where

$$\Phi(x) = \frac{2-n}{\omega_n}|x|^{2-n}$$

with $u \in C^2$*. In particular, for* $n = 3$ *we have*

$$u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|} g(y)\,dy$$

We may write

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)g(y)\,dy = \int_{\mathbb{R}^n} \Phi(y)g(x-y)\,dy$$

for suitable $g(x)$ of compact support.

B.3 Relation to Navier-Stokes Equations

We will connect Poisson's equation to Navier-Stokes equation. Recall the divergence-free condition $\nabla \cdot u = 0$. It will be shown that upon taking the divergence of both sides of the Navier-Stokes , that is, by applying the divergence operator $\nabla\cdot$ to both sides of the Navier-Stokes equation, where $u \equiv u(x,t)$, $p \equiv p(x,t)$, to show that

$$-\Delta p(x,t) = g(x,t)$$

where $g(x,t) = \sum_{i,j=1}^{3} D_i D_j (u_i u_j)(x,t)$ (see [29]). In the language of Poisson's equation g is the source of the pressure field, and p is the pressure field. Consider now, with $\nabla \cdot u = 0$:

$$\nabla \cdot (u_t + u \cdot \nabla u + \nabla p) = \nabla \cdot \nu \Delta u$$

We determine that the first computation is 0.

$$\begin{aligned}
\nabla \cdot (u_t) &= \sum_{i=1}^{3} \nabla \cdot u_t \\
&= \sum_{i=1}^{3} u_{it} \\
&= \frac{\partial}{\partial t}(\nabla \cdot u) \\
&= \frac{\partial}{\partial t}(0) \\
&= 0
\end{aligned}$$

For the second term:

$$
\begin{aligned}
\nabla \cdot (u \cdot \nabla u) &= \sum_{i,j=1}^{3} D_i(u_j D_j u_i) \\
&= \sum_{i,j=1}^{3} u_j D_i D_j(u_i) + (D_i u_j)(D_j u_i) \\
&= \sum_{i,j=1}^{3} u_j D_j D_i(u_i) + (D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3} u_j D_j \sum_{i=1}^{3} D_i(u_i) + \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3} u_j D_j(0) + \sum_{i=1}^{3} (D_i u_j)(D_j u_i) \\
&= \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) \\
&= \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i)
\end{aligned}
$$

We note here that

$$\sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) = \sum_{i,j=1}^{3} D_i D_j(u_i u_j) \qquad \text{(B.3.1)}$$

from the following computation:

$$\begin{aligned}
\sum_{i,j=1}^{3} D_i D_j(u_i u_j) &= \sum_{i=1}^{3}\sum_{j=1}^{3} D_i D_j(u_i u_j) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} D_i((u_i D_j u_j) + u_j(D_j u_i)) \\
&= \sum_{i=1}^{3} D_i\Big(u_i \sum_{j=1}^{3} D_j u_j + \sum_{j=1}^{3} u_j(D_j u_i)\Big) \\
&= \sum_{i=1}^{3} D_i\Big(u_i \cdot 0 + \sum_{j=1}^{3} u_j(D_j u_i)\Big) \\
&= \sum_{i=1}^{3} D_i\Big(\sum_{j=1}^{3} u_j(D_j u_i)\Big) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} D_i(u_j(D_j u_i)) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} u_j D_i D_j u_i + (D_i u_j)(D_j u_i) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} u_j D_j D_i u_i + (D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3} u_j D_j D_i u_i + (D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3} u_j D_j\Big(\sum_{i=1}^{3} D_i u_i\Big) + \sum_{j=1}^{3}\sum_{i=1}^{3}(D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3} u_j D_j(0) + \sum_{j=1}^{3}\sum_{i=1}^{3}(D_i u_j)(D_j u_i) \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}(D_i u_j)(D_j u_i) \\
&= \sum_{i,j=1}^{3}(D_i u_j)(D_j u_i)
\end{aligned}$$

For the third term, we simply have

$$\nabla \cdot \nabla p = \Delta p$$

For the last term on the right hand side we obtain:

$$\begin{aligned}
\nabla \cdot (\nu \Delta u) &= \nu(\nabla \cdot \sum_{i=1}^{3} u_{ii}) \\
&= \nu \sum_{i=1}^{3} \nabla \cdot u_{ii} \\
&= \nu \sum_{i=1}^{3} \sum_{j=1}^{3} D_j (u_{ii})_j \\
&= \nu \sum_{i=1}^{3} \sum_{j=1}^{3} D_j (D_i (D_i (u_j))) \\
&= \nu \sum_{i=1}^{3} D_i \Big(D_i \Big(\sum_{j=1}^{3} D_j (u_j) \Big) \Big) \\
&= \nu \sum_{i=1}^{3} D_i (D_i (0)) \\
&= \nu \cdot 0 \\
&= 0
\end{aligned}$$

So the first and last terms are 0. This yields:

$$\begin{aligned}
\nabla \cdot (u_t + u \cdot \nabla u + \nabla p) &= \nabla \cdot \nu \Delta u \\
&\Rightarrow \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) + \Delta p = 0 \\
&\Rightarrow \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i) + \Delta p = 0 \\
&\Rightarrow -\Delta p = \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i)
\end{aligned}$$

Thus

$$-\Delta p = \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i)$$

We may rewrite this as:

$$-\Delta p = \sum_{i,j=1}^{3} D_i D_j (u_i u_j) \tag{B.3.2}$$

Thus we have

$$-\Delta p(x,t) = g(x,t) \tag{B.3.3}$$

where $g(x,t) = \sum_{i,j=1}^{3} D_i D_j (u_i u_j)(x,t) = \sum_{i,j=1}^{3} (D_i u_j)(D_j u_i)$

Appendix C

The Heat (Diffusion) Equation and Solution

C.1 The Heat Equation with Solution

The homogeneous heat equation is given by

$$u_t = \Delta u \tag{C.1.1}$$

If $u(x, 0) = g(x)$, then the function

$$u(x,t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \tag{C.1.2}$$

solves (C.1.1) with $u(x, 0) = g(x)$. The non-homogeneous heat equation

$$u_t = \Delta u + F(x,t) \tag{C.1.3}$$

can be solved by

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y,s)\, dyds \tag{C.1.4}$$

with $u = 0$ at $t = 0$. We will write

$$e^{\Delta t} g = u(x,t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \tag{C.1.5}$$

for a simplified expression of the solution. Thus (C.1.4) becomes

$$u(x,t) = \int_0^t e^{\Delta(t-s)} F(s)\, ds \tag{C.1.6}$$

Finally, through the use of *Duhamel's principle*, we may consider a problem

$$u_t - \Delta u = F(x,t) \quad \text{with} \quad u(x,0) = g(x) \tag{C.1.7}$$

and write the solution as

$$u(x,t) = e^{\Delta t} g + \int_0^t e^{\Delta(t-s)} F(s)\, ds \tag{C.1.8}$$

where

$$e^{\Delta t} g = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

C.2 Relation to Navier-Stokes Equations

Let $x \in \mathbb{R}^3$ be a space variable, let $t \in \mathbb{R}$, where $t \geq 0$ is the time variable. Then the *homogeneous heat equation* is given by:

$$u_t = \nu \Delta u \tag{C.2.1}$$

while the *inhomogeneous heat equation*is given by:

$$u_t = \nu \Delta u + F(x,t) \tag{C.2.2}$$

Now, recall the incompressible Navier-Stokes . We have

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \nu \Delta u \Rightarrow u_t = \nu \Delta u - (u \cdot \nabla u + \nabla p) \\ &= \nu \Delta u + Q \end{aligned}$$

Thus the Navier-Stokes equation becomes

$$u_t = \nu\Delta u - (u \cdot \nabla u + \nabla p) \tag{C.2.3}$$

where $Q \equiv Q(x,t) = -u \cdot \nabla u - \nabla p$, $u \equiv u(x,t)$, and $p \equiv p(x,t)$. Thus the Navier-Stokes equations are essentially a non-linear form of the three-dimensional heat equation. Using (C.1.8) we may write (for $\nu = 1$)

$$u(x,t) = e^{\Delta t} g + \int_0^t e^{\Delta(t-s)} Q(s)\, ds \tag{C.2.4}$$

where

$$Q \equiv Q(x,t) = -u \cdot \nabla u - \nabla p, \quad u \equiv u(x,t), \quad \text{and} \quad p \equiv p(x,t). \tag{C.2.5}$$

Appendix D

Bounds Involving a Cutoff Function

D.1 The Bounds on the Derivatives of ϕ

In this section we will produce several calculations crucial to results in the main paper.

Lemma D.1.1. *Let ϕ be a C^∞ cut-off function where $\phi(r) = 1$ for $0 \le r \le 1$, $\phi(r) = 0$ for $r \ge 2$, and $0 \le \phi(r) \le 1$. Then for $\delta > 0$, if $0 \le |x - y| \le 2\delta$, and*

$$\phi \equiv \phi\Big(\frac{|x-y|}{\delta}\Big)$$

then for all i, j

$$|D_i(\phi)| \le \frac{C}{\delta} \quad \text{and} \quad |D_iD_j(\phi)| \le \frac{C}{\delta^2} \tag{D.1.1}$$

where the constant C depends on $\|\phi'\|_\infty$ for $|D_i(\phi)|$, and depends on $\|\phi'\|_\infty, \|\phi''\|_\infty$ for $|D_iD_j(\phi)|$

Proof. We use equation (B.1.1) and define $D_i = \frac{\partial}{\partial y_i}$. Then trivially

$$D_i(\phi) = \frac{-\phi'(x_i - y_i)}{\delta|x-y|}$$

so that

$$|D_i(\phi)| \leq \Big|\frac{\phi'(x_i - y_i)}{\delta|x-y|}\Big|_\infty \leq \frac{\|\phi'\|_\infty}{\delta} = \frac{C}{\delta}$$

where C depends on ϕ'. Next, for $i \neq j$, we use (B.1.2):

$$\begin{aligned} D_iD_j(\phi) &= D_i\Big(-\frac{\phi'(x_i - y_i)}{\delta|x-y|}\Big) \\ &= \frac{1}{\delta}\Big[\frac{-\phi''(x_i - y_i)(x_j - y_j)}{|x-y|^3} + \frac{(x_i - y_i)(x_j - y_j)\phi'}{\delta|x-y|^2}\Big] \\ &= \frac{1}{\delta}\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2}\Big[\frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}\Big] \\ &= \frac{1}{\delta}\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2}[\rho] \end{aligned}$$

Thus

$$D_iD_j(\phi) = \frac{1}{\delta}\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^2}[\rho] \tag{D.1.2}$$

where

$$\rho = \frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}$$

Now, if $|x-y| \leq 2\delta$,

$$-\frac{1}{|x-y|} \leq -\frac{1}{2\delta}$$

so that

$$\begin{aligned}
\|\rho\|_\infty &= \Big\|\frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}\Big\|_\infty \\
&\leq \Big\|\frac{\phi''}{\delta} - \frac{\phi'}{2\delta}\Big\|_\infty \\
&= \|\frac{2\phi''}{2\delta} - \frac{\phi'}{2\delta}\Big\|_\infty \\
&\leq \frac{1}{2\delta}\|2\phi'' - \phi'\|_\infty \\
&\leq \frac{1}{2\delta}(\|2\phi''\|_\infty + \|\phi'\|_\infty) \\
&= \frac{C}{\delta}
\end{aligned}$$

where $C = \|2\phi''\|_\infty + \|\phi'\|_\infty$; that is C depends on ϕ' and ϕ''. Thus

$$\begin{aligned}
|D_iD_j(\phi)| &= \Big|\frac{1}{\delta}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}[\rho]\Big| \\
&\leq \frac{1}{\delta}\Big|\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\Big|\|\rho\|_\infty \\
&\leq \frac{1}{\delta}\|\rho\|_\infty \\
&\leq \frac{1}{\delta}\cdot\frac{C}{\delta} \\
&= \frac{C}{\delta^2}
\end{aligned}$$

and the constant C is as described above. If $i = j$, we use (B.1.3) and obtain

$$D_jD_j(\phi) = -\frac{1}{\delta}\Big[\frac{(x_j-y_j)^2}{|x-y|^2}\cdot\frac{\phi''}{\delta} + \phi'\cdot\frac{|x-y|^2-(x_j-y_j)^2}{|x-y|^3}\Big]$$

Using a similar approach to the case $i \neq j$ we again obtain

$$|D_jD_j(\phi)| \leq \frac{C}{\delta^2}$$

and the required result is obtained. □

Lemma D.1.2. *Let ϕ be a C^∞ cut-off function where $\phi(r) = 1$ for $0 \leq r \leq 1$, $\phi(r) = 0$ for $r \geq 2$, and $0 \leq \phi(r) \leq 1$. Then for $\delta > 0$, if $0 \leq |x-y| \leq 2\delta$, and*

$$\phi = \phi\Big(\frac{|x-y|}{\delta}\Big)$$

then for all i, j, k

$$|D_i D_j D_{k,x}(\phi)| \leq \frac{C}{\delta^3} \tag{D.1.3}$$

where C depends on ϕ', ϕ'', and ϕ'''.

Proof. From (D.1.1)and (B.1.2) we may write

$$D_j D_{k,x}(\phi) = -\frac{1}{\delta}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2}\Big[\frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}\Big] = AB \tag{D.1.4}$$

where

$$A = \frac{1}{\delta}\frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} \quad \text{and} \quad B = \frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}$$

Here we will use (B.1.2), (B.1.3), and (B.1.4). We write $z_i = x_i - y_i$ and first compute $D_i(A)$:

$$D_i(A) = \frac{2(x_i-y_i)(x_j-y_j)(x_k-y_k)}{\delta|x-y|^4} = \frac{2z_iz_jz_k}{\delta|z|^4}$$

Next, for the more complicated $D_i(B)$, and $D_i(AB)$.

$$\begin{aligned} D_iB &= D_i\Big[\frac{\phi''}{\delta} - \frac{\phi'}{|x-y|}\Big] \\ &= \Big[-\frac{\phi'''}{\delta^2}\cdot\frac{x_i-y_i}{|x-y|} - \frac{|x-y|\cdot\phi''\cdot\frac{-(x_i-y_i)}{\delta|x-y|} - \phi'\cdot\frac{-x_i-y_i}{|x-y|}}{|x-y|^2}\Big] \\ &= \frac{x_i-y_i}{|x-y|}\Big[-\frac{\phi'''}{\delta^2} + \frac{\phi''}{\delta|x-y|} - \frac{\phi'}{|x-y|^2}\Big] \\ &= \frac{z_i}{|z|}\Big[-\frac{\phi'''}{\delta^2} + \frac{\phi''}{\delta|z|} - \frac{\phi'}{|z|^2}\Big] \end{aligned}$$

$$\begin{aligned}
D_i(AB) &= A(D_iB) + (D_iA)B \\
&= -\frac{1}{\delta}\frac{(z_jz_k)}{|z|^2} \cdot \frac{z_i}{|z|}\Big[-\frac{\phi'''}{\delta^2} + \frac{\phi''}{\delta|z|} - \frac{\phi'}{|z|^2}\Big] \\
&- \frac{2z_iz_jz_k}{\delta|z|^4}\Big[\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big] \\
&= \frac{z_iz_jz_k}{\delta|z|^3}\Big[\frac{\phi'''}{\delta^2} - \Big(\frac{\phi''}{\delta|z|} - \frac{\phi'}{|z|^2}\Big) + \frac{2}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big)\Big] \\
&= \frac{z_iz_jz_k}{\delta|z|^3}\Big[\frac{\phi'''}{\delta^2} - \frac{1}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big) + \frac{2}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big)\Big] \\
&= \frac{z_iz_jz_k}{\delta|z|^3}\Big[\frac{\phi'''}{\delta^2} + \frac{1}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big)\Big]
\end{aligned}$$

Finally with $|z| = |x-y| \leq 2\delta$ we compute

$$\begin{aligned}
|D_iD_jD_{k,x}(\phi)| &= D_i(AB) \\
&= |A(D_iB) + (D_iA)B| \\
&= \Big| -\frac{z_iz_jz_k}{\delta|z|^3}\Big[\Big(-\frac{\phi'''}{\delta^2} + \frac{1}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{|z|}\Big)\Big]\| \\
&\leq \frac{1}{\delta}\Big\| -\frac{\phi'''}{\delta^2} + \frac{1}{|z|}\Big(\frac{\phi''}{\delta} - \frac{\phi'}{2\delta}\Big)\Big]\Big\|_\infty \\
&\leq \frac{1}{\delta}\Big\| -\frac{\phi'''}{\delta^2} + \frac{1}{2\delta|z|}(2\phi'' - \phi')\Big\|_\infty \\
&\leq \frac{1}{\delta^2}\Big\|\frac{\phi'''}{\delta} - \frac{1}{2|z|}(2\phi'' - \phi')\Big\|_\infty \\
&\leq \frac{1}{2\delta^2}\Big\|\frac{2\phi'''}{\delta} - \frac{1}{|z|}(2\phi'' - \phi')\Big\|_\infty \\
&\leq \frac{1}{2\delta^2}\Big\|\frac{2\phi'''}{\delta} - \frac{1}{2\delta}(2\phi'' - \phi')\Big\|_\infty \\
&\leq \frac{1}{2\delta^2}\Big\|\frac{4\phi'''}{2\delta} - \frac{1}{2\delta}(2\phi'' - \phi')\Big\|_\infty \\
&= \frac{1}{4\delta^3}\|4\phi''' - 2(\phi'' - \phi')\|_\infty \\
&= \frac{1}{4\delta^3}(\|4\phi'''\|_\infty + \|2\phi''\|_\infty + \|\phi'\|_\infty) \\
&= \frac{C}{\delta^3}
\end{aligned}$$

We note here that C is a combination of the maximum norms of $\|\phi'\|_\infty$,$\|\phi''\|_\infty$, and $\|\phi'''\|_\infty$. If $i = j$, or $i = j = k$, we use (B.1.4), (B.1.5), or (B.1.6), and again the bound

$$|D_i D_j D_{k,x}(\phi)| \leq \frac{C}{\delta^3}$$

is obtained. □

D.2 The Bounds on the Derivatives of u

We now turn our attention to the derivatives in maximum norm of the integrands in Chapter 4. We recall (4.7):

$$\|\mathcal{D}^j v(x,t)\|_\infty = \max_{|\alpha|=j} \|D^\alpha v(x,t)\|_\infty$$

where $\|\mathcal{D}^j v(x,t)\|_\infty$ measures all space derivatives of order j in maximum norm. We first prove:

Theorem D.2.1. *Let v be a C^∞ function, and consider the function u from the Navier-Stokes equations, with the divergence-free condition $\nabla \cdot u = 0$. Then for $1 \leq i, j \leq 3$*

$$|D_i(v u_i u_j)| \leq \|v\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_i v| \|u\|_\infty^2 \tag{D.2.1}$$

Proof. We have $D_i(vu_iu_j) = v(D_i(u_iu_j)) + (D_iv)(u_iu_j)$. Then

$$\begin{aligned}
|D_i(vu_iu_j)| &\leq |\sum_{i,j} D_i(vu_iu_j)| \\
&= |\sum_{i,j} v(D_i(u_iu_j)) + (D_iv)(u_iu_j)| \\
&= \sum_{i,j} |v[u_iD_iu_j + u_jD_iu_i] + (D_iv)(u_iu_j)| \\
&\leq \sum_{i,j} |v[u_iD_iu_j] + (D_iv)(u_iu_j)| \\
&\leq \sum_{i,j} |v||u_iD_iu_j| + |(D_iv)|(u_iu_j)| \\
&\leq \|v\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iv|\|u\|_\infty^2
\end{aligned}$$

□

We now prove

Theorem D.2.2. *Let v be a C^∞ function, and consider the function u from the Navier-Stokes equations, with the divergence-free condition $\nabla \cdot u = 0$. Then for for all i, j with $1 \leq i, j \leq 3$*

$$|D_iD_j(vu_iu_j)| \leq \|v\|_\infty \|\mathcal{D}u\|_\infty^2 + |D_iv|\|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iD_jv|\|u\|_\infty^2 \qquad \text{(D.2.2)}$$

Proof. Recalling from (B.3.1) that $\sum_{i,j=1}^3 D_iD_j(u_iu_j)(x,t) = \sum_{i,j=1}^3 (D_iu_j)(D_ju_i)$ we compute

$$\begin{aligned}
D_iD_j(vu_iu_j) &= D_i[v(D_i(u_iu_j)) + (D_jv)(u_iu_j)] \\
&= v(D_iD_ju_iu_j) + (D_iv)[u_iD_ju_j + u_jD_ju_i] \\
&+ (D_jv)[u_iD_iu_j + u_jD_iu_i] + (D_iD_jv)(u_iu_j)
\end{aligned}$$

Again, using the divergence-free condition we now obtain

$$\begin{aligned} |D_iD_j(vu_iu_j)| &\leq |\sum_{i,j} D_iD_j(vu_iu_j)| \\ &= |\sum_{i,j} v(D_iD_ju_iu_j) + (D_iv)[u_iD_ju_j + u_jD_ju_i] \\ &+ (D_jv)[u_iD_iu_j + u_jD_iu_i] + (D_iD_jv)(u_iu_j)| \\ &\leq \|v\|_\infty \|Du\|_\infty^2 \\ &+ (|D_iv| + |D_jv|)(\|u\|_\infty \|\mathcal{D}u\|_\infty) + |D_iD_jv|\|u\|_\infty^2 \\ &\leq |v|_\infty \|\mathcal{D}u\|_\infty^2 + C|D_iv|\|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iD_jv|\|u\|_\infty^2 \end{aligned}$$

□

Finally, we will write the bounds of $|D_i(vu_iu_j)|$ and $|D_iD_j(vu_iu_j)|$ in terms of $\|u\|_\infty$, $\|\mathcal{D}u\|_\infty$, and δ.

Corollary D.2.1. *Let v be defined by the C^∞ cut-off function from D.1.1, where*

$$v = \phi(r) = \left(\frac{|x-y|}{\delta}\right)$$

Then

$$|D_i(\phi u_iu_j)| \leq C(\delta^{-1}\|u\|_\infty^2 + \|u\|_\infty \|\mathcal{D}u\|_\infty) \tag{D.2.3}$$

where C is a constant depending on the maximum norm of ϕ,ϕ'.

Proof. From Theorem D.2.1 we have, for all i, j:

$$|D_i(vu_iu_j)| \leq \|v\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iv|\|u\|_\infty^2$$

For $v = \phi$, we have

$$|D_i(\phi u_iu_j)| \leq \|\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_i\phi|\|u\|_\infty^2$$

Now, suppose $C_1 = \|\phi\|_\infty$. Note from Lemma D.1.1, that there exists a constant C_2 where

$$|D_i(\phi)| \leq \frac{C_2}{\delta}$$

C_2 is a constant depending on the maximum norm of ϕ', ϕ'' Then

$$\begin{aligned} |D_i(\phi u_i u_j)| &\leq \|\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + \|D_i\phi\|_\infty \|u\|_\infty^2 \\ &\leq \|\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + \frac{C_1}{\delta}\|u\|_\infty^2 \\ &= C_2\|u\|_\infty \|\mathcal{D}u\|_\infty + \frac{C_1}{\delta}\|u\|_\infty^2 \\ &\leq C(\delta^{-1}\|u\|_\infty^2 + \|u\|_\infty \|\mathcal{D}u\|_\infty) \end{aligned}$$

We note here that C is a constant depending on the maximum norm of ϕ and ϕ' in maximum norm. That is

$$C_1, C_2 \leq C = \max\{\|\phi\|_\infty, \|\phi'\|_\infty\}$$

□

Next we have

Corollary D.2.2. *Let v be defined by the C^∞ cut-off function from D.1.1, where*

$$v = \phi(r) = \left(\frac{|x-y|}{\delta}\right)$$

Then, for all i, j:

$$|D_iD_j(\phi u_i u_j)| \leq C[\|Du\|_\infty^2 + \delta^{-1}\|u\|_\infty \|\mathcal{D}u\|_\infty + \delta^{-2}\|u\|_\infty^2] \tag{D.2.4}$$

where C is a constant depending on the maximum norms of ϕ, ϕ', and ϕ''.

Proof. We begin with theorem D.2.2 and equation (D.2.2)

$$|D_iD_j(vu_iu_j)| \leq \|v\|_\infty \|\mathcal{D}u\|_\infty^2 + |D_iv|\|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iD_jv|\|u\|_\infty^2$$

Then, with $v = \phi$

$$|D_iD_j(\phi u_iu_j)| \le \|\phi\|_\infty \|\mathcal{D}u\|_\infty^2 + |D_i\phi| \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iD_j\phi| \|u\|_\infty^2$$

Now, let $C_1 = \|\phi\|_\infty$, and recall from Lemma (D.1.1) (equation (D.1.1)) that there exists constants C_2 and C_3 depending on ϕ' and ϕ'' such that

$$\|D_i(\phi)\|_\infty \le \frac{C_2}{\delta} \quad \text{and} \quad \|D_iD_j(\phi)\|_\infty \le \frac{C_3}{\delta^2}$$

We now compute

$$\begin{aligned} |D_iD_j(vu_iu_j)| &\le \|\phi\|_\infty \|\mathcal{D}u\|_\infty^2 + \|D_i\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty \\ &+ |D_j\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + \|D_iD_j\phi\|_\infty \|u\|_\infty^2 \\ &\le C_1 \|\mathcal{D}u\|_\infty^2 + C_2\delta^{-1} \|u\|_\infty \|\mathcal{D}u\|_\infty + C_3\delta^{-2} \|u\|_\infty^2 \\ &\le C[\|\mathcal{D}u\|_\infty^2 + \delta^{-1} \|u\|_\infty \|\mathcal{D}u\|_\infty + \delta^{-2} \|u\|_\infty^2] \end{aligned}$$

where $C = \max\{C_1, C_2, C_3\}$, and depends on ϕ, ϕ', ϕ''. □

Finally, we let $v = \phi_{k,x}$ to obtain the final corollary.

Corollary D.2.3. *Let v be defined by the C^∞ cut-off function from D.1.1, where $v = D_{k,x}\phi(r)$, and*

$$v = \phi(r) = \left(\frac{|x-y|}{\delta}\right)$$

Then for all i, j, k:

$$|D_iD_jD_{k,x}(u_iu_j)| \le C[\delta^{-1} \|Du\|_\infty^2 + \delta^{-2} \|u\|_\infty \|\mathcal{D}u\|_\infty + \delta^{-3} \|u\|_\infty^2] \tag{D.2.5}$$

Proof. In this case we begin with theorem D.2.2 and equation (D.2.2):

$$|D_iD_j(vu_iu_j)| \le \|v\|_\infty \|\mathcal{D}u\|_\infty^2 + |D_iv| \|u\|_\infty \|\mathcal{D}u\|_\infty + |D_iD_jv| \|u\|_\infty^2$$

We note that for $v = D_{k,x}\phi$ we have from Lemma D.1 that there is a C_1 dependent on ϕ' and a C_2 dependent on ϕ', ϕ'' such that

$$|D_{k,x}\phi| \leq \frac{C_1}{\delta} \quad \text{and} \quad \mathcal{D}_j D_{k,x}(\phi)| \leq \frac{C_2}{\delta^2}$$

Additionally, we use lemma D.1.2 and equation (D.1.3)

$$|D_i D_j D_{k,x}(\phi)| \leq \frac{C_3}{\delta^3}$$

Then, with $v = D_{k,x}\phi$

$$\begin{aligned} |D_i D_j D_{k,x}\phi(u_i u_j)| &\leq \|D_{k,x}\phi\|_\infty \|\mathcal{D}u\|_\infty^2 + \|D_j D_{k,x}\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty \\ &+ \|D_i D_j D_{k,x}\phi\|_\infty \|u\|_\infty^2 \end{aligned}$$

We now produce

$$\begin{aligned} \|D_i D_j D_{k,x}(u_i u_j)\|_\infty &\leq \|D_{k,x}\phi\|_\infty \|\mathcal{D}u\|_\infty^2 + \|D_j D_{k,x}\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty \\ &+ \|D_i D_{k,x}\phi\|_\infty \|u\|_\infty \|\mathcal{D}u\|_\infty + \|D_i D_j D_{k,x}\phi\|_\infty \|u\|_\infty^2 \\ &\leq C_1\delta^{-1}\|\mathcal{D}u\|_\infty^2 + 2C_2\delta^{-2}\|u\|_\infty \|\mathcal{D}u\|_\infty + C_3\delta^{-3}\|u\|_\infty^2 \\ &\leq C[\delta^{-1}\|\mathcal{D}u\|_\infty^2 + \delta^{-2}\|u\|_\infty \|\mathcal{D}u\|_\infty + \delta^{-3}\|u\|_\infty^2] \end{aligned}$$

Where $C = \max\{C_1, C_2, C_3\}$ and depends on the maximum norms of ϕ', ϕ'', ϕ''' □

This completes the necessary computations in terms of the maximum norm.

D.3 Ancillary Results

In this section we note various technical results needed in Chapter 4 and 5. We display here the Leibnitz differentiation rule that will be used in Chapter 4.

Lemma D.3.1. *The n^{th} derivative of the product $f \cdot g$ is given by*

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \tag{D.3.1}$$

Recall the definition of the Beta function

Definition D.3.1. *The Beta function is defined to be*

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt \tag{D.3.2}$$

One may express the Beta function as

$$B(x, y) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

The *incomplete* Beta function is defined to be

Definition D.3.2. *The incomplete Beta function is defined to be*

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt \tag{D.3.3}$$

We now prove the following lemma

Lemma D.3.2. *Let t be a variable, $t > 0$. Then*

$$I = \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2}\, ds = t^{(1-j)/2} B_{1/2}(1/2, 1 + j/2) \tag{D.3.4}$$

where $B_x(a, b)$ is the incomplete Beta function.

Proof. Let $u = t - s$, $s = t - u$. At $s = t$, $u = 0$, while at $s = t/2$, $u = t/2$, with $du = -ds$. Then

$$\begin{aligned} I &= \int_{t/2}^{t} (t-s)^{-1/2} s^{-j/2}\, ds \\ &= -\int_{t/2}^{0} u^{-1/2}(t-u)^{-j/2}\, du \\ &= \int_0^{t/2} u^{-1/2}(t-u)^{-j/2}\, du \end{aligned}$$

Now let $r = \frac{u}{t}$. Then $dr = \frac{du}{t}$. At $u = t/2$, $r = 1/2$ while at $u = 0$, $r = 0$. We now compute

$$
\begin{aligned}
I &= \int_0^{t/2} u^{-1/2}(t-u)^{-j/2}\,du \\
&= \int_0^{t/2} (tr)^{-1/2}(t-u)^{-j/2}\,\frac{dr}{t} \\
&= \int_0^{t/2} (tr)^{-1/2}\Big(1-\frac{u}{t}\Big)^{-j/2} t\,dr \\
&= t^{-1/2}\cdot t\cdot t^{-j/2}\int_0^{1/2} r^{-1/2}(1-r)^{-j/2}\,dr \\
&= t^{1/2}\cdot t^{-j/2}\int_0^{1/2} r^{-1/2}(1-r)^{-j/2}\,dr \\
&= t^{(1-j)/2}\int_0^{1/2} r^{-1/2}(1-r)^{-j/2}\,dr \\
&= t^{(1-j)/2}B_{1/2}(1/2, 1+j/2)
\end{aligned}
$$

The required result is proved. □

Appendix E

Theorems for Pressure Derivatives

E.1 Bounds on the Local Pressure Derivative

In this section we prove the bounds on the local pressure derivative of order $j-1$. That is, $D^{j-1}(D_{q,x})(p_{loc}(x))$. This is used in determining bounds on the derivatives $\mathcal{D}^j u$ in maximum norm. The theorem is listen in the main paper as Theorem 5.1.1. It is used to prove Proposition 5.1.1. Finally, it will be noted here that $C_0 = \frac{1}{4\pi}$.

Theorem E.1.1. *Consider the Navier-Stokes equation*

$$v_t = \Delta v + D^j Q, \quad v = D^j u$$

u a solution, and where

$$Q = -\nabla p - u \cdot \nabla u$$

Let $j \geq 1$ and assume that for $0 \leq k \leq j-1$ there are constants K_k independent of t and f such that

$$t^{k/2} \|\mathcal{D}^k u(t)\|_\infty \leq K_k \|f\|_\infty \quad \textit{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2} \tag{E.1.1}$$

Then there exists a constant C *independent of* t *and* f *such that*

$$\|D^{j-1}(D_{q,x})(p_{loc}(x))\|_\infty \leq C(\|f\|_\infty \|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2) \quad \text{(E.1.2)}$$

Proof. As in the previous theorems we will let $\delta = \sqrt{t}$. Applying D^{j-1} to both sides of

$$u_t = \Delta u + Q, \quad v = D^j u$$

we obtain

$$v_t = \Delta v + D^{j-1}Q, \quad v = D^{j-1}u$$

By taking the divergence of

$$v_t = \Delta v + D^{j-1}Q, \quad v = D^{j-1}u$$

we have $-\Delta D^{j-1}p = \sum_{i,k} D_i D_k(D^{j-1}(u_i u_k))$. The solution is given by $D^{j-1}p_{loc} + D^{j-1}p_{glb}$ where

$$D^{j-1}p_{loc}(x) = \sum_{i,k} C_0 \int_{0<|x-y|<2\delta} |x-y|^{-1} D_i D_k(\phi(D^{j-1}(u_i u_k)))\, dy$$

and

$$D^{j-1}p_{glb}(x) = \sum_{i,k} C_0 \int_{|x-y|>\delta} |x-y|^{-1} D_i D_k((1-\phi)(D^{j-1}(u_i u_k)))\, dy$$

We begin with $D_{q,x}(D^{j-1}p_{loc}(x))$.

$$\begin{aligned} D_{q,x}(D^{j-1}p_{loc}(x)) &= \sum_{i,j} C_0 \int_{B(x,2\delta)} D_{q,x}(|x-y|^{-1}) D_i D_k(\phi D^{j-1}(u_i u_k))\, dy \\ &+ \sum_{i,j} C_0 \int_{B(x,2\delta)} |x-y|^{-1}(D_i D_j(D_{q,x}(\phi)) D^{j-1}(u_i u_k))\, dy \\ &= I_1 + I_2 \end{aligned}$$

Using Theorem D.2.2, replacing $u_i u_j$ with $D^{j-1}(u_i u_j)$, and rearranging we have

$$\begin{aligned}
\|D_i D_k(vD^{j-1}(u_i u_k))\|_\infty &= \sum_{i,k} \|vD^{j-1}(D_i u_k D_k u_i)\|_\infty \\
&+ \|(D_i v)D^{j-1}(u_k D_k u_i)\|_\infty \\
&+ \sum_{i,k} \|(D_k v)(u_i D_i u_k)\|_\infty \\
&+ \|(D_i D_k v)D^{j-1}(u_i u_k)\|_\infty \\
&= J_1 + J_2 + J_3 + J_4
\end{aligned}$$

For J_1 we use the Leibnitz differentiation theorem. We note that this term is quadratic in Du. Now for any maximum norm derivatives of order l, with the assumption on $\|\mathcal{D}^k u\|_\infty$:

$$\begin{aligned}
\|D^l(u_i u_k)\|_\infty &= \| \sum_{m=0}^{l} \binom{l}{m} D^m u_i D^{l-m} u_k \|_\infty \\
&\leq \| \sum_{m=0}^{l} C_m D^m u_i D^{l-m} u_k \|_\infty \\
&\leq C_m \| \sum_{m=0}^{l} \|D^m u_i D^{l-m} u_k\|_\infty \\
&\leq C_m \| \sum_{m=0}^{l} \|D^m u_i\|_\infty \|D^{l-m} u_k\|_\infty \\
&\leq C_m \| \sum_{m=0}^{l} \|\mathcal{D}^m u\|_\infty \|\mathcal{D}^{l-m} u\|_\infty \\
&\leq C_m(\|u\|_\infty \|\mathcal{D}^l u\|_\infty + \sum_{m=1}^{l-1} \|\mathcal{D}^m u\|_\infty \|\mathcal{D}^{l-m} u\|_\infty) \\
&\leq C_m(\|u\|_\infty \|\mathcal{D}^l u\|_\infty + \sum_{m=1}^{l-1} t^{-m/2} K_m \|f\|_\infty t^{(m-l)/2} K_{m-l} \|f\|_\infty) \\
&\leq C_m K_{max}(\|u\|_\infty \|\mathcal{D}^l u\|_\infty + t^{-l/2} \|f\|_\infty^2) \\
&\leq C_2(\|u\|_\infty \|\mathcal{D}^l u\|_\infty + t^{-l/2} \|f\|_\infty^2)
\end{aligned}$$

Thus

$$\|D^l(u_iu_k)\|_\infty \le C(\|u\|_\infty\|\mathcal{D}^l u\|_\infty + t^{-l/2}\|f\|_\infty^2) \tag{E.1.3}$$

with C independent of t and f. Replacing the u with Du, and letting $l = j-1$, in equation (E.1.3) we obtain

$$\|D^{j-1}(D_iu_kD_ku_i)\|_\infty = C(\|\mathcal{D}u\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2)$$

Then, for $v = \phi$, and using the assumption on $\|\mathcal{D}^k u\|_\infty$

$$\begin{aligned}
J_1 &= \|vD^{j-1}(D_iu_kD_ku_i)\|_\infty \\
&\le \|v\|_\infty\|D^{j-1}(D_iu_kD_ku_i)\|_\infty \\
&\le C_1C_2(\|\|\mathcal{D}u\|_\infty\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2) \\
&= C_3(\|f\|_\infty t^{-1/2}\|\mathcal{D}^j u\|_\infty + K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2)
\end{aligned}$$

Next, for J_2, and $\|(D_iv)\|_\infty = \|(D_i\phi)\|_\infty \le C\delta^{-1}$

$$\begin{aligned}
J_2 &= \|(D_i\phi)D^{j-1}(u_kD_ku_i)\|_\infty \\
&\le C_2C\delta^{-1}(\|u\|_\infty\|\mathcal{D}^{j-1}u\|_\infty + K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2) \\
&\le C_3\delta^{-1}t^{-(j-1)/2}\|f\|_\infty^2
\end{aligned}$$

J_3 is the same. For J_4, $\|D_iD_kv\|_\infty = \|D_iD_k\phi\|_\infty \le C\delta^{-2}$ and

$$\begin{aligned}
J_4 &= \|(D_iD_k\phi)D^{j-1}(u_kD_ku_i)\|_\infty \\
&\le C_2C\delta^{-2}(\|u\|_\infty\|\mathcal{D}^{j-1}u\|_\infty + K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2) \\
&\le C_3\delta^{-2}t^{-(j-1)/2}\|f\|_\infty^2
\end{aligned}$$

Finally

$$\begin{aligned}
\|D_iD_k(\phi D^{j-1}(u_iu_k))\|_\infty &= J_1+J_2+J_3+J_4 \\
&\le C_3(\|f\|_\infty t^{-1/2}\|\mathcal{D}^j u\|_\infty + K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2) \\
&+ 2C_3\delta^{-1}t^{-(j-1)/2}\|f\|_\infty^2 \\
&+ C_3\delta^{-2}t^{-(j-1)/2}\|f\|_\infty^2
\end{aligned}$$

Now for I_1: The computation is similar to the one in section 4.3

$$\begin{aligned}
\|I_1\|_\infty &= \Big\|\sum_{i,j} C_0\int_{B(x,2\delta)} D_{q,x}(|x-y|^{-1})D_iD_k(\phi D^{j-1}(u_iu_k))\,dy\Big\|_\infty \\
&\le C_0\|D_iD_k(\phi D^{j-1}(u_iu_k))\|_\infty\int_{B(x,2\delta)}|x-y|^{-2}\,dy \\
&\le C_0\|D_iD_k(\phi D^{j-1}(u_iu_k))\|_\infty 2\delta \\
&\le 2C_0\delta(C_3(\|f\|_\infty t^{-1/2}\|\mathcal{D}^j u\|_\infty + K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2) + C_3\delta^{-1}t^{-(j-1)/2}\|f\|_\infty^2 \\
&+ C_3\delta^{-2}t^{-(j-1)/2}\|f\|_\infty^2 \\
&= C_5(\delta\|f\|_\infty t^{-1/2}\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + \delta^{-1}t^{-(j-1)/2}\|f\|_\infty^2
\end{aligned}$$

For $\delta = t^{1/2}$ we have

$$\|I_1\|_\infty \le \|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2$$

Consider now I_2. First, from equation (D.2.5) and $v = D_{q,x}\phi$, the structure is exactly the same as I_1.

$$\begin{aligned}
\|(D_iD_j(D_{q,x}(\phi))D^{j-1}(u_iu_k))\|_\infty &= \sum_{i,k}\|vD^{j-1}(D_iu_kD_ku_i)\|_\infty \\
&+ \|(D_iv)D^{j-1}(u_kD_ku_i)\|_\infty \\
&+ \|(D_kv)(u_iD_iu_k)\|_\infty \\
&+ \|(D_iD_kv)D^{j-1}(u_iu_k)\|_\infty \\
&= J_5+J_6+J_7+J_8
\end{aligned}$$

We note here that $J_1 \equiv J_5, J_2 \equiv J_6, J_3 \equiv J_7, J_4 \equiv J_8$. The only change is that

$$\|D_{k,x}\phi\|_\infty \le C\delta^{-1} \quad \|D_i D_{k,x}\phi\|_\infty \le C\delta^{-2} \quad \|D_i D_k D_{q,x}\phi\|_\infty \le C\delta^{-3}$$

Thus

$$\begin{aligned}
\|I_2\|_\infty &\le C_0\|D_iD_k(D_{q,x}\phi D^{j-1}(u_iu_k))\|_\infty \int_{B(x,2\delta)} |x-y|^{-1}\,dy \\
&\le C_0\|D_iD_k(D_{q,x}\phi D^{j-1}(u_iu_k))\|_\infty C_1\delta^2 \\
&\le 2C_0C_1\delta^2(C_3(\|f\|_\infty \delta^{-1}t^{-1/2}\|\mathcal{D}^j u\|_\infty \\
&+ \delta^{-2}K_{j-1}t^{-(j-1)/2}\|f\|_\infty^2) \\
&+ C_3\delta^{-2}t^{-(j-1)/2}\|f\|_\infty^2 \\
&+ C_3\delta^{-3}t^{-(j-1)/2}\|f\|_\infty^2 \\
&= C(\delta\|f\|_\infty t^{-1/2}\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + \delta^{-1}t^{-(j-1)/2}\|f\|_\infty^2)
\end{aligned}$$

For $\delta = t^{1/2}$ we have

$$\|I_2\|_\infty \le \|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2$$

Thus we have

$$\begin{aligned}
\|D_{q,x}p_{loc}\|_\infty &= \|I_1 + I_2\|_\infty \\
&= \|I_1\|_\infty + \|I_2\|_\infty \\
&= C_5(\|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2) \\
&+ C_6(\|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2) \\
&\le C_7(\|f\|_\infty\|\mathcal{D}^j u\|_\infty + t^{-(j-1)/2}\|f\|_\infty^2 + t^{-j/2}\|f\|_\infty^2)
\end{aligned}$$

This is the required result. □

E.2 Bounds on the Global Pressure Derivative

Our next theorem concerns $\|D_{q,x}p_{glb}\|_\infty$. Again, this is used in Proposition 5.1.1 to prove bounds on the derivative $\mathcal{D}^j u$. In the main paper it is listed as Theorem 5.1.2.

Theorem E.2.1. *Consider the Navier-Stokes equation*

$$v_t = \Delta v + D^j Q, \quad v = D^j u$$

u a solution, and where

$$Q = -\nabla p - u \cdot \nabla u$$

Let $j \geq 1$ and assume that for $0 \leq k \leq j-1$ there are constants K_k independent of t and f such that

$$t^{k/2}\|\mathcal{D}^k u(t)\|_\infty \leq K_k \|f\|_\infty \quad \textit{for} \quad 0 < t \leq \frac{c_0}{\|f\|_\infty^2}$$

Then there exists a constant C independent of t and f such that

$$\|D^{j-1}(D_{q,x}(p_{glb}(x))\|_\infty \leq C\|f\|_\infty^2 t^{-j/2} \tag{E.2.1}$$

Proof. We begin be applying integration by parts to

$$D^{j-1}p_{glb}(x) = \sum_{i,k} C_0 \int_{|x-y|>\delta} |x-y|^{-1} D_i D_k((1-\phi)(D^{j-1}(u_i u_k)))\,dy$$

We find that

$$D^{j-1}p_{glb}(x) = \sum_{i,k} C_0 \int_{|x-y|>\delta} D_i D_k(|x-y|^{-1})((1-\phi)(D^{j-1}(u_i u_k)))\,dy$$

We now apply $D_{q,x}$ under the integral sign.

$$\begin{aligned}
D^{j-1}(D_{q,x}(p_{glb}(x)) &= D_{q,x}(\sum_{i,k} C_0 \int_{|x-y|>\delta} D_i D_k(|x-y|^{-1})((1-\phi)(D^{j-1}(u_i u_k)))\,dy) \\
&= \sum_{i,k} C_0 \int_{|x-y|>\delta} D_{q,x}(D_i D_k(|x-y|^{-1})((1-\phi)))(D^{j-1}(u_i u_k)))\,dy) \\
&= \sum_{i,k} C_0 \int_{|x-y|>\delta} (D_{q,x} D_i D_k(|x-y|^{-1})((1-\phi)))(D^{j-1}(u_i u_k)))\,dy) \\
&+ \sum_{i,k} C_0 \int_{|x-y|>\delta} D_i D_k(|x-y|^{-1})(D_{q,x}(1-\phi)))(D^{j-1}(u_i u_k)))\,dy) \\
&= I_3 + I_4
\end{aligned}$$

As in the proof of Lemma 4.3.1, we note that

$$\begin{aligned}
\Big| \int_{|x-y|>\delta} D_{q,x} D_i D_k(|x-y|^{-1})\,dy \Big| &\leq C_1 \int_{|x-y|>\delta} |x-y|^{-4}\,dy \\
&= C_1 \delta^{-1}
\end{aligned}$$

Note C_1 is independent of t and f. We now must estimate $\|D^{j-1}(u_i u_k)\|_\infty$, We use the Leibnitz differentiation theorem with $l = j-1$. Additionally, we use equation (E.2.1):

$$\begin{aligned}
\|D^{j-1}(u_i u_k)\|_\infty &\leq \|\sum_{m=0}^{j-1} D^m u D^{j-1-m} u\|_\infty \\
&\leq C_2 \sum_{m=0}^{j-1} \|\mathcal{D}^m u\|_\infty \|\mathcal{D}^{j-1-m} u\|_\infty \\
&\leq C_2 \|f\|_\infty^2 t^{-(1-j)/2}
\end{aligned}$$

By construction, C_2 is independent of t and f. Then

$$\begin{aligned}
\|I_3\|_\infty &= \Big\| \sum_{i,k} C_0 \int_{|x-y|>\delta} (D_{q,x} D_i D_k(|x-y|^{-1})((1-\phi)))(D^{j-1}(u_i u_k)))\, dy) \Big\|_\infty \\
&\le C_0 \Big\| \int_{|x-y|>\delta} (D_{q,x} D_i D_k(|x-y|^{-1})((1-\phi)))(D^{j-1}(u_i u_k)))\, dy) \Big\|_\infty \\
&\le C_0 \|1-\phi\|_\infty \|D^{j-1}(u_i u_k)\|_\infty \int_{|x-y|>\delta} |(D_{q,x} D_i D_k(|x-y|^{-1})|\, dy \\
&\le C_0 C_3 C_1 \delta^{-1} \|D^{j-1}(u_i u_k)\|_\infty \\
&\le C_0 C_3 C_1 \delta^{-1} C_2 \|f\|_\infty^2 t^{-(1-j)/2}
\end{aligned}$$

Replacing $\delta = t^{1/2}$, we have

$$\begin{aligned}
\|I_3\|_\infty &\le C_0 C_3 C_1 \delta^{-1} C_2 \|f\|_\infty^2 t^{-(1-j)/2} \\
&= C_0 C_3 C_1 t^{-1/2} C_2 \|f\|_\infty^2 t^{(1-j)/2} \\
&= C_4 \|f\|_\infty^2 t^{-j/2}
\end{aligned}$$

C_4 is then independent of t and f by construction. Now for I_4. We first consider $\|(D_{q,x}(1-\phi)\|_\infty$. We note that by construction, $\phi' = 0$ for $r < 1$, and $r > 2$. So $\phi' \neq 0$ on $[1, 2]$. We then find that I_4 becomes

$$I_4 = \sum_{i,k} C_0 \int_{\delta<|x-y|<2\delta} D_i D_k(|x-y|^{-1})(D_{q,x}(1-\phi)))(D^{j-1}(u_i u_k)))\, dy$$

Again as in the proof of Lemma 3.2.1 we find that

$$\begin{aligned}
\Big| \int_{\delta<|x-y|<2\delta} D_i D_k(|x-y|^{-1})\, dy \Big| &\le \int_{\delta<|x-y|<2\delta} |x-y|^{-3}\, dy \\
&= \ln 2 \\
&= C_3
\end{aligned}$$

We also note that

$$\|(D_{q,x}(1-\phi)\|_\infty \le C_1 \delta^{-1}$$

Again, as in the proof for I_3, we find that

$$\|D^{j-1}(u_i u_k)\|_\infty \leq C_2 \|f\|_\infty^2 t^{-(1-j)/2}$$

We now estimate $\|I_4\|_\infty$:

$$\begin{aligned}
\|I_4\|_\infty &= \Big\| \sum_{i,k} C_0 \int_{\delta<|x-y|<2\delta} D_i D_k(|x-y|^{-1})(D_{q,x}(1-\phi)))(D^{j-1}(u_i u_k)))\, dy \Big\|_\infty \\
&\leq C_0 \|(D_{q,x}(1-\phi)\|_\infty \|D^{j-1}(u_i u_k)\|_\infty \Big| \int_{\delta<|x-y|<2\delta} D_i D_k(|x-y|^{-1})\, dy \Big| \\
&\leq C_0 C_1 \delta^{-1} C_3 \|D^{j-1}(u_i u_k)\|_\infty \\
&\leq C_0 C_1 \delta^{-1} C_3 C_2 \|f\|_\infty^2 t^{(1-j)/2} \\
&= C_5 \delta^{-1} \|f\|_\infty^2 t^{(1-j)/2}
\end{aligned}$$

Using $\delta = t^{1/2}$ we have

$$\begin{aligned}
\|I_4\|_\infty &\leq C_4 \delta^{-1} \|f\|_\infty^2 t^{(1-j)/2} \\
&= C_4 t^{-1/2} \|f\|_\infty^2 t^{(1-j)/2} \\
&= C_5 \|f\|_\infty^2 t^{-j/2}
\end{aligned}$$

We now may estimate $|D^{j-1} p_{glb}\|_\infty$

$$\begin{aligned}
\|D^{j-1} D_{q,x} p_{glb}\|_\infty &= \|I_3 + I_4\|_\infty \\
&\leq \|I_3\|_\infty + \|I_4\|_\infty \\
&\leq C_4 \|f\|_\infty^2 t^{-j/2} + C_5 \|f\|_\infty^2 t^{-j/2} \\
&\leq C_6 \|f\|_\infty^2 t^{-j/2}
\end{aligned}$$

By the same reasoning as in the case of I_3, C_5, and C_6 are independent of t and f. Thus the required result is proved. □

References

[1] H. Amann, *On the Strong Solvability of the NavierStokes Equations*, J. Math. Fluid Mech. 2 (2000) pp. 16-98

[2] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), pp. 289-309

[3] A.P.Calderón A. Zygmund, *On the Existence of Certain Singular Integrals*, Acta Mathematica Volume **88**, Number 1, pp. 85-139

[4] J.R. Cannon, G.H. Knightly, *A note on the Cauchy problem for the NavierStokes equations*, SIAM J. Appl. Math. 18 (1970) pp. 641-644

[5] Der-Chen Chang, Cora Sardosky, *On Functions of Bounded Mean Oscillation*, Taiwanese Journal of Mathematics, Vol. 10, No. 3, March 2006 pp. 573-601

[6] Charles R. Doering, J.D.Gibbon, *Applied Analysis of the Navier Stokes Equations*, Cambridge University Press New York,New York 2004. 0-521-44568-X

[7] Lawrence C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics vol. 19 American Mathematical Society Providence, Rhode Island 1998. ISBN 0-8218-0772-2

[8] Fefferman, Charles, *Characterizations of bounded mean oscillation*, Bulletin of the American Mathematical Society 77 (1971) (4): pp. 587-588

[9] Fefferman, C.; Stein, E.M., *H^p spaces of several variables*, Acta Mathematica 129:(1972) pp. 137-193

[10] Gerald B. Folland, *Introduction to Partial Differential Equations*, Second edition, Princeton University Press Princeton, New Jersey 1995. ISBN 0-691-04361-2

[11] Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second edition, John Wiley ans Sons, Inc. New York 1999.

[12] G.P. Galdi, P. Maremonti, *A Uniqueness Theorem for Viscous Fluid Motions in Exterior Domains*, Arch. Rational Mech. Anal. 91 (1986) pp. 375-384

[13] Y. Giga, K. Inui, J. Kato, S. Matsui, *Remarks on the Uniqueness of Bounded Solution of the Navier-Stokes Equations*, Hokido University Preprint Series in Mathematics **497** October, 2000

[14] Y. Giga, K. Inui, J. Kato, S. Matsui, *On the Cauchy Problem for the Navier-Stokes Equations with Non-Decaying Initial Data*,Quaderni di Mathematica (**4**) 1999, pp. 28-68

[15] Y. Giga, *Weak and Strong Solutions of the Navier-Stokes Initial Value Problem*, RIMS Kyoto University (**19**) 1983, pp. 887-910

[16] David Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of 1998 ed., Springer-Verlag New York 2001.ISBN 3-540-41160-7

[17] John, F. (1961), *Partial Differential Equations*, Applied Mathematical Sciences (1),Springer-Verlag New York 1982. ISBN 0-387-90609-6

[18] John, F. (1961), *Rotation and strain*, Communications on Pure and Applied Mathematics 14 (3): pp. 391-413

[19] John, F. and Nirenberg, L., *On functions of bounded mean oscillation.* Communications on Pure and Applied Mathematics, 14:(August 1961) pp. 415-426

[20] Daniel D. Joseph *Potential flow of viscous fluids: Historical Notes* International Journal of Multiphase Flow 32 (2006) pp. 285-310

[21] T. Kato *Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions*, Math Z. **187**,(1984), pp. 471-480

[22] Kato J., *The uniqueness of nondecaying solutions for the Navier-Stokes equations.* Arch. Rational Mech. Anal. 169 (2003), pp. 159-175

[23] N. Kim, D. Chae, *On the Uniqueness of the Unbounded Classical Solutions of the NavierStokes and Associated Equations*, J. Math. Anal. Appl. 186 (1994) pp. 91-96

[24] J.R. Cannon, G.H. Knightly, *On the Cauchy problem for the NavierStokes equations*, SIAM J. Math. Anal 3 (1971) pp. 506-511

[25] Heinz-Otto Kreiss and Jens Lorenz, *Apriori Estimates in Terms of the Maximum Norm for the Solutions of the Navier-Stokes Equations.* **J. Diff. Eqn.**, 2004 pp. 216-231

[26] Heinz-Otto Kreiss and Jens Lorenz, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, Classics in Applied Mathematics. 247. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). (2004). ISBN 0-89871-565-2

[27] H.-O Kreiss, J. Lorenz, M.J.Naughton, *Convergence of the Solutions of the Compressible to the Solutions of the Incompressible Navier-Stokes Equations*, Advances in Pure and Applied Mathematics **12**, 1991, pp. 187-214

[28] J. Leray *Essai sur le Mouvement D'un Fluide Visqueux Emplissant L'Espace*, Acta Math. **63** (1934), pp. 193-248

[29] Jens Lorenz and Oksana Guba, In *Advanced Differential Equations* Lecture Notes (unpublished) , Albuquerque, New Mexico Spring 2003 pp. 14-36

[30] Jens Lorenz Verbal Communication, Albuquerque, New Mexico May 16,2012.

[31] Andrea L. Bertozzi and Andrew J Majda, *Vorticity and Incompressible Flow*, Cambidge Texts in Applied Mathematics, Cambidge University Press New York 2002. ISBN 0-521-63948-4

[32] Navier, C.L.M.H., *On the laws of motion of fluids taking into consideration the adhesion of the molecules.*, 1822. Ann. Chim. Phys. 19, pp. 234-245

[33] Umberto Neri, *Some properties of functions with bounded mean oscillation.* U Neri - Studia Math, 1977, pp. 63-75

[34] Poisson, S.D., 1829. *Memoire sur les equations generales de lequilibre et du mouvement des corps solides elastiques et des fluides.* J.E cole Poly. 13, pp. 1-174

[35] Cora Sadosky *Interpolation of Operators and Singular Integrals: An Introduction to Harmonic Analysis*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Decker, Inc., New York, 1979. ISBN 0-8427-6883-3

[36] Saint-Venant, A.-J.-C.-B. De., 1843. Note to be added to the memoir on the dynamics of fluids. Compt. Rend. 17, pp. 1240-1243

[37] Saint-Venant, A.-J.-C.-B., De., 1869. *Proble'me des mouvements que peuvent prendre les divers points dune masse liquide, ou solideductile, continue dans un vase a' parois verticales, pendant son ecoulement par un orifice horizontal interieur.* C.R. Acad. Sci. Paris 68, pp. 221-237

[38] Ivar Stakgold, *Boundary Value Problems of Mathematical Physics*, Classics in Applied Mathematics. 29. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). (2000). ISBN 0-89871-456-7

[39] Elias M. Stein, *Harmonic Analysis*, Princeton University Press Princeton, New Jersey 1993. ISBN-13:978-0-691-03216-0

[40] Elias M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press Princeton, New Jersey 1970. ISBN-13:978-0-691-08079

[41] Elias M. Stein, *Singular Integrals:The Roles of Calderón and Zygmund* Notices Amer. Math. Soc. 45 (1998) pp. 1130-1140

[42] Elias M. Stein, *Singular Integrals, Harmonic Functions and Differentiability Properties of Functions of Several Variables*,Proc. Symp. in Pure Math 10(1967) pp. 316-335

[43] Stokes, G.G., 1851(read on December 9, 1850). *On the effect of the internal friction of fluids on the motion of pendulums.* Trans. Cambridge Phil. Soc. IX, 8106, Mathematical and Physical Papers 3, pp. 1-141

[44] Stokes, G.G., 1880. *Considerations relative to the greatest height of oscillatory irrotational waves which can be propagated without change of form.* Mathematical and Physical Papers I, pp. 225-228

[45] Stokes, G.G., 1845. *On the theories of the internal friction of fluids in motion.* Trans. Cambridge Philos. Soc. 8, 287-319, Mathematical and Physical Papers 1, pp. 75-129

[46] Uchiyama, A., *A constructive proof of the Fefferman-Stein decomposition of BMO (*$\mathbb{R}^n$ Acta Mathematica 148:(1982) pp. 215-241

[47] M. Wiegner, *The Navier–Stokes equations — a neverending challenge?*, DMV Jahresbericht, 101. Band Heft 1, pp. 1-25