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Standard Closure Operations on Several Rings of Dimension One

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Dedication

To my mother, Jeannette.

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Abstract

There are numerous results regarding the cardinality of the set of star and semistar operations on a domain R. However the cardinality of the set of such operations has only just begun to be examined in the non-domain setting. Epstein [Ep2] has shown a correspondence between the set of *finite-type* semistar operations of a ring and the *finite-type* standard closures of a ring. I will classify the set of *finite-type* standard closures on several rings of dimension one.

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Glossary

R	a commutative ring with unity
$\mathcal{I}(R)$	the set of ideals of R
$\mathcal{I}_f(R)$	the set of finitely generated ideals of R
Q	the total ring of fractions of R
$\overline{\mathcal{F}}(R)$	the R-submodules of Q
$\mathcal{F}(R)$	the fractional ideals of R
$\mathcal{F}_f(R)$	the finitely generated fractional ideals of R
$S(R)$	the set of semistar operations on R
$S_f(R)$	the set of *finte-type* semistar operations on R
$S'(R)$	the set of star operations on R
$\text{rref}(A)$	the *reduced row echelon* form of a matrix A

Chapter 1

Introduction

1.1 Closure Operations

Let R be a commutative ring with unity. We denote the set of ideals of R by $\mathcal{I}(R)$ and the set of finitely generated ideals of R by $\mathcal{I}_f(R)$. Let Q be the total ring of fractions of R. A fractional ideal of R is an R-submodule A of Q satisfying the property that there exists a regular element $x \in R$ such that $xA \subseteq R$. We denote the R-submodules of Q by $\overline{\mathcal{F}}(R)$, the fractional ideals of R by $\mathcal{F}(R)$ and the finitely generated fractional ideals by $\mathcal{F}_f(R)$.

Definition 1.1.1. *A* **closure operation** *on the set of ideals of R is a function $c : \mathcal{I}(R) \to \mathcal{I}(R)$ which for $I, J \in \mathcal{I}(R)$ satisfies:*

- *(Extension) $I \subseteq I^c$.*
- *(Order Preservation) If $I \subseteq J$ then $I^c \subseteq J^c$.*
- *(Idempotence) $(I^c)^c = I^c$.*

We say that c is of finte-type if $I^c = \bigcup\{J^c | J \subseteq I \text{ and } J \in \mathcal{I}_f(R)\}$. Closure operations $c : \mathcal{F}(R) \to \mathcal{F}(R)$ or $c : \overline{\mathcal{F}}(R) \to \overline{\mathcal{F}}(R)$ are defined similarly.

Example 1.1.2. *Suppose R has only three proper ideals, I, J and (0) and the ideal lattice is as shown in figure 1.1.2. Let $c : \mathcal{I}(R) \to \mathcal{I}(R)$ be the map such that $(0)^c = I, J^c = R$ and $I^c = I$. Then c is a closure operation.*

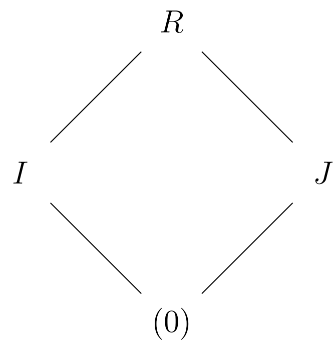

Figure 1.1: Example 1.1.2: Ideal lattice of R.

(Actually there are a total of 7 closure operations on the set of ideals of R.)

Example 1.1.3. *Trivial closures*

- *The identity closure, $I^c = I$ for all $I \in \mathcal{I}(R)$.*
- *The indiscrete closure, $I^c = R$ for all $I \in \mathcal{I}(R)$.*

Example 1.1.4. *Some non-trivial closures*

- *The radical of an ideal $I \subseteq R$, $\operatorname{rad}(I) = \sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \geq 1\}$ or equivalently $\sqrt{I} := \cap\{\mathfrak{p} \in SpecR | I \subseteq \mathfrak{p}\}$.*
- *The integral closure, $\overline{I} := \{r \in R \mid \exists n \in \mathbb{N} \text{ and } a_i \in I^i \text{ s.t. } r^n + \sum_{i=1}^{n} a_i r^{n-i} = 0\}$.*
- *Suppose R is a Noetherian ring of prime characteristic $p > 0$. If $x \in R$ then x is in the tight closure of $I = (x_1, \dots, x_n)$ if there exists a c not in any minimal prime such that $cx^q \in I^{[q]}$ for all large q, where $I^{[q]} = (x_1^q, \dots, x_n^q)$.*

- *Fix an ideal* $\mathfrak{a}$ *of* R. *Then the* $\mathfrak{a}$*-saturation,* $\cup_{n\in\mathbb{N}}(I : \mathfrak{a}^n) = \{r \in R | \exists n \in \mathbb{N} \text{ such that } \mathfrak{a}^n r \subseteq I\}$, *is a closure operation.*

- *Let* R *be a ring of prime characteristic. The Frobenius closure* F *is defined as follows: for an ideal* I *and an element* $x \in R$, $x \in I^F$ *if there exists an* $n \in \mathbb{N}$ *such that* $x^{p^n} \in I^{[p^n]}$.

1.2 Semistar and Standard Closure Operations

Several types of closure operations have been of considerable interest to mathematicians including star and semistar operations. Star operations were introduced by Krull in his 1935 book Idealtheorie [Kr]. Star operations were generalized to semistar operations by Matsuda and Okabe [MO]. Research with regards to both star and semistar operations has been conducted primarily in the domain setting. Huckaba defines the $*$-operation in [Huc]. This is a semistar operation over Marot rings. Epstein gives a slightly different viewpoint on star operations in [Ep1] and semistar operations in [Ep2] over more general commutative rings. Epstein also introduced standard closure operations and weakly prime operations in [Ep2]. He then proved an important correspondence between the set of *finite-type* semistar operations of a ring and the *finite-type* standard closures of a ring.

Definition 1.2.1. *A set map* $\star : \overline{\mathcal{F}}(R) \to \overline{\mathcal{F}}(R)$ *is a* ***semistar operation*** *provided it is a closure operation and it satisfies the divisibility property:* $uA_\star = (uA)_\star$ *for all* $A \in \overline{\mathcal{F}}(R)$ *and all units* u *of* Q.

Definition 1.2.2. *A set map* $\star : \mathcal{F}(R) \to \mathcal{F}(R)$ *is a* ***star*** *operation provided it has the following properties:*

- $\star$ *is a closure operation*

- *(Divisibility)* $uA_\star = (uA)_\star$ *for all* $A \in \mathcal{F}(R)$ *and all units* u *of* Q
- $R_\star = R$

A *star operation* $\star$ extends to a *semistar operation* by defining $A_\star = Q$ for any R-module $A \in \overline{\mathcal{F}}(R)/\mathcal{F}(R)$.

There are *semistar operations* which are not obtained by extending *star operations*.

Example 1.2.3. *Let* $R = k[[x^3, x^4, x^5]]$. *Thus* $Q = k((x))$. *Consider the overring* $N = k[[x^2, x^3]]$. *Let* M *be an* R*-submodule of* $k((x))$. *Define* $M_\star := MN$. *This is a semistar operation. However it is not the extension of a star operation since* $R_\star = RN = N$.

Definition 1.2.4. *[Ep2] A closure* $c : \mathcal{I}(R) \to \mathcal{I}(R)$ *is* ***standard*** *if for all ideals* I,

$$((xI)^c : x) = I^c$$

for any regular element $x \in R$.

Definition 1.2.5. *[Ep2] A closure* $c : \mathcal{I}(R) \to \mathcal{I}(R)$ *is* ***weakly prime*** *if for all ideals* I *and for any regular element* $x \in R$, $xI^c \subseteq (xI)^c$.

Remark 1.2.6. *Any standard closure is also weakly prime. There are weakly prime closures that are not standard.*

Example 1.2.7. *Let* $R = \frac{k[[X,Y]]}{(XY)}$. *The radical closure is weakly prime. However, it is not standard. For instance, let* $w = x+y$ *and* $I = (x^2, y^3)$. *We have* $(\sqrt{(wI)} : w) = R$ *but* $\sqrt{I} = (x, y)$.

Let $S(R)$ denote the set of semistar operations on R and $S'(R)$ denote the set of star operations on R. In the domain case (for not necessarily *finite-type* operations) there are numerous results regarding the cardinality of $|S(R)|$ or $|S'(R)|$. I will mention a few of them here:

Theorem. *[MO] For any integral domain D, we have $|\text{overrings of } D| \leq |S(D)|$.*

Theorem. *[MS] Let R be a conducive domain. Then*

1. *Every star-operation on R has a unique extension to a semistar operation;*
2. *$|S(R)| = 1 + \sum_{T\in[R,L[} |S'(T)|$ where $[R, L[$ is the set of overings of R properly contained in L the total ring of fractions.*

Theorem. *[MS] Let R be an integral domain and T a proper overring of R. Then $|S'(R)| + |S(T)| \leq |S(R)|$, and equality holds if and only if R is a conducive domain which is local and each proper overring of R contains T.*

Theorem. *[HMP] If R is a Noetherian domain, not a field, $|S'(R)| < \infty$, then* $\dim R = 1$.

Theorem. *[HMP] If R is a Noetherian domain, then $|S'(R)| = \prod_{\mathfrak{m}\in\operatorname{Max} R} |S'(R_{\mathfrak{m}})|$.*

Theorem. *[HMP] Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian domain such that $R/\mathfrak{m}$ is finite and the integral closure $\overline{R}$ of R is a finitely generated R-module. Then $|S'(R)| < \infty$.*

Theorem. *[Wh] Let $R = k + x^n k[[x]]$ be a conductive numerical semigroup ring with finite base field k. Then R admits only finitely many star operations.*

White also classifies all star operations on R in the case $n = 4$.

For more general rings (not necessarily domains) Epstein proved the following previously mentioned result:

Theorem 1.2.8. *[Ep2] There is a one to one order preserving correspondence between the set of finite-type standard closure operations on R and the finite-type semistar operations on R.*

Thus by classifying the set of *finite-type* standard closures of a ring we will have classified the *finite-type* semistar operations of that ring.

1.3 Dimension zero, one and two

Let $S_f(R)$ denote the set of *finite-type* semistar operations on R. Under certain assumpations we can show that for rings R with dimension zero $|S_f(R)| < \infty$ and that for rings R with dimension 2 or greater $|S_f(R)| = \infty$.

Throughout, we use the term local ring to mean any ring with unique maximal ideal.

Lemma 1.3.1. *Suppose $(R, \mathfrak{m})$ is a local ring and $\dim R = 0$. Then every closure operation is a standard closure.*

Proof. Note that if x is a unit of R then for all ideals I and closure operations c we have $((xI)^c : x) = (I^c : x) = I^c$. Thus if every regular element $x \in R$ is a unit every closure operation is standard.

If $\mathfrak{m} = (0)$ then R is a field and we are done. Suppose $\mathfrak{m} \neq (0)$. Let $m \in \mathfrak{m}$ such that $m \neq 0$. Since $\dim R = 0$ by Thm 3.1 [Huc] there exists $y \in R$ such that $m^{2n}y = m^n$. This implies $m^n(m^n y - 1) = 0$. Suppose $m^n y - 1 \in \mathfrak{m}$ then $1 \in \mathfrak{m}$. This is a contradiction so $m^n y - 1$ is a unit. Hence $m^n = 0$. Thus every $m \in \mathfrak{m}$ is nilpotent. Since the only regular elements are units every closure operation on R is a standard closure. □

Example 1.3.2. *There are zero dimensional local rings with infinitely many standard closures. Let $R = \mathbb{C}[x, y]/(x, y)^2$. Let $a \in \mathbb{C}^\times$. Define $\star_a : \mathcal{I}(R) \to \mathcal{I}(R)$ by $(x + ay)^{\star_a} = (x, y)$ and $I^{\star_a} = I$, for all $I \neq (x + ay)$. So $\star_a$ is a closure operation and by Lemma 1.3.1 it is a standard closure. Since we have infinitely many $a \in \mathbb{C}^\times$, R has infinitely many standard closures. See Figure 1.2.*

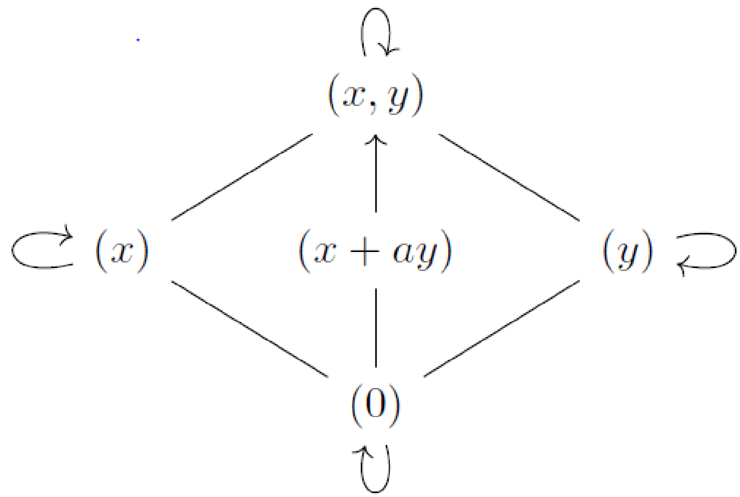

Figure 1.2: Example 1.3.2: Ideal lattice and closure operation.
A dash indicates inclusion and an arrow indicates the closure of the ideal.

Proposition 1.3.3. *Suppose* $(R, \mathfrak{m})$ *is a local ring,* $\dim R = 0$ *and* $\mathfrak{m}$ *is principle. Then* R *has* 2^{s-1} *closure operations (where* s *is the number of ideals in* R*) and every closure operation is standard.*

Proof. By Lemma 1.3.1 every closure operation on R is a standard closure operation.

Let $I \neq 0$ be an ideal in R. Since $\mathfrak{m} = (t)$ every $a \in I$ is of the form $a = r_1 t^{m_1} + \cdots + r_n t^{m_n}$, $m_j \in \mathbb{N}$ for $j \in \{1, \ldots, n\}$ with $m_i \neq m_j$ for $i \neq j$. Without loss of generality suppose $m_1 < m_i$ for all $i \neq 1$. So $a = (r_1 + r_2 t^{m_2 - m_1} + \cdots + r_n t^{m_n - m_1}) t^{m_1}$. Since $r_1 + r_2 t^{m_2 - m_1} + \cdots + r_n t^{m_n - m_1} \notin \mathfrak{m}$ it is a unit. Thus $I = (t^l)$ for some l. Since every element of $\mathfrak{m}$ is nilpotent, $t^s = 0$ for some s. So the ideals of R form a chain,

$$0 \subseteq (t^{s-1}) \subseteq (t^{s-2}) \subseteq (t^{s-3}) \subseteq \cdots \subseteq (t) = \mathfrak{m}.$$

Thus R has finitely many closure operations. In fact by Proposition 2.1 [MV] it has exactly 2^{s-1} closure operations. □

Lemma 1.3.4. *Let* R *be a ring and* P *a prime ideal of* R *containing only zero* *divisors. Define $\star_P : \mathcal{I}(R) \to \mathcal{I}(R)$ as $I^{\star_P} = I + P$. Then $\star$ is a standard closure operation.*

Proof. Let $I, J \subseteq R$ be ideals. We will show directly that the extension, order preservation and idempotence properties hold for $\star_P$.

Extension: $I \subseteq I + P = I^{\star_P}$.
Order Preservation: $I \subset J \Rightarrow I + P \subseteq J + P \Rightarrow I^{\star_P} \subseteq J^{\star_P}$.
Idempotence: $(I^{\star_P})^{\star_P} = (I+P)^{\star_P} = I + P + P = I + P = I^{\star_P}$.

Thus $\star_P$ is a closure operation. It remains to show that $\star_P$ is standard. Suppose $w \in R$ is a regular element. Since $((wI)^{\star_P} : w) = (wI + P : w)$ we need to show $(wI + P : w) = I + P$. Let $x \in (wI + P : w)$. So $wx \in wI + P$. Thus $wx = wi + p$, $i \in I$ and $p \in P$. This gives $w(x - i) = p$. Since w is a regular element, $w \notin P$ thus $x - i \in P$. So $x - i = p'$, $p' \in P$. Hence $x = i + p' \in I + P$. Therefore $\star_P$ is standard. □

Theorem 1.3.5. *Suppose R is a Noetherian ring and there exists $P \in \operatorname{Ass} R$ such that $\operatorname{ht} P \geq 2$. Then $|S_f(R)| = \infty$.*

Proof. Let $\{P_i\}_{i\in\lambda}$ be the set of prime ideals properly contained in P. By Lemma 1.3.4 $\star_{P_i}$ is a standard closure operation for all $i \in \lambda$ and since R is Noetherian it is also a *finite-type* standard closure operation. Then by Lemma 1.2.8 $\star_{P_i}$ corresponds to a *finte-type* semistar operation on R. Suppose $i, j \in \lambda$, $i \neq j$. Thus $(0)^{\star_{P_i}} = P_i \neq P_j = (0)^{\star_{P_j}}$. Hence $\star_{P_i} \neq \star_{P_j}$. If $|\{P_i\}_{i\in\lambda}| = \infty$ we are done.

Since $\operatorname{ht} P \geq 2$ there exist distinct prime ideals P_0 and P_1 such that $P_0 \subseteq P_1 \subseteq P$. Now let T be the quotient of R by P_0 localized at the image of P. Thus T is a Noetherian local ring with $\dim T \geq 2$. Any prime ideal of T will correspond to prime ideal of R contained in P. By the following claim T contains infinitely many height 1 or 0 prime ideals.

Claim. *If R is a Noetherian local ring with $\dim R \geq 2$ then R contains infinitely many prime ideals $\mathfrak{p}$ such that $\operatorname{ht}\mathfrak{p} \leq 1$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R. Suppose to the contrary that R contains only finitely many prime ideals $\mathfrak{p}$ such that $\operatorname{ht}\mathfrak{p} \leq 1$. Choose $a_1 \in \mathfrak{m}$. By Krull's Prinicpal Ideal Theorem a_1 is contained in a prime ideal $\mathfrak{p}_1$ such that $\operatorname{ht}\mathfrak{p}_1 \leq 1$. Now choose $a_2 \in \mathfrak{m}\backslash\mathfrak{p}_1$. Again by Krull's PIT a_2 is contained in a prime ideal $\mathfrak{p}_2$ such that $\operatorname{ht}\mathfrak{p}_2 \leq 1$. By assumption we can only repeat this process finitely many times. Thus $\mathfrak{m}\backslash \cup_1^n \mathfrak{p}_i = \emptyset$ with each $\mathfrak{p}_i$ prime such that $\operatorname{ht}\mathfrak{p}_i \leq 1$. This implies $\cup_1^n\mathfrak{p}_i = \mathfrak{m}$. By prime avoidance $\mathfrak{m} \subseteq \mathfrak{p}_i$ for some $i \in \{1, \ldots, n\}$. This is a contradiction since $\operatorname{ht}\mathfrak{m} \geq 2$. □

□

The explanation in the proof of Theorem 1.3.5 that $|\{P_i\}_{i\in\lambda}| = \infty$ is from [Se] and is included for completeness.

Example 1.3.6. *Let $R = k[[x, y, z, w]]/(xyzw, y^2zw, yz^2w)$. We have the chain of prime ideals $(y) \subseteq (x, y) \subseteq (x, y, z)$. The ideal $\operatorname{Ann}(yzw) = (x, y, z)$ is an associated prime and $\operatorname{ht}(x, y, z) = 2$. Hence by Theorem 1.3.5 $|S_f(R)| = \infty$.*

Okabe and Matsuda show for integral domains D that the number of semistar operations on D is greater than or equal to the number of overrings of D [MO]. We will use a similar method of proof for the next theorem.

Theorem 1.3.7. *Suppose a ring R contains an integral domain E and $S = R[[u_1, \ldots, u_v]]$ with u_i indeterminate for $i \in \{1, \ldots, v\}$. If $v \geq 2$ then $|S_f(S)| = \infty$.*

Proof. Let $T = E[[u_1, \ldots, u_v]]$. Let $P \subseteq T$ be a prime ideal. So T_P is an overring of T. Let S_P be the ring $(T - P)^{-1}S$. Thus S_P is an overring of S. Now define $\star_{S_P}$:

$\overline{\mathcal{F}}(S) \to \overline{\mathcal{F}}(S)$ as $A_{\star_{S_P}} = AS_P$. We claim this is a *finite-type* semistar operation. Let $A, B \in \overline{\mathcal{F}}(S)$, let u be a unit if the total ring of fractions of S. We will show directly that the extension, order preservation, idempotence and divisibility properties hold for $\star_{S_P}$.

Extension: $A \subseteq AS_P = A_{\star_{S_P}}$.
Order Preservation: If $A \subseteq B \Rightarrow AS_P \subseteq BS_P \Rightarrow A_{\star_{S_P}} \subseteq B_{\star_{S_P}}$.
Idempotence: $(A_{\star_{S_P}})_{\star_{S_P}} = (AS_P)_{\star_{S_P}} = (AS_P)S_P = AS_P$.
Divisibility: $u(A_{\star_{S_P}}) = u(AS_P) = (uA)S_P = (uA)_{\star_{S_P}}$.

Thus $\star_{S_P}$ is a semistar operation. It remains to show that $\star_{S_P}$ is of *finite-type*. Let $a \in A_{\star_{S_P}}$. So $a = \sum_{i=1}^{n} a_i r_i$, $a_i \in A$, $r_i \in S_P$. Let $B = Sa_1 + \cdots + Sa_n$. So $B \in \mathcal{F}_f(S)$ and $a \in BS_P = B_{\star_{S_P}}$.

Since $\dim T \geq 2$, T has infinitely many prime ideals P. Let P_1 and P_2 be prime ideals of T such that $P_1 \neq P_2$. Thus $SS_{P_1} \neq SS_{P_2} \Rightarrow S_{\star_{S_{P_1}}} \neq S_{\star_{S_{P_1}}}$. Hence $|S_f(S)| = \infty$. $\square$

For rings R such that $\dim R = 1$ it is not clear under what conditions $|S_f(R)| < \infty$. However, for the dimension one rings S, R_2 and R_3, which will be described later, the set of *finite-type* standard closures is finite. In fact, we can count set of *finite-type* standard closures on these rings.

Chapter 2

Counting Standard Closures on S

In this chapter we will exhibit all of the standard closures on the ring S described below. In order to do this we will begin by classifying all of the ideals of S by their generators.

2.1 The ring S

Let $R = k[x]/(x^2)$, k a field and let $S = R[[t]]$.

Claim 2.1.1. *All of the ideals of S can be expressed in one of the following forms.*

1. (0)
2. (x)
3. (t^l), $l \in \mathbb{N}$
4. (xt^l), $l \in \mathbb{N}$
5. (xt^l, t^n), $l, n \in \mathbb{N}$, $n > l$
6. (x, t^l), $l \in \mathbb{N}$
7. $J = (t^s + \sum_{i=0}^{s-1} s_i t^i)$ *with* $s_0 \neq 0$, $s_i \in R \backslash k$

8. $I = (t^s + \sum_{i=1}^{s-1} s_i t^i)$, $s_i \in R\backslash k$, *such that* $s_i \neq 0$ *for some* i
9. (xt^l, J), $l \in \mathbb{N}$, $l < s$, $s_l = 0$
10. (xt^l, I), $l \in \mathbb{N}$, $l < s$, $s_l = 0$

Proof. Suppose A is a proper ideal of S and $a \in A$. So $a = \sum_{i=0}^{\infty} r_i t^i$, $r_i \in R$. Thus we can write $a = \sum_{i=0}^{\infty} c_i t^i + \sum_{i=0}^{\infty} s_i t^i$, $c_i \in k$, $s_i \in R\backslash k \cup \{0\}$. Since A is a proper ideal a is not invertible. Thus we have $a = \sum_{i=m}^{\infty} c_i t^i + \sum_{i=0}^{\infty} s_i t^i$, where $m = \min\{i|c_i \neq 0\}$, $m > 0$.

Case: At least one $c_i \neq 0$. So $a = t^m \left(\sum_{i=m}^{\infty} c_i t^{i-m}\right) + \sum_{i=0}^{\infty} s_i t^i$. The element $c = \sum_{i=m}^{\infty} c_i t^{i-m} \in k[[t]]$ is invertible. We have $c^{-1}a = t^m + c^{-1}\sum_{i=0}^{\infty} s_i t^i = t^m + \sum_{i=0}^{\infty} s'_i t^i$, $s'_i \in R\backslash k \cup \{0\}$. Since $x(c^{-1}a) = xt^m \in A$ we have $t^m + \sum_{i=0}^{m-1} s'_i t^i \in A$.

Case: $c_i = 0$ for all i. So $a = \sum_{i=0}^{\infty} s_i t^i$.

Subcase: $s_0 \neq 0$. So $a = x\sum_{i=0}^{\infty} \tilde{s}_i t^i$, $\tilde{s}_i \in k$. Thus $s = \sum_{i=0}^{\infty} \tilde{s}_i t^i$ is an invertible element. Hence $s^{-1}a = x \in A$

Subcase: $s_i = 0$ for all i such that $0 \leq i < l$ and $s_l \neq 0$. So $a = xt^l \sum_{i=l}^{\infty} \tilde{s}_i t^{i-l}$, $\tilde{s}_i \in k$. Thus $s = \sum_{i=l}^{\infty} \tilde{s}_i t^{i-l}$ is an invertible element. So $s^{-1}a = xt^l \in A$

Therefore the only possible generators of A are of the form $t^m + \sum_{i=0}^{m-1} s_i t^i$, $s_i \in R\backslash k$, x, and xt^l.

Suppose A contains x and $t^m + \sum_{i=0}^{m-1} s_i t^i$ such that some $s_i \neq 0$. Then A contains both x and t^m. And these elements generate $t^m + \sum_{i=0}^{m-1} s_i t^i$.

Suppose A contains t^n and $t^m + \sum_{i=0}^{m-1} s_i t^i$ such that some $s_i \neq 0$. Now let $c =$

$\min\{i|s_i \neq 0\}$. If $n \leq m$ then $\sum_{i=0}^{m-1} s_i t^i = xt^c \sum_{i=c}^{m-1} \tilde{s}_i t^{i-c} \in A$ where $\tilde{s}_i \in k$. Since $\sum_{i=c}^{m-1} \tilde{s}_i t^{i-c}$ is invertible $xt^c \in A$. And $t^m + \sum_{i=0}^{m-1} s_i t^i$ is generated by xt^c and t^n. If $n > m$ then $t^{n-m}(t^m + \sum_{i=0}^{m-1} s_i t^i) - t^n = t^{n-m} \sum_{i=0}^{m-1} s_i t^i = xt^{c+n-m} \sum_{i=c}^{m-1} \tilde{s}_i t^{i-c} \in A$. Since $\sum_{i=c}^{m-1} \tilde{s}_i t^{i-c} \in A$ is invertible $xt^{c+n-m} \in A$. And t^n is generated by $xt^{c+n-m} \in A$ and $t^m + \sum_{i=0}^{m-1} s_i t^i$. Now suppose $s = t^m + \sum_{i=0}^{m-1} s_i t^i$ and $r = t^n + \sum_{i=0}^{n-1} r_i t^i$ are contained in A with $s_u \neq 0$ for some u such that $0 \leq u \leq m-1$ and $r_v \neq 0$ for some v such that $0 \leq v \leq n-1$. Let $c = \min\{i|s_i \neq 0 \text{ or } r_i \neq 0\}$. If $m > n$ we have $s - t^{m-n}r = \sum_{i=0}^{m-1}(s_i - t^{m-n}r_i)t^i = xt^c \sum_{i=c}^{m-1}(\tilde{s}_i - t^{m-n}\tilde{r}_i)t^{i-c}$. If $\tilde{s}_c \neq 0$, then $\sum_{i=c}^{m-1}(\tilde{s}_i - t^{m-n}\tilde{r}_i)t^{i-c}$ is invertible ($\tilde{r}_i \in k$). Thus $xt^c \in A$ which implies $t^m \in A$. And these elements generate s. If $\tilde{s}_c = 0$ then we can suppose $u = \min\{i|s_i \neq 0\}$. Now suppose $m - n \leq u - c$ then

$$s - t^{m-n}r = xt^{c+m-n}\left(-\sum_{i=c}^{u-1} \tilde{r}_i t^{i-c} + \sum_{i=u}^{m-1}(\tilde{s}_i - t^{m-n}\tilde{r}_i)t^{(i-c)-(m-n)}\right) \in A.$$

And $-\sum_{i=c}^{u-1} \tilde{r}_i t^{i-c} + \sum_{i=u}^{m-1}(\tilde{s}_i - t^{m-n}\tilde{r}_i)t^{(i-c)-(m-n)}$ is invertible. Thus $xt^{c+m-n} \in A$ which implies $t^m \in A$. And these elements generate s. If $m = n$ we get the same result. □

2.2 Standard Closure Operations on S

Lemma 2.2.1. *Let R be a ring with proper ideal I. Suppose that I contains a regular element. If $\star$ is a standard closure on R then $(0)^\star \neq I$.*

Proof. Suppose on the contrary that $(0)^\star = I$. Suppose $r \in I$ is a regular element. We get the following equation

$$(((r)(0))^\star : r) = ((0)^\star : r) = (I : r) = R.$$

Since $\star$ is standard this implies $(0)^\star = R$ which is a contradiction. $\square$

Lemma 2.2.2. *Suppose $\star$ is a standard closure on S and $(t^m)^\star = S$ for some $m \in \mathbb{N}$. Then $(t^l)^\star = S$ for all $l \in \mathbb{N}$.*

Proof. So $S = (t^m)^\star \subseteq (t^r)^\star$ implies $(t^r)^\star = S$ for all $r \in \mathbb{N}$ such that $r \leq m$. Since $\star$ is weakly prime

$$(t)(t^m)^\star \subseteq ((t)(t^m))^\star \Rightarrow (t) \subseteq (t^{m+1})^\star \Rightarrow (t^{m+1})^\star = (t)^\star = S.$$

Inductively we have $(t^l)^\star = S$ for all $l \in \mathbb{N}$. $\square$

Lemma 2.2.3. *Suppose $\star$ is a standard closure on S, $m, r \in \mathbb{N}$ and $r < m$. Then $(t^m)^\star \neq (t^r, A)$ for any proper ideal A.*

Proof. Suppose on the contrary that $(t^m)^\star = (t^r, A)$ for some proper ideal A, $m, r \in \mathbb{N}$ with $r < m$. The inclusion $(t^r, A) = (t^m)^\star \subseteq (t^n)^\star \subseteq (t^r, A)$ implies $(t^n)^\star = (t^r, A)$ for $n \in \mathbb{N}$ such that $r \leq n \leq m$. The equation

$$(((t^r)(t))^\star : t^r) = ((t^{r+1})^\star : t^r) = ((t^r, A) : t^r) = S$$

implies $(t)^\star = S$. By Lemma 2.2.2 $(t^l)^\star = S$ for all $l \in \mathbb{N}$ which is a contradiction. $\square$

Lemma 2.2.4. *Suppose $\star$ is a standard closure on S and $(t^m)^\star = (t^r)$ for some $m, r \in \mathbb{N}$. Then $(t^l)^\star = (t^l)$ for all $l \in \mathbb{N}$.*

Proof. By Lemma 2.2.3 $(t^m)^\star = (t^m)$. The inclusion $(t^{m+1})^\star \subseteq (t^m)^\star = (t^m)$ implies $(t^{m+1})^\star = (t^m)$ or $(t^{m+1})^\star = (t^{m+1})$. However by Lemma 2.2.3 we must have $(t^{m+1})^\star = (t^{m+1})$. Inductively $(t^l)^\star = (t^l)$ for all $l, m \in \mathbb{N}$ such that $l > m$. Consider the ideal (t^l) for some $l \in \mathbb{N}$ and

$$(((t^m)(t^l))^\star : t^m) = ((t^{m+l})^\star : t^m) = ((t^{m+l}) : t^m) = (t^l).$$

This implies $(t^l)^\star = (t^l)$ for all $l \in \mathbb{N}$. $\square$

Lemma 2.2.5. *Let J and I be as above. Suppose $\star$ is a standard closure on S then*

1. $(t^m)^\star \neq J$ *for any* J *or* $m \in \mathbb{N}$.
2. $(t^m)^\star \neq I$ *for any* I *or* $m \in \mathbb{N}$.

Proof. (1) Suppose on the contrary that $(t^m)^\star = J$ for some J and $m \in \mathbb{N}$. The inclusion $(tJ)^\star \subseteq J^\star = J$ implies $(tJ)^\star = tJ$ or $(tJ)^\star = J$. Suppose the latter. Consider $((tJ)^\star : t) = (J : t) = (t^{2s-1}, xt^{s-1}, J)$. Since $\star$ is standard this implies $J^\star = (t^{2s-1}, xt^{s-1}, J)$. Since J is $\star$ closed this is a contradiction. Thus $(tJ)^\star = tJ$. Inductively we get $(t^m J)^\star = t^m J$. Let $j = t^s + \sum_{i=0}^{s-1} s_i t^i$ the generator of J. Consider

$$(((j)(t^m))^\star : j) = ((t^m J)^\star : j) = (t^m J : j) = (t^m).$$

This implies $(t^m)^\star = (t^m)$ which is a contradiction.
(2) Here we get a similar contradiction.

□

Lemma 2.2.6. *Suppose $\star$ is a standard closure on S and $(x,t)^\star = (x,t)$ then either*

1. $(x)^\star = (x)$, $(t^l)^\star = (t^l)$, $(xt^l, t^n)^\star = (xt^l, t^n)$, $(x, t^l)^\star = (x, t^l)\,\forall\, l, n \in \mathbb{N}$ *or*
2. $(x)^\star = (x)$, $(t)^\star = (x, t)$, $(t^{l+1})^\star = (xt^l, t^{l+1})$, $(xt^l, t^n)^\star = (xt^l, t^n)$, $(x, t^l)^\star = (x, t^l)\,\forall\, l, n \in \mathbb{N}$

Proof. First we will show the following claim:

Claim. *Suppose $\star$ is a standard closure on S, for some $m \in \mathbb{N}$, $(x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$ such that $l \leq m$ and $(x, t^{m+1})^\star = (x, t^m)$ then $(x, t^l)^\star = (x, t^m)$ for all $l > m$.*

Proof. Suppose $(x, t^{m+1})^\star = (x, t^m)$. Further suppose there exists $n \in \mathbb{N}$ such that $n > m + 1$ and $(x, t^n)^\star = (x, t^r)$, for some $r \in \mathbb{N}$ such that $m < r \leq n$, and

$(x,t^j)^\star = (x,t^m)$ for all $j \in \mathbb{N}$ such that $m \leq j < n$. Since $(xt,t^n)^\star \subseteq (x,t^n)^\star = (x,t^r)$ either $(xt,t^n)^\star = (xt,t^v)$ or $(xt,t^n)^\star = (x,t^v)$ for some $v \in \mathbb{N}$ such that $r \leq v \leq n$. Consider

$$((t)(x,t^{n-1}))^\star : t) = ((xt,t^n)^\star : t) = \begin{cases} ((xt,t^v) : t) = (x,t^{v-1}) \\ ((x,t^v) : t) = (x,t^{v-1}) \end{cases}.$$

This implies $(x,t^{n-1})^\star = (x,t^{v-1})$. Since $(x,t^{n-1})^\star = (x,t^m)$ we have $v-1 = m \Rightarrow v = m+1$. So $m < r \leq m+1$ implies $r = m+1$. However then we have $(x,t^n)^\star = (x,t^{m+1})$ which is a contradiction since (x,t^{m+1}) is not $\star$ closed. Hence for all $l,m \in \mathbb{N}$ such that $l > m$, $(x,t^l)^\star = (x,t^m)$. This concludes the proof of the claim. □

The inclusion $(t)^\star \subseteq (x,t)^\star = (x,t)$ implies $(t)^\star = (t)$ or $(t)^\star = (x,t)$.

Case: $(t)^\star = (t)$. By Lemma 2.2.4 $(t^l)^\star = (t^l)$ for all l. Since $(xt,t^2)^\star \subseteq (t)^\star = (t)$ either $(xt,t^2)^\star = (t)$ or $(xt,t^2)^\star = (xt,t^2)$.

Subcase: $(xt,t^2)^\star = (t)$. Consider

$$(((t)(x,t))^\star : t) = ((xt,t^2)^\star : t) = ((t)^\star : t) = ((t) : t) = S.$$

This implies $(x,t)^\star = S$ which is a contradiction.

Subcase: $(xt,t^2)^\star = (xt,t^2)$. With the subcase as our base case suppose $(xt^r,t^{r+1})^\star = (xt^r,t^{r+1})$ for some $r \in \mathbb{N}$. The inclusion $(xt^{r+1},t^{r+2})^\star \subseteq (xt^r,t^{r+1})^\star = (xt^r,t^{r+1})$ implies either $(xt^{r+1},t^{r+2})^\star = (xt^{r+1},t^{r+2})$, $(xt^{r+1},t^{r+2})^\star = (xt^r,t^{r+2})$, $(xt^{r+1},t^{r+2})^\star = (t^{r+1})$ or $(xt^{r+1},t^{r+2})^\star = (xt^r,t^{r+1})$. Consider the colon ideal

$$(((t)(xt^r,t^{r+1}))^\star : t) = ((xt^{r+1},t^{r+2})^\star : t).$$

If $(xt^{r+1},t^{r+2})^\star = (xt^r,t^{r+2})$ then since $\star$ is standard

$$(xt^r,t^{r+1})^\star = \begin{cases} (xt^{r-1},t^{r+1}), \text{ for } r > 1 \\ (x,t^2), \text{ for } r = 1 \end{cases}.$$

A contradiction. If $(xt^{r+1}, t^{r+2})^\star = (t^{r+1})$ then $(xt^r, t^{r+1})^\star = (t^r)$. A contradiction. If $(xt^{r+1}, t^{r+2})^\star = (xt^r, t^{r+1})$ then $(xt^r, t^{r+1})^\star = (xt^{r-1}, t^r)$. Another contradiction. Thus we are left with $(xt^r, t^{r+1})^\star = (xt^r, t^{r+1})$ for all $r \in \mathbb{N}$. Now suppose for some $m \in \mathbb{N}$, $(x, t^{m+1})^\star = (x, t^m)$. Hence by the claim $(x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$ such that $l \leq m$ and $(x, t^l)^\star = (x, t^m)$ for all $l > m$. Suppose now that $n \in \mathbb{N}$ and $n > m$. Since $\star$ is weakly prime

$$(t^r)(x, t^n)^\star \subseteq ((t^r)(x, t^n))^\star \Rightarrow (xt^r, t^{m+r}) \subseteq (xt^r, t^{n+r})^\star \Rightarrow (xt^r, t^{n+r})^\star = (xt^r, t^{m+r})^\star.$$

The inclusions $(xt, t^{m+1})^\star \subseteq (xt, t^2)^\star = (xt, t^2)$ and $(xt, t^{m+1})^\star \subseteq (x, t^m)^\star = (x, t^m)$ imply $(xt, t^{m+1})^\star = (xt, t^{m+1})$ or $(xt, t^{m+1})^\star = (xt, t^m)$. Suppose the latter and consider

$$(((t)(x, t^m))^\star : t) = ((xt, t^{m+1})^\star : t) = ((xt, t^m) : t) = \begin{cases} (x, t^{m-1}) \text{ if } m > 1 \\ S \text{ if } m = 1 \end{cases}.$$

This is a contradiction so $(xt, t^{m+1})^\star = (xt, t^{m+1})$. Inductively we see $(xt^r, t^{m+r})^\star = (xt^r, t^{m+r})$. Thus $(xt^r, t^{n+r})^\star = (xt^r, t^{m+r})$ for all $r \in \mathbb{N}$. Now consider

$$(((t^{m+2} + xt)(x, t^{m+1}))^\star : t^{m+2} + xt) = ((xt^{m+2}, t^{2m+3} + xt^{m+2})^\star : t^{m+2} + xt)$$

$$= ((xt^{m+2}, t^{2m+3})^\star : t^{m+2} + xt) = ((xt^{m+2}, t^{2m+2}) : t^{m+2} + xt) = (x, t^{m+1}).$$

This implies $(x, t^{m+1})^\star = (x, t^{m+1})$ which is a contradiction. Thus $(x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$.

The inclusion $(x)^\star \subseteq (x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$ implies $(x)^\star \subseteq \cap_{l=1}^{\infty}(x, t^l) = (x)$. Thus $(x)^\star = (x)$.

The inclusions $(xt^l, t^n)^\star \subseteq (xt^l, t^{l+1})^\star = (xt^l, t^{l+1})$ and $(xt^l, t^n)^\star \subseteq (x, t^n)^\star = (x, t^n)$ imply $(xt^l, t^n)^\star = (xt^l, t^n)$ for all $l, n \in \mathbb{N}$. This proves (1).

Case: $(t)^\star = (x, t)$. Since $\star$ is weakly prime

$$(t)(t)^\star \subseteq ((t)(t))^\star \Rightarrow (xt, t^2) \subseteq (t^2)^\star \Rightarrow (t^2)^\star = (xt, t^2)^\star \subseteq (x, t)^\star = (x, t).$$

This implies $(t^2)^\star = (xt, t^2)$ or $(t^2)^\star = (x, t)$. However $(t^2)^\star = (x, t)$ contradicts Lemma 2.2.3. So $(t^2)^\star = (xt, t^2)$. Now suppose $(t^m)^\star = (xt^{m-1}, t^m)$ for some $m \in \mathbb{N}$ such that $m \geq 2$. We have

$$(t)(t^m)^\star \subseteq ((t)(t^m))^\star \Rightarrow (xt^m, t^{m+1}) \subseteq (t^{m+1})^\star$$

$$\Rightarrow (t^{m+1})^\star = (xt^m, t^{m+1})^\star \subseteq (xt^{m-1}, t^m)^\star = (xt^{m-1}, t^m).$$

This implies $(t^{m+1})^\star = (xt^m, t^{m+1})$, $(t^{m+1})^\star = (xt^{m-1}, t^{m+1})$ or $(t^{m+1})^\star = (xt^{m-1}, t^m)$. However $(t^{m+1})^\star = (xt^{m-1}, t^m)$ contradicts Lemma 2.2.3.
Suppose $(t^{m+1})^\star = (xt^{m-1}, t^{m+1})$. Consider

$$(((t)(t^m))^\star : t) = ((t^{m+1})^\star : t) = ((xt^{m-1}, t^{m+1}) : t) = \begin{cases} (xt^{m-2}, t^m) \text{ if } m > 2 \\ (x, t^m) \text{ if } m = 2 \end{cases}.$$

Since $\star$ is standard both cases are contradictions. Thus we are left with $(t^{m+1})^\star = (xt^m, t^{m+1})$. Hence $(t^{l+1})^\star = (xt^l, t^{l+1})$ for all $l \in \mathbb{N}$.

The inclusion $(xt^2, t^4)^\star \subseteq (xt^2, t^3)^\star = (xt^2, t^3)$ implies $(xt^2, t^4)^\star = (xt^2, t^3)$ or $(xt^2, t^4)^\star = (xt^2, t^4)$.

Subcase: $(xt^2, t^4)^\star = (xt^2, t^3)$. Consider

$$(((t^2)(x, t^2))^\star : t^2) = ((xt^2, t^4)^\star : t^2) = ((xt^2, t^3) : t^2) = (x, t).$$

This implies $(x, t^2)^\star = (x, t)$. By the claim, $(x, t^l)^\star = (x, t)$ for all $l \in \mathbb{N}$. Also $(t^n)(x, t^l)^\star \subseteq ((t^n)(x, t^l))^\star \Rightarrow (xt^n, t^{n+1}) \subseteq (xt^n, t^{n+l})^\star \Rightarrow (xt^n, t^{n+l})^\star = (xt^n, t^{n+1})^\star = (xt^n, t^{n+1})$ for all $l, n \in \mathbb{N}$.

Now consider

$$(((t^{m+2} + xt)(x, t^{m+1}))^\star : t^{m+2} + xt) = ((xt^{m+2}, t^{2m+3} + xt^{m+2})^\star : t^{m+2} + xt)$$

$$= ((xt^{m+2}, t^{2m+3})^\star : t^{m+2} + xt) = ((xt^{m+2}, t^{m+3}) : t^{m+2} + xt) = (x, t^{m+1}).$$

This implies $(x, t^{m+1})^\star = (x, t^{m+1})$ which is a contradiction.

Subcase: $(xt^2, t^4)^\star = (xt^2, t^4)$. Now suppose for some $m \in \mathbb{N}$, $(x, t^{m+1})^\star = (x, t^m)$. We obtain the same contradiction as we did in case $(t)^\star = (t)$, subcase $(xt, t^2)^\star = (xt, t^2)$. Thus $(x, t^l)^\star = (x, t^l)$ for all l. Just as in case $(t)^\star = (t)$, subcase $(xt, t^2)^\star = (xt, t^2)$ we have $(x)^\star = (x)$ and $(xt^l, t^n)^\star = (xt^l, t^n)$ for all $l, n \in \mathbb{N}$. This proves (2). □

Lemma 2.2.7. *Suppose $\star$ is a standard closure on S and $(x, t)^\star = S$, then $(x, t^l)^\star = (xt^r, t^{l+r})^\star = (t^r)^\star = J^\star = I^\star = (xt^l, J)^\star = (xt^l, I)^\star = S$ for all $l, r \in \mathbb{N}$ and I and J.*

Proof. With the assumption as the base case suppose $(x, t^n)^\star = S$ for some $n \in \mathbb{N}$. Since $\star$ is weakly prime $(t)(x, t)^\star \subseteq ((t)(x, t))^\star \Rightarrow (t) \subseteq (xt, t^{n+1})^\star$. Since $t \in (xt, t^{n+1})^\star \subseteq (x, t^{n+1})^\star$, $(x, t^{n+1})^\star = S$. Thus $(x, t^l)^\star = S$ for all $l \in \mathbb{N}$.

Again since $\star$ is weakly prime

$$(t^r)(x, t^l)^\star \subseteq ((t^r)(x, t^l))^\star \Rightarrow (t^r) \subseteq (xt^r, t^{l+r})^\star \Rightarrow (xt^r, t^{l+r})^\star = (t^r)^\star.$$

Since $(t^r)^\star \subseteq (x, t^r)^\star = S$ and Lemma 2.2.5 $(xt^r, t^{l+r})^\star = (xt^j, t^r)$, for some $j \in \mathbb{N}$ such that $j < r$ or $(xt^r, t^{l+r})^\star = S$. Suppose the former.

$$(((t^j)(xt^{r-j}, t^{l+r-j}))^\star : t^j) = ((xt^r, t^{l+r})^\star : t^j) = ((xt^j, t^r) : t^j) = (x, t^{r-j}).$$

This implies $(xt^{r-j}, t^{l+r-j})^\star = (x, t^{r-j})$. This is a contradiction since (x, t^{r-j}) is not $\star$ closed. Thus $(xt^r, t^{l+r})^\star = (t^r)^\star = S$ for all $l, r \in \mathbb{N}$. Since $S = (t^{2s})^\star \subseteq J^\star$, $J^\star = S$ for all $r \in \mathbb{N}$. Similarly $I^\star = (xt^l, J)^\star = (xt^l, I)^\star = S$ for all $l \in \mathbb{N}$, J and I. □

Theorem 2.2.8. *There are exactly six standard closures on S which are given below.*

1. $\star_1 : A^{\star_1} = S \ \forall$ *ideals* A.
2. $\star_2 : (0)^{\star_2} = (xt^l)^{\star_2} = (x)^{\star_2} = (x)$, $(x, t^l)^{\star_2} = (xt^n, t^{l+n})^{\star_2} = (t^n)^{\star_2} = J^{\star_2} = I^{\star_2} = (xt^l, J)^{\star_2} = (xt^l, I)^{\star_2} = S \ \forall \ l, n \in \mathbb{N}$ *and I and J.*

3. $\star_3 : (0)^{\star_3} = (0)$, $A^{\star_3} = S\ \forall$ *ideals* A *such that* $A \neq 0$.
4. $\star_4 : (0)^{\star_4} = (0)$, $(xt^l)^{\star_4} = (x)^{\star_4} = (x)\ \forall\ l \in \mathbb{N}$ *and* $A^{\star_4} = S\ \forall$ *other ideals* A.
5. $\star_5 : (0)^{\star_5} = (0)$, $(x)^{\star_5} = (x)$, $(t)^{\star_5} = (x,t)$, $(t^{l+1})^{\star_5} = (xt^l, t^{l+1})$, $(xt^l)^{\star_5} = (xt^l)$, $(xt^l, t^n)^{\star_5} = (xt^l, t^n)$, $(x, t^l)^{\star_5} = (x, t^l)$, $J^{\star_5} = (xt^{s-1}, J)$, $I^{\star_5} = (xt^{s-1}, I)$, $(xt^l, J)^{\star_5} = (xt^l, J)$, $(xt^l, I)^{\star_5} = (xt^l, I)\ \forall\ l, n \in \mathbb{N}$ *and* I *and* J.
6. $\star_6 : A^{\star_6} = A\ \forall$ *ideals* A.

Proof. By Lemma 2.2.1 $(0)^\star = S$, $(0)^\star = (x)$ or $(0)^\star = (0)$.

Suppose $(0)^\star = S$. Since for all ideals A, $S = (0)^\star \subseteq A$ we have $A^\star = S$ for all A. This is $\star_1$.

Suppose $(0)^\star = (x)$. The inclusion $(x) = (0)^\star \subseteq (t^m)^\star$ and Lemma 2.2.5, $(t^m)^\star = (x, t^j)$ or $(t^m)^\star = S$.

Case: $(t^m)^\star = (x, t^j)$ for some $j, m \in \mathbb{N}$. By Lemma 2.2.3 $(t^m)^\star = (x, t^m)$. Suppose $(x,t)^\star = S$ by Lemma 2.2.7 $(x, t^m)^\star = S$ which is a contradiction since (x, t^m) is $\star$ closed. Thus $(x,t)^\star = (x,t)$ and by Lemma 2.2.6 $(t^l)^\star = (t^l)$ for all $l \in \mathbb{N}$ or $(t)^\star = (t)$ and $(t^{l+1})^\star = (xt^l, t^{l+1})$ for all $l \in \mathbb{N}$ which is a contradiction.

Case: $(t^m)^\star = S$ for some $m \in \mathbb{N}$. Since $S = (t^m)^\star \subseteq (x,t)^\star$ we have $(x,t)^\star = S$. By Lemma 2.2.7 $(x, t^l)^\star = (xt^r, t^{l+r})^\star = (t^r)^\star = J^\star = I^\star = (xt^l, J)^\star = (xt^l, I)^\star = S$ for all $l, r \in \mathbb{N}$ and I and J. Since $(x) = (0)^\star \subseteq (xt^l)^\star \subseteq (x)^\star = (x)$, $(xt^l)^\star = (x)$ for all $l \in \mathbb{N}$. This is $\star_2$.

Suppose $(0)^\star = (0)$. Either $(x,t)^\star = S$ or $(x,t)^\star = (x,t)$.

Case: $(x,t)^\star = S$. By Lemma 2.2.7 $(x, t^l)^\star = (xt^r, t^{l+r})^\star = (t^r)^\star = J^\star = I^\star = (xt^l, J)^\star = (xt^l, I)^\star = S$ for all $l, r \in \mathbb{N}$ and I and J. Since $(x)^\star \subseteq (x, t^l)^\star = S$ for all $l \in \mathbb{N}$ either $(x)^\star = S$ or $(x)^\star = (x)$.

Subcase: $(x)^\star = S$. Since $\star$ is weakly prime $(t^l)(x)^\star \subseteq ((t^l)(x))^\star \Rightarrow (t^l) \subseteq (xt^l)^\star \Rightarrow (xt^l)^\star = (t^l)^\star = S$ for all $l \in \mathbb{N}$. This is $\star_3$.

Subcase: $(x)^\star = (x)$. The inclusion $(xt)^\star \subseteq (x)^\star = (x)$ implies $(xt)^\star = (xt)$ or $(xt)^\star = (x)$. Suppose the former. Since $\star$ is weakly prime

$$(xt)(t^{l-1})^\star \subseteq ((xt)(t^{l-1}))^\star \Rightarrow (xt) \subseteq (xt^l)^\star \Rightarrow (xt^l)^\star = (xt)^\star = (xt)$$

for all $l \in \mathbb{N}$. Consider $(((t)(xt^{l-1}))^\star : t) = ((xt^l)^\star : t) = ((xt) : t) = (x)$. This implies $(xt^{l-1})^\star = (x)$ which is a contradiction. Thus we must have $(xt)^\star = (x)$. Then $(t)(x)^\star \subseteq ((t)(x))^\star \Rightarrow (xt) \subseteq (xt^2)^\star \Rightarrow (xt^2)^\star = (xt)^\star = (x)$. Inductively $(xt^l)^\star = (x)$ for all $l \in \mathbb{N}$. This is $\star_4$.

Case: $(x,t)^\star = (x,t)$. By Lemma 2.2.6 there are two possibilities.

Subcase: $(x)^\star = (x)$, $(t)^\star = (x,t)$, $(t^{l+1})^\star = (xt^l, t^{l+1})$, $(xt^l, t^n)^\star = (xt^l, t^n)$ for all $l, n \in \mathbb{N}$, $(x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$. The inclusion $(xt)^\star \subseteq (x)^\star = (x)$ implies $(xt)^\star = (xt)$ or $(xt)^\star = (x)$. Suppose the latter. We have $(t)(xt)^\star \subseteq ((t)(xt))^\star \Rightarrow (xt) \subseteq (xt^2)^\star \Rightarrow (xt^2)^\star = (xt)^\star = (x)$. However $(x) = (xt^2)^\star \subseteq (xt^2, t^3)^\star = (xt^2, t^3)$ is a contradiction. Thus $(xt)^\star = (xt)$. The inclusion $(xt^2)^\star \subseteq (xt)^\star = (xt)$ implies $(xt^2)^\star = (xt^2)$ or $(xt^2)^\star = (xt)$. Suppose the latter and consider

$$(((t)(xt))^\star : t) = ((xt^2)^\star : t)((xt) : t) = (x).$$

This implies $(xt)^\star = (x)$ which is a contradiction. Thus $(xt^2)^\star = (xt^2)$. Inductively $(xt^l)^\star = (xt^l)$ for all $l \in \mathbb{N}$. This leaves the ideals containing J and I. Consider

$$(((t^s - \sum_{i=0}^{s-1} s_i t^i)J)^\star : t^s - \sum_{i=0}^{s-1} s_i t^i) = ((t^{2s})^\star : t^s - \sum_{i=0}^{s-1} s_i t^i)$$

$$= ((xt^{2s-1}, t^{2s}) : t^s - \sum_{i=0}^{s-1} s_i t^i) = (xt^{s-1}, J).$$

Thus $J^\star = (xt^{s-1}, J)$. Similarly $I^\star = (xt^{s-1}, I)$. Consider

$$(((t^s - \sum_{i=0}^{s-1} s_i t^i)(xt^l, J))^\star : t^s - \sum_{i=0}^{s-1} s_i t^i) = ((xt^{l+s}, t^{2s})^\star : t^s - \sum_{i=0}^{s-1} s_i t^i)$$

$$= ((xt^{l+s}, t^{2s}) : t^s - \sum_{i=0}^{s-1} s_i t^i) = (xt^l, J).$$

Thus $(xt^l, J)^\star = (xt^l, J)$ for all $l \in \mathbb{N}$. Similarly $(xt^l, I)^\star = (xt^l, I)$ for all $l \in \mathbb{N}$. This is $\star_5$.

Subcase: $(x)^\star = (x)$, $(t^l)^\star = (t^l)$ for all $l \in \mathbb{N}$, $(xt^l, t^n)^\star = (xt^l, t^n)$ for all $l, n \in \mathbb{N}$, $(x, t^l)^\star = (x, t^l)$ for all $l \in \mathbb{N}$. The inclusion $(xt^l)^\star \subseteq (t^l)^\star = (t^l)$ implies that for each $l \in \mathbb{N}$ either $(xt^l)^\star = (xt^l)$ or $(xt^l)^\star = (t^l)$. Suppose the latter for some $l \in \mathbb{N}$ and consider

$$(((t^l)(x))^\star : t^l) = ((xt^l)^\star : t^l) = ((t^l) : t^l) = S.$$

This implies $(x)^\star = S$ which is a contradiction. Thus $(xt^l)^\star = (xt^l)$ for all l.

This leaves the ideals containing J and I. Consider

$$(((t^s - \sum_{i=0}^{s-1} s_i t^i)J)^\star : t^s - \sum_{i=0}^{s-1} s_i t^i) = ((t^{2s})^\star : t^s - \sum_{i=0}^{s-1} s_i t^i) = ((t^{2s}) : t^s - \sum_{i=0}^{s-1} s_i t^i) = J.$$

Thus $J^\star = J$. Similarly $I^\star = I$.

$$(((t^s - \sum_{i=0}^{s-1} s_i t^i)(xt^l, J))^\star : t^s - \sum_{i=0}^{s-1} s_i t^i) = ((xt^{l+s}, t^{2s})^\star : t^s - \sum_{i=0}^{s-1} s_i t^i)$$

$$= ((xt^{l+s}, t^{2s}) : t^s - \sum_{i=0}^{s-1} s_i t^i) = (xt^l, J).$$

Thus $(xt^l, J)^\star = (xt^l, J)$ for all $l \in \mathbb{N}$. Similarly $(xt^l, I)^\star = (xt^l, I)$ for all $l \in \mathbb{N}$. This is $\star_6$, the identity closure. □

Corollary 2.2.9.

$$|S_f(S))| = 6$$

Proof. By Theorem 2.2.8 and Lemma 1.2.8. □

Chapter 3

Standard Closures on R_t

In this chapter we will develop many tools for classifying the standard closures on R_t (described below) for any $t \geq 2$. We will then exhibit and count all of the standard closures on R_2 and R_3. As with the ring S we must begin by classifying all of the ideals of R_t.

3.1 The ring R_t

Throughout k is a field and $R_t = k[[X_1, \ldots, X_t]]/(X_iX_j | i < j)$. We use lower case letters $x_1, \ldots, x_t$ to denote the images of $X_1, \ldots, X_t$ in R_t.

Lemma 3.1.1. *Every proper ideal of R_t is generated by polynomials of the form $x_{i_1}^{m_{i_1}} + a_2 x_{i_2}^{m_{i_2}} + \cdots + a_v x_{i_v}^{m_{i_v}}$ with $a_l \in k^\times$ for $2 \leq l \leq v$ and $1 \leq i_1 < \cdots < i_v \leq t$.*

Proof. Suppose f is an element of a proper ideal J. Thus f is a nonunit with the form $f = \sum_{j_1=m_1}^{\infty} a_{j_1}^1 x_{i_1}^{j_1} + \cdots + \sum_{j_v=m_v}^{\infty} a_{j_v}^v x_{i_v}^{j_v}$ with each $a_{j_n}^n$ a non-zero power series in $x_{i_1}, \ldots, \hat{x}_{i_n}, \ldots, x_{i_v}$ and $m_s \geq 1$ for $s = 1, \ldots, v$. However since $x_{i_\alpha} \cdot x_{i_\beta} = 0$ for

$\alpha \neq \beta$ we can assume each $a^n_{j_n} \in k^\times$. So

$$f = x^{m_1} \sum_{j_1=m_1}^{\infty} a^1_{j_1} x_{i_1}^{j_1-m_1} + \cdots + x^{m_v} \sum_{j_v=m_v}^{\infty} a^v_{j_v} x_{i_v}^{j_v-m_v}.$$

So each $\sum_{j_n=m_n}^{\infty} a^n_{j_n} x_{i_n}^{j_n-m_n}$ is an invertible element in the subring $k[[x_n]]$ with inverse $u_n = \sum_{l_n=0}^{\infty} c^n_{l_n} x_{i_n}^{l_n}$, $c^n_{l_n} \in k^\times$. Now

$$u_1 f = x_{i_1}^{m_1} + c^1_0 x_{i_2}^{m_2} \sum_{j_2=m_2}^{\infty} a^2_{j_2} x_{i_2}^{j_2-m_2} + \cdots + c^1_0 x_{i_v}^{m_v} \sum_{j_v=m_v}^{\infty} a^v_{j_v} x_{i_v}^{j_v-m_v}$$

$$u_2 u_1 f = c^2_0 x_{i_1}^{m_1} + c^1_0 x_{i_2}^{m_2} + c^1_0 c^2_0 x_{i_3}^{m_3} \sum_{j_3=m_3}^{\infty} a^3_{j_3} x_{i_3}^{j_3-m_3} + \cdots + c^1_0 c^2_0 x_{i_v}^{m_v} \sum_{j_v=m_v}^{\infty} a^v_{j_v} x_{i_v}^{j_v-m_v}$$

$$\vdots$$

$$u_v \dots u_1 f = c^2_0 \dots c^t_0 x_{i_1}^{m_1} + c^1_0 c^3_0 \dots c^t_0 x_{i_1}^{m_1} + \cdots + c^1_0 c^2_0 \dots c^{t-1}_0 x_{i_v}^{m_v}$$

Multilplying the last equation by $(c^2_0 \dots c^t_0)^{-1}$ we have the result. □

Lemma 3.1.2. *Let I be a proper ideal of R_t. Then $I = (f_1, \dots, f_l)$ where $f_j = \sum_{i=1}^{t} a_{i,j} x_i^{m_{i,j}}$ with $a_{i,j} \in k$ and $m_{i,j} \in \mathbb{N}$ for $1 \leq j \leq l$ where each f_j is monic.*

Proof. Since R_t is Noetherian and Lemma 3.1.1. □

The following algorithm gives an ordering for the generators of an ideal $I \subseteq R_t$.

Algorithm 3.1.3. *Let I be a proper ideal of R_t. By Lemma 3.1.2 $I = (f_1, \dots, f_l)$ where $f_j = \sum_{i=1}^{t} a_{i,j} x_i^{m_{i,j}}$ with $a_{i,j} \in k$ and $m_{i,j} \in \mathbb{N}$ for $1 \leq j \leq l$ (each f_j is also monic). Let $W = \{f_i\}_{i=1}^{l}$. Let $1 \leq n \leq t$. Define*

$$\alpha_n : W \to \mathbb{N} \cup \{0\} \text{ as } \alpha_n(f_j) = \begin{cases} 0, & \text{if } a_{n,j} = 0 \\ m_{n,j}, & \text{if } a_{n,j} \neq 0 \end{cases}$$

Consider the generators f_r and f_s, $1 \leq r \leq l$, $1 \leq s \leq l$.

1. *Let $n = 1$*
2. *If $\alpha_n(f_r)\alpha_n(f_s) = 0$ go to (3) otherwise go to (4).*
3. *If $\alpha_n(f_r) \neq 0$ then $f_r > f_s$. If $\alpha_n(f_s) \neq 0$ then $f_s > f_r$. Otherwise go to (5).*
4. *If $\alpha_n(f_r) > \alpha_n(f_s)$ then $f_r < f_s$. If $\alpha_n(f_s) > \alpha_n(f_r)$ then $f_s < f_r$. Otherwise go to (5).*
5. *If $n < t$ let $n := n + 1$ and go to (2). If $n = t$ go to (6).*
6. *If $(a_{1,r} - a_{1,s}, \ldots, a_{t,r} - a_{t,s}) = (0.\ldots, 0)$ then $f_r = f_s$. Otherwise f_r and f_s are not comparable.*

Note f_r and f_s are equal or not comparable if and only if $\alpha_n(f_r) = \alpha_n(f_s)$ for all n.

Definition 3.1.4. *Let I and W be as in Algorithm 3.1.3. If $f_r = f_s$ or f_r is not comparable to f_s then $f_r \sim f_s$.*

Proposition 3.1.5. *The relation $\sim$ is an equivalence relation.*

Proof. Let f_r,f_s and $f_u \in W$. Since $f_r = f_r$ then under Algorithm 3.1.3 $f_r = f_r$. So $f_r \sim f_r$. Suppose $f_r \sim f_s$. Then $\alpha_n(f_r) = \alpha_n(f_s)$ for all n. So $f_s \sim f_r$. Suppose $f_r \sim f_s$ and $f_s \sim f_u$. Then $\alpha_n(f_r) = \alpha_n(f_s)$ for all n and $\alpha_n(f_s) = \alpha_n(f_u)$ for all n. Thus $\alpha_n(f_r) = \alpha_n(f_u)$ for all n. So $f_s \sim f_u$. □

Definition 3.1.6. *Let I and W be as in Algorithm 3.1.3. Let $a, b \in W/\sim$. Suppose f_r is a representative of a and f_s a representative of b. We say $a > b$ if $f_r > f_s$.*

Proposition 3.1.7. *The relation given in 3.1.6 is well defined.*

Proof. Suppose f_{r_1} and f_{r_2} are both representatives of a and f_s is a representative of b such that $f_{r_1} > f_s$. We need to show that $f_{r_2} > f_s$. Since f_{r_1} and f_{r_2} are representative of a, $\alpha_n(f_{r_1}) = \alpha_n(f_{r_2})$ for all n. Thus $f_{r_2} > f_s$. □

We will denote the *reduced row echelon form* of a matrix A by $\text{rref}(A)$.

Algorithm 3.1.8. *Let I be an ideal of R_t*

1. *Let $I = (f_1, \dots, f_l)$ where $f_j = \sum_{i=1}^{t} a_{i,j} x_i^{m_{i,j}}$ with $a_{i,j} \in k$ and $m_{i,j} \in \mathbb{N}$ for $1 \le j \le l$ where each f_j is monic which we can do by Lemma 3.1.2.*
2. *Let J be the smallest monomial ideal generated by a subset of the generators of the maximal ideal of R that contains I. Thus $J = (x_{j_1}, \dots, x_{j_v})$ with $1 \le j_1 < \cdots < j_v \le t$.*
3. *Let $W := \{f_s\}_{s=1}^{l}$.*
4. *Partition W with respect to $\sim$ such that $W = \amalg_{u=1}^{d} W_u$ and $W_1 > W_2 > \cdots > W_d$.*
5. *Reindex the f_s with respect to their equivalence class W_u so that for $u = 1, \dots, d$, $W_u := \{f_{u_\beta}\}_{\beta=1}^{l_u}$. So each $f_{u_\beta} = \sum_{c=1}^{v} a_{u_{\beta_c}} x_{j_c}^{m_{u_{\beta_c}}}$, $a_{u_{\beta_c}} \in k$, for $1 \le \beta \le l_u$.*
6. *For each u such that $1 \le u \le d$, let $A_u = (a_{u_{\beta_c}})$ a $l_u \times v$ matrix.*
7. *Compute $\operatorname{rref}(A_u)$ for each u such that $1 \le u \le d$. So $\operatorname{rref}(A_u) = (b_{u_{\beta_c}})$, $b_{u_{\beta_c}} \in k$.*
8. *Let $f_{u_\beta} := \sum_{c=1}^{v} b_{u_{\beta_c}} x_{i_c}^{m_{u_{\beta_c}}}$.*
9. *Arrange all the nonzero f_{u_β} by their indices in dictionary order and reindex them in that order with the natural numbers in the usual order and let $W := \{f_s\}_{s=1}^{l'}$.*

Since elementary row operations are reversible we can recover each A_u from $\operatorname{rref}(A_u)$. Thus W is still a generating set for I.

Algorithm 3.1.9. *Let I be an ideal of R_t.*

1. *Run Algorithm 3.1.8 for I to obtain $W := \{f_s\}_{s=1}^{l}$, $i := 0$, $n := 0$.*
2. *Let $i := i + 1$.*
3. *Let $j := min(\{1, \dots, l\} \backslash \{i\})$.*
4. *Let $n := n + 1$.*

5. *If there exists $a \in k$ and $m \in \mathbb{N}$ such that $ax_n^m f_i$ is a term of f_j replace f_j with $f_j - ax_n^m f_i$.*
6. *If $n < t$ go to step 4.*
7. *If $j < l$ and $j + 1 \neq i$ then let $j := j + 1$, $n := 0$ and go to step 4.*
8. *If $j < l$ and $j + 1 = i$ then let $j := j + 2$, $n := 0$. If $j \leq l$ go to step 4.*
9. *If $i < l$ go to step 2.*
10. *Let $W := \{f_s\}_{s=1}^{l} \backslash \{f_s | f_s = 0\}$. So W now has $r \leq l$ elements. Let $l := r$. Arrange all of the f_s in dictionary order and reindex them in that order with the natural numbers in the usual order. Let $W := \{f_s\}_{s=1}^{r}$.*
11. *Now for $s \in \{1, \ldots, l\}$, $f_s = \sum\limits_{n=1}^{t} a_{s,n} x_n^{m_n}$, $a_{s,n} \in k$. Let $A = (a_{s,n})$. Compute $\text{rref}(A) = (b_{s,n})$, $b_{s,n} \in k$. If $A = \text{rref}(A)$ we are done. If not for each $s \in \{1, \ldots, l\}$ let $f_s = \sum\limits_{n=1}^{t} b_{s,n} x_n^{m_n}$.*
12. *Go to step 10.*

Definition 3.1.10. *Let I be an ideal of R_t. We will call the set W that results from applying Algorithm 3.1.9 to I a reduced generating set of I.*

Lemma 3.1.11. *Regardless of how the polynomials obtained in step (1) of Algorithm 3.1.8 are indexed Algorithm 3.1.9 produces a unique reduced generating set of I.*

Proof. Let I be an ideal of R_t. Step (1) of Algorithm 3.1.9 is to run Algorithm 3.1.8 for I. Suppose we index the generators of I obtained in step (1) of Algorithm 3.1.8 in two different ways $I = (f_1, \ldots, f_l)$ and $I = (f_{r_1}, \ldots, f_{r_l})$. We now continue algorithm 3.1.8 for I with the former indexing and then I with the later indexing. In each case we obtain $J = (x_{j_1}, \ldots, x_{j_v})$. In step (3) we have $W := \{f_s\}_{s=1}^{l}$ and $W' := \{f_{r_s}\}_{s=1}^{l}$. Since W and W' contain the same polynomials after step (4) $W_u = W_u^{'}$ for $1 \leq u \leq d$. So the matrices A_u and $A_u^{'}$ obtained in step (6) are such that $\text{rref}(A_u) = \text{rref}(A_u^{'})$. Thus W and W' obtained in step (9) are equal. □

Lemma 3.1.12. *Let I be an ideal of R_t and $W = \{f_s\}_{s=1}^{l}$ be a reduced generating set of I. The elements of the set W have order $f_1 > f_2 > \cdots > f_l$.*

Proof. For each $s \in \{1, \ldots, l\}$, $f_s = \sum_{n=1}^{t} b_{s,n} x_n^{m_n}$, $b_{s,n} \in k$ where by step (11) of Algorithm 3.1.9 the matix $(b_{s,n})$ is in reduced row echelon form. Let $c = \min\{n | b_{s,n} \neq 0\}$. Thus $b_{s+1,c} = 0$. So $\alpha_c(f_s)\alpha_c(f_{s+1}) = 0$ and $\alpha_c(f_s) \neq 0$. This implies $f_s > f_{s+1}$. □

Lemma 3.1.13. *Let I be an ideal of R_t and $W = \{f_s\}_{s=1}^{l}$ be a reduced generating set of I. Then*

(1) no f_i generates a term of f_j for any $j \neq i$

(2) W is k-linearly independent set

(3) no proper subset of W generates I

Proof. (1) This is a result of steps (2) through (9) of Algorithm 3.1.9. (2) This is a result of steps (10) through (12) of Algorithm 3.1.9. (3) Suppose not. Suppose W/f_i generates I. By (2) W is k-linearly independent set. Thus f_i is not a k-linear combination of elements of W/f_i. Since $f_i \in I$, f_i must be an R_t-linear combination of elements of W/f_i but this is impossible by (1). □

Proposition 3.1.14. *Suppose I is an ideal of R_t. Then I has a unique reduced generating set.*

Proof. Run algorithm 3.1.9 for I. In step (1) we apply Algorithm 3.1.8 to I. In step (1) of Algorithm 3.1.8 we use Lemma 3.1.2 to get a polynomial generating set for I, $I = (f_1, \ldots, f_l)$ where $f_j = \sum_{i=1}^{t} d_{i,j} x_i^{m_{i,j}}$ with $d_{i,j} \in k$ and $m_{i,j} \in \mathbb{N}$ for $1 \leq j \leq l$ where each f_j is monic. Upon completion of the algorithm we have a reduced generating set $W = \{h_j\}_{j=1}^{r}$ for I. Now suppose we run Algorithm 3.1.9 again for I except this time in step (1) we obtain a different polynomial generating set for I, $I = (g_1, \ldots, g_m)$ where $g_j = \sum_{i=1}^{t} e_{i,j} x_i^{m_{i,j}}$ with $e_{i,j} \in k$ and $m_{i,j} \in \mathbb{N}$ for $1 \leq j \leq m$ where each g_j is monic. And upon completion of the algorithm we have a reduced generating set $W' = \{h'_j\}_{j=1}^{s}$ for I. We need to show $W = W'$.

Consider $h'_j \in W'$. By Lemma 3.1.12 W is totally ordered, thus $h_1 > h_2 > \cdots > h_r$. There are two cases.

Case 1: ($h'_j \sim h_n$ for exactly one n, $1 \leq n \leq r$) Thus $h'_j = \sum_{i=1}^{v} a_{j_i} x_{j_i}^{m_{j_i}}$, $a_{j_i} \in k^{\times}$, and $h_n = \sum_{i=1}^{v} b_{j_i} x_{j_i}^{m_{j_i}}$, $b_{j_i} \in k^{\times}$. Suppose $h'_j \neq h_n$. Suppose for some $1 \leq m \leq r$, h_m generates a term of h'_j. So for some $1 \leq c \leq v$, $a \in k^{\times}$, $d \in \mathbb{N}$, $ax_{j_c}^{d} h_m = a_{j_c} x_{j_c}^{m_{j_c}} \Rightarrow \frac{b_{j_c}}{a_{j_c}}(ax_{j_c}^{d} h_m) = b_{j_c} x_{j_c}^{m_{j_c}}$. This is a contradiction since by Lemma 3.1.13 h_m cannot generate a term of h_n. Thus no h_m generates a term of h'_j. Since $h'_j \neq h_n$ and both h'_j and h_n are monic h'_j is not a multiple of h_n. So h'_j is not generated by $\{h_1, \ldots, h_r\}$ which is a contradiction since $h'_j \in I$. Thus $h'_j = h_n$.

Case 2: ($h'_j \nsim h_n$ for any n such that $1 \leq n \leq r$) Suppose h'_j is in the k-linear vector space generated by W. So $h'_j = \sum_{i=1}^{r} a_i h_i$, $a_i \in k$ implies $h_1 = a_1^{-1} h'_j - a_1^{-1} \sum_{i=2}^{r} a_i h_i$ which implies h_1 is a k-linear combination of elements of W. This is a contradiction since by Lemma 3.1.13 W is k-linearly independent. So h'_j is not in the k-linear vector space generated by W. Suppose h'_j has term $\beta x_u^{m_u}$, $\beta \in k^{\times}$, but no h_n has term $\alpha x_u^{r_u}$, $r_u < m_u$, $\alpha \in k^{\times}$. This is a contradiction since W generates I. Suppose h'_j has term $\beta x_u^{m_u}$ and for some n, h_n has term $\alpha x_u^{r_u}$, $r_u < m_u$. Since $h'_j \in W'$, no element of W' has term $\mu x_u^{s_u}$, $s_u < m_u$, $\mu \in k^{\times}$. This is a contradiction since $h_n \in I$ and W' generates I. Thus $W' \subseteq W$. By similar argument we can show $W \subseteq W'$. Thus $W' = W$.

Also since both W and W' are totally ordered by Lemma 3.1.12 we must have

$$h_1 = h'_1, h_2 = h'_2, \ldots, h_r = h'_s$$

with $r = s$.

□

3.2 Standard Closure Operations on R_t

Lemma 3.2.1. *Suppose $1 \le i_1 < \cdots < i_v \le t$. If $\star$ is a standard closure on R_t and $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$ for some $m_{i_j} \in \mathbb{N}$ with each $m_{i_j} > 1$ then $(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})^\star = (x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})$ for all $r_{i_j} \in \mathbb{N}$.*

Proof. Suppose $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$ for some $m_{i_j} \in \mathbb{N}$ with each $m_{i_j} > 1$. Let $(s_{i_1}, \ldots, s_{i_v}) \in \mathbb{N}^v$ with each $s_{i_j} < m_{i_j}$ for $j = 1, \ldots, v$. Let

$$w = x_1 + \cdots + x_{i_1 - 1} + x_{i_1}^{m_{i_1} - s_{i_1}} + x_{i_1+1} + \cdots + x_{i_v - 1} + x_{i_v}^{m_{i_v} - s_{i_v}} + x_{i_v+1} + \cdots + x_t.$$

Consider

$$\begin{aligned} ((w(x_{i_1}^{s_{i_1}}, \ldots, x_{i_v}^{s_{i_v}}))^\star : w) &= ((x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star : w) \\ &= ((x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}) : w) = (x_{i_1}^{s_{i_1}}, \ldots, x_{i_v}^{s_{i_v}}). \end{aligned}$$

Since $\star$ is standard this implies $(x_{i_1}^{s_{i_1}}, \ldots, x_{i_v}^{s_{i_v}})^\star = (x_{i_1}^{s_{i_1}}, \ldots, x_{i_v}^{s_{i_v}})$ for all $(s_{i_1}, \ldots, s_{i_v}) \in \mathbb{N}^v$ with each $s_{i_j} < m_{i_j}$.

We need to show

$$\begin{aligned} &(x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}-1}, x_{i_j}^{m_{i_j}}, x_{i_{j+1}}^{m_{i_{j+1}}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star \\ &= (x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}-1}, x_{i_j}^{m_{i_j}}, x_{i_{j+1}}^{m_{i_{j+1}}-1}, \ldots, x_{i_v}^{m_{i_v}-1}). \end{aligned}$$

Without loss of generality we will show

$$(x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

Let $I = (x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$. So

$$I^\star \subseteq (x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

So either $I^\star = I$ or $I^\star = (x_{i_1}^{m_{i_1}-1}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$. Suppose the latter. Since $\star$ is weakly prime

$$(x_1 + \cdots + x_t)I^\star \subseteq ((x_1 + \cdots + x_t)I)^\star \Rightarrow (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}) \subseteq (x_{i_1}^{m_{i_1}+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star$$

$$\Rightarrow (x_{i_1}^{m_{i_1}+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}).$$

Let $u = x_1 + \cdots + x_{i_1-1} + x_{i_1}^2 + x_{i_1+1} + \cdots + x_t$.

Consider

$$\begin{aligned}((u(x_{i_1}^{m_{i_1}-1}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}))^\star : u) &= ((x_{i_1}^{m_{i_1}+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star : u) \\ &= ((x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}) : u) \\ &= \begin{cases} (x_{i_1}^{m_{i_1}-2}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}) \text{ if } m_{i_1} > 2 \\ (x_{i_1}, x_{i_v}^{m_{i_2}-1}, \ldots, x_{i_2}^{m_{i_v}-1}) \text{ if } m_{i_1} = 2 \end{cases}.\end{aligned}$$

Since $\star$ is standard we have a contradiction unless $m_{i_1} = 2$. Thus

$$(x_{i_1}^2, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^2, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}}) = (x_{i_1}^3, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star$$

and

$$(x_{i_1}^2, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

Let $r = x_1 + \cdots + x_{i_1-1} + x_{i_1}^2 + x_{i_1+1} + \cdots + x_{i_2-1} + x_{i_2}^2 + x_{i_2+1} + \cdots + x_t$.

Using again that $\star$ is weakly prime,

$$\begin{aligned} r(x_{i_1}^2, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star &\subseteq (r(x_{i_1}^2, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}))^\star \\ \Rightarrow (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}}) &\subseteq (x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star \\ \Rightarrow (x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star &= (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star. \end{aligned}$$

Since $(x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star \subseteq (x_{i_1}^3, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star$,

$$\begin{aligned}(x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star &= (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star \\ &= (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})\end{aligned}$$

or

$$(x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}} \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^3, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$$

Notice $(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star \subseteq (x_{i_1}^2, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star \Rightarrow$

$$(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$$

or

$$(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$$

Now consider

$$(((x_1 + \cdots + x_t)(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1}))^\star : x_1 + \cdots + x_t)$$

$$= ((x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}} \ldots, x_{i_v}^{m_{i_v}})^\star : x_1 + \cdots + x_t)$$

is either

$$((x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}} \ldots, x_{i_v}^{m_{i_v}}) : x_1 + \cdots + x_t) = (x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$$

or

$$((x_{i_1}^3, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}} \ldots, x_{i_v}^{m_{i_v}}) : x_1 + \cdots + x_t) = (x_{i_1}^2, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

Since $\star$ is standard the latter is not possible. So we must have

$$(x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star$$

$$= (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})$$

and

$$(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star = (x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$$

Now

$$((u(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1}))^\star : u) = ((x_{i_1}^4, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}})^\star : u)$$

$$= ((x_{i_1}^3, x_{i_2}^{m_{i_2}+1}, x_{i_3}^{m_{i_3}}, \ldots, x_{i_v}^{m_{i_v}}) : u) = (x_{i_1}, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

This is a contradiction since $(x_{i_1}^2, x_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}-1}, \ldots, x_{i_v}^{m_{i_v}-1})$ is $\star$ closed.

Hence

$$(x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}-1}, x_{i_j}^{m_{i_j}}, x_{i_{j+1}}^{m_{i_{j+1}}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^\star$$
$$= (x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}-1}, x_{i_j}^{m_{i_j}}, x_{i_{j+1}}^{m_{i_{j+1}}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

We need to show

$$(x_{i_1}^{m_{i_1}}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}}, x_{i_j}^{m_{i_j}+r}, x_{i_{j+1}}^{m_{i_{j+1}}}, \ldots, x_{i_v}^{m_{i_v}})^\star$$
$$= (x_{i_1}^{m_{i_1}}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}}, x_{i_j}^{m_{i_j}+r}, x_{i_{j+1}}^{m_{i_{j+r}}}, \ldots, x_{i_v}^{m_{i_v}})$$

for all $r \in \mathbb{N}$. Without loss of generality we will show $(x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$ for all $r \in \mathbb{N}$.

Suppose for some $r \in \mathbb{N}$,

$$(x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}}).$$

Since

$$(x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star \subseteq (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$$

either

$$(x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$$

or

$$(x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}}).$$

Suppose the latter. Consider

$$(((x_1 + \cdots + x_{i_1-1} + x_{i_1}^{r+1} + x_{i_1+1} \cdots + x_t)(x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}))^\star$$
$$: x_1 + \cdots + x_{i_1-1} + x_{i_1}^{r+1} + x_{i_1+1} \cdots + x_t)$$
$$= ((x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^\star : x_1 + \cdots + x_{i_1-1} + x_{i_1}^{r+1} + x_{i_1+1} \cdots + x_t)$$
$$= ((x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}}) : x_1 + \cdots + x_{i_1-1} + x_{i_1}^{r+1} + x_{i_1+1} \cdots + x_t)$$

$$= (x_{i_1}^{m_{i_1}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

Which is a contradiction since

$$(x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1})^{\star} = (x_{i_1}^{m_{i_1}}, x_{i_2}^{m_{i_2}-1}, \ldots, x_{i_v}^{m_{i_v}-1}).$$

Thus

$$(x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^{\star} = (x_{i_1}^{m_{i_1}+r+1}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$$

So

$$(x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})^{\star} = (x_{i_1}^{m_{i_1}+r}, x_{i_2}^{m_{i_2}}, \ldots, x_{i_v}^{m_{i_v}})$$

for all $r \in \mathbb{N}$. Hence

$$(x_{i_1}^{m_{i_1}}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}}, x_{i_j}^{m_{i_j}+r}, x_{i_{j+1}}^{m_{i_{j+1}}}, \ldots, x_{i_v}^{m_{i_v}})^{\star}$$

$$= (x_{i_1}^{m_{i_1}}, \ldots, x_{i_{j-1}}^{m_{i_{j-1}}}, x_{i_j}^{m_{i_j}+r}, x_{i_{j+1}}^{m_{i_{j+r}}}, \ldots, x_{i_v}^{m_{i_v}})$$

for all $r \in \mathbb{N}$.

Suppose $r_{i_j} > m_{i_j}$ for $1 \leq j \leq v$,

$$(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})^{\star} \subseteq (x_{i_1}^{m_{i_1}}, \ldots, x_{i_j-1}^{m_{i_j}}, x_{i_j}^{r_{i_j}}, x_{i_j+1}^{m_{i_j}} \ldots, x_{i_v}^{m_{i_v}})^{\star}$$

$$= (x_{i_1}^{m_{i_1}}, \ldots, x_{i_j-1}^{m_{i_j}}, x_{i_j}^{r_{i_j}}, x_{i_j+1}^{m_{i_j}} \ldots, x_{i_v}^{m_{i_v}}).$$

So

$$(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})^{\star} \subseteq \cap_1^v (x_{i_1}^{m_{i_1}}, \ldots, x_{i_j-1}^{m_{i_j}}, x_{i_j}^{r_{i_j}}, x_{i_j+1}^{m_{i_j}} \ldots, x_{i_v}^{m_{i_v}}) = (x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}}).$$

Thus

$$(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})^{\star} = (x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})$$

for all $r_{i_j} > m_{i_j}$ for $1 \leq j \leq v$.

Now suppose $r_{i_j} \in \mathbb{N}$ for $1 \leq j \leq v$. Let $s = x_1 + \cdots + x_{i_1-1} + x_{i_1}^{m_{i_1}} + x_{i_1+1} + \cdots + x_{i_v-1} + x_{i_v}^{m_{i_v}} + x_{i_v+1} + \cdots + x_t$. Consider

$$((s(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}}))^{\star} : s) = ((x_{i_1}^{m_{i_1}+r_{i_1}}, \ldots, x_{i_v}^{m_{i_v}+r_{i_v}}))^{\star} : s) = (x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})$$

Hence $(x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})^\star = (x_{i_1}^{r_{i_1}}, \ldots, x_{i_v}^{r_{i_v}})$ for all $r_{i_j} \in \mathbb{N}$.

□

Lemma 3.2.2. *If $\star$ is a standard closure on R_t and $(x_1, \ldots, x_t)^\star = R_t$ then $(x_1^{m_1}, \ldots, x_t^{m_t})^\star = R_t$ for all $(m_1, \ldots, m_t) \in \mathbb{N}^t$. Moreover if I is an $(x_1, \ldots, x_t)$-primary ideal then $I^\star = R_t$.*

Proof. Suppose $(x_1, \ldots, x_t)^\star = R_t$. Since $\star$ is weakly prime

$$(x_1^{m_1} + \cdots + x_t^{m_t})(x_1, \ldots, x_t)^\star \subseteq ((x_1^{m_1} + \cdots + x_t^{m_t})(x_1, \ldots, x_t))^\star$$

$\Rightarrow (x_1^{m_1}+\cdots+x_t^{m_t}) \subseteq (x_1^{m_1+1}, \ldots, x_t^{m_t+1})^\star \Rightarrow (x_1^{m_1+1}, \ldots, x_t^{m_t+1})^\star = (x_1^{m_1}+\cdots+x_t^{m_t})^\star$.

Note that

$$(x_1^2, x_2^3, \ldots, x_t^3)^\star \supseteq (x_1 + x_2^2 + \cdots + x_t^2)$$

$$(x_1^3, x_2^2, x_3^3, \ldots, x_t^3)^\star \supseteq (x_1^2 + x_2 + x_3^2 \cdots + x_t^2)$$

$$\vdots$$

$$(x_1^3, \ldots, x_{t-1}^3, x_t^2)^\star \supseteq (x_1^2 + \cdots + x_{t-1}^2 + x_t)$$

And Algorithm 3.1.9 implies $(x_1 + x_2^2 + \cdots + x_t^2, x_1^2 + x_2 + x_3^2 \cdots + x_t^2, \ldots, x_1^2 + \cdots + x_{t-1}^2 + x_t) = (x_1, \ldots, x_t)$. Thus we have the following inclusion,

$$(x_1, \ldots, x_t) = (x_1 + x_2^2 + \cdots + x_t^2, x_1^2 + x_2 + x_3^2 \cdots + x_t^2, \ldots, x_1^2 + \cdots + x_{t-1}^2 + x_t)$$

$$\subseteq (x_1^2, x_2^3, \ldots, x_t^3)^\star + (x_1^3, x_2^2, x_3^3, \ldots, x_t^3)^\star + \cdots + (x_1^3, \ldots, x_{t-1}^3, x_t^2)^\star \subseteq (x_1^2, \ldots, x_t^2)^\star$$

$$\subseteq (x_1, \ldots, x_t)^\star.$$

This implies $R_t = (x_1, \ldots, x_t)^\star = (x_1^2, \ldots, x_t^2)^\star$. Furthermore $(x_1^2, \ldots, x_t^2) \subseteq (x_1 + \cdots + x_t)$ this implies $(x_1 + \cdots + x_t)^\star = R_t$. Now we can show by induction that $(x_1^n, \ldots, x_t^n)^\star = R_t$ for all $n \in \mathbb{N}$ (our hypothesis is our base case). So assume $(x_1^n, \ldots, x_t^n)^\star = R_t$ for some $n \in \mathbb{N}$. Since $\star$ is weakly prime,

$$(x_1 + \cdots + x_t)(x_1^n, \ldots, x_t^n)^\star \subseteq ((x_1 + \cdots + x_t)(x_1^n, \ldots, x_t^n))^\star$$

$$\Rightarrow (x_1 + \cdots + x_t) \subseteq (x_1^{n+1}, \ldots, x_t^{n+1})^\star.$$

$$\Rightarrow (x_1^{n+1}, \ldots, x_t^{n+1})^\star = (x_1 + \cdots + x_t)^\star = R_t.$$

So $(x_1^n, \ldots, x_t^n)^\star = R_t$ for all $n \in \mathbb{N}$. Let $N = max\{m_1, \ldots, m_t\}$.

$$R_t = (x_1^N, \ldots, x_t^N)^\star \subseteq (x_1^{m_1}, \ldots, x_t^{m_t})^\star$$

$$\subseteq R_t \Rightarrow (x_1^{m_1}, \ldots, x_t^{m_t})^\star = R_t \text{ for all } (m_1, \ldots, m_t) \in \mathbb{N}^t.$$

□

Lemma 3.2.3. *If $\star$ is a standard closure on R_t and $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = R_t$ for some $(m_{i_1}, \ldots, m_{i_v}) \in \mathbb{N}^v$ then $(x_{i_1}^{n_{i_1}}, \ldots, x_{i_v}^{n_{i_v}})^\star = R_t$ for all $(n_{i_1}, \ldots, n_{i_v}) \in \mathbb{N}^v$. Moreover if J is an $(x_{i_1}, \ldots, x_{i_v})$-primary ideal then $J^\star = R_t$.*

Proof. Suppose $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = R_t$ for some $(m_{i_1}, \ldots, m_{i_v}) \in \mathbb{N}^v$. The inclusion $R_t = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star \subseteq (x_{i_1}, \ldots, x_{i_v})^\star$ implies $(x_{i_1}, \ldots, x_{i_v})^\star = R_t$. And the inclusion $R_t = (x_{i_1}, \ldots, x_{i_v})^\star \subseteq (x_1, \ldots, x_t)^\star$ implies $(x_1, \ldots, x_t)^\star = R_t$. Thus by Lemma 3.2.2 $(x_1 + \cdots + x_t)^\star = R_t$. Since $\star$ is weakly prime we have

$$(x_1 + \cdots + x_t)(x_{i_1}, \ldots, x_{i_v})^\star \subseteq ((x_1 + \cdots + x_t)(x_{i_1}, \ldots, x_{i_v}))^\star$$

$$(x_1 + \cdots + x_t) \subseteq (x_{i_1}^2, \ldots, x_{i_v}^2)^\star.$$

This implies $(x_{i_1}^2, \ldots, x_{i_v}^2)^\star = (x_1 + \cdots + x_t)^\star = R_t$. Inductively we have $(x_{i_1}^n, \ldots, x_{i_v}^n)^\star = R_t$ for all $n \in \mathbb{N}$. Consider $(x_{i_1}^{n_1}, \ldots, x_{i_v}^{n_v})$, $(n_1, \ldots, n_v) \in \mathbb{N}^v$. Let $N = \max\{n_1, \ldots, n_v\}$. The inclusion $R_t = (x_{i_1}^N, \ldots, x_{i_v}^N)^\star \subseteq (x_{i_1}^{n_1}, \ldots, x_{i_v}^{n_v})^\star$ implies $(x_{i_1}^{n_1}, \ldots, x_{i_v}^{n_v})^\star = R_t$ for all $(n_1, \ldots, n_v) \in \mathbb{N}^v$. □

Lemma 3.2.4. *Let $R_t^{\hat{n}} = k[[x_1, \ldots, x_{n-1}, \hat{x}_n, x_{n+1}, \ldots, x_t]]/(x_i x_j | i < j)$ and $\phi : R_t^{\hat{n}} \to R_t$ be defined $\phi(a) = \bar{a}$. Suppose $\star_T$ is a standard closure operation on $R_t^{\hat{n}}$. Define the following operation*

$\star_S : \mathcal{I}(R_t) \to \mathcal{I}(R_t)$ where $B^{\star_S} = \phi(A^{\star_T})$ if $B = \phi(A)$ for some proper ideal A in $R_t^{\hat{n}}$ or $B^{\star S} = R_t$ if $B \neq \phi(A)$ for any proper ideal A in $R_t^{\hat{n}}$.

Then ϕ is an injective ring homomorphism and $\star_S$ is a standard closure operation.

Proof. If a is non-zero then $\bar{a}$ is non-zero. Thus ϕ is injective. Since elements of $R_t^{\hat{n}}$ do not contain any terms which are multiples of x_n their images under ϕ are identical. Thus ϕ is a ring homomorphism. We will show $\star S$ is a closure operation. Let $B \in \mathcal{I}(R_t)$.

(i) (extension) Case: $B = \phi(A)$ for some proper $A \in \mathcal{I}(R_t^{\hat{n}})$. Thus $B = \phi(A) \subseteq \phi(A^{\star T}) = B^{\star S}$.

Case: $B \neq \phi(A)$ for any proper ideal $A \subseteq R_t^{\hat{n}}$. Thus $B \subseteq R_t = B^{\star S}$.

(ii) (order preservation) Suppose $B \subseteq C$, C an ideal in R_t. Case: $B = \phi(A)$ for some proper $A \in \mathcal{I}(R_t^{\hat{n}})$. If $C = \phi(I)$ for some proper $I \in \mathcal{I}(R_t^{\hat{n}})$ then it contains A. Thus $B^{\star S} = \phi(A^{\star T}) \subseteq \phi(I^{\star T}) = C^{\star S}$. If $C \neq \phi(I)$ for any $I \in \mathcal{I}(R_t^{\hat{n}})$ then $B^{\star S} = \phi(A^{\star T}) \subseteq R_t = C^{\star S}$. Case: $B \neq \phi(A)$ for any proper ideal $A \subseteq R_t^{\hat{n}}$. Thus B contains an element b such that $b \notin \operatorname{Im}\phi$. Since C also contains b, $B^{\star S} = R_t = C^{\star S}$.

(iii) (idempotence) Case: $B = \phi(A)$ for some proper $A \in \mathcal{I}(R_t^{\hat{n}})$.

$(B^{\star S})^{\star S} = (\phi(A^{\star T}))^{\star S} = \phi((A^{\star T})^{\star T}) = \phi(A^{\star T}) = B^{\star S}$. Case: $B \neq \phi(A)$ for any proper ideal $A \subseteq R_t^{\hat{n}}$. So $(B^{\star S})^{\star S} = (R_t)^{\star S} = R_t = B^{\star S}$.

Now we will show $\star S$ is a standard closure operation, $B^{\star S} = ((sB)^{\star S} : s)$ where s is a non-unit regular element of R_t. Thus $s = \sum_{i=1}^{t} a_i x_i^{m_i}$, $a_i \neq 0$ for $i = 1, \ldots, t$. Let $r = \sum_{\substack{i=1 \\ i \neq n}}^{t} a_i x_i^{m_i}$. So r is a regular element in the subring $R_t^{\hat{n}}$. Let $f \in B^{\star S}$. Case: $B \neq \phi(A)$ for any proper ideal $A \subseteq R_t^{\hat{n}}$. Thus B contains an element $g = h + ux_n^m$, $h \in \operatorname{Im}\phi$, and $u \in R_t^{\times}$. So $sg = (\sum_{\substack{i=0 \\ i \neq n}}^{t} a_i x_i^{m_i})g + (a_n x_n^{m_n})g = r(h + ux_n^m) + (a_n x_n^{m_n})(h + ux_n^m) = rh + a_n u x_n^{m+m_n}$. Thus $sB \neq \phi(A)$ for any proper ideal $A \subseteq R_t^{\hat{n}}$. So $(sB)^{\star S} = R_t \Rightarrow ((sB)^{\star S} : s) = (R_t : s) = R_t$ which contains f. Case: $B = \phi(A)$ for some proper $A \in \mathcal{I}(R_t^{\hat{n}})$. Thus $B^{\star S} = \phi(A^{\star T})$. Now $sB = s\phi(A) = r\phi(A) + a_n x_n^{m_n}\phi(A) = r\phi(A) = \phi(r)\phi(A) = \phi(rA) \Rightarrow (sB)^{\star S} = \phi((rA)^{\star T})$. Since $f \in B^{\star S}$ there exists $g \in A^{\star T}$ such that $f = \phi(g)$. Hence $rg \in rA^{\star T} \subseteq (rA)^{\star T}$ (since $\star T$ is weakly prime). So $\phi(rg) \in \phi((rA)^{\star T}) = (sB)^{\star S}$. Thus

$$\phi(rg) = \phi(r)\phi(g) = r\phi(g) = s\phi(g) = sf \Rightarrow sf \in (sB)^{\star S} \Rightarrow f \in ((sB)^{\star S} : s).$$

To show the other inclusion let $f \in ((sB)^{\star S} : s)$.
Case: $B \neq \phi(A)$ for any proper ideal $A \subsetneq R_t^{\hat{n}}$. Thus there exists $b \in B$ such that $b \notin \operatorname{Im}\phi$. Hence $sb \notin \operatorname{Im}\phi$. So $sB \neq \phi(A)$ for any proper ideal $A \subsetneq R_t^{\hat{n}}$ which implies $(sB)^{\star S} = R_t$.

$$sB \subseteq B \Rightarrow R_t = (sB)^{\star S} \subseteq B^{\star S} \Rightarrow B^{\star S} = R_t \Rightarrow f \in B^{\star S}.$$

Case: $B = \phi(A)$ for some proper $A \in \mathcal{I}(R_t^{\hat{n}})$. First we will show there exists $g \in R_t^{\hat{n}}$ such that $\phi(g) = f$.

$$B = \phi(A) \Rightarrow sB = s\phi(A) = r\phi(A) = \phi(r)\phi(A) = \phi(rA)$$

$$\Rightarrow (sB)^{\star S} = \phi((rA)^{\star T}) \subseteq \phi(A^{\star T}).$$

Since $f \in ((sB)^{\star S} : s)$, $sf \in \phi(A^{\star T}) \subseteq \operatorname{Im}\phi$. Now suppose $f \notin \operatorname{Im}\phi$. So $f = \sum_{k=1}^{t} c_k x_k^{w_k}$, $c_k \in R_t^{\times}$ or $c_k = 0$ for k such that $1 \leq k \leq t$ and $c_n \neq 0$. Thus

$$sf = rf + (a_n x_n^{m_n})f = r\sum_{\substack{k=0 \\ k\neq n}}^{t} c_k x_k^{w_k} + a_n c_n x_n^{m_n+w_n} \notin \operatorname{Im}\phi.$$

This is a contradiction. So there exists $g \in R_t^{\hat{n}}$ such that $\phi(g) = f$. So $sf \in \phi((rA)^{\star T}) \Rightarrow rf \in \phi((rA)^{\star T})$. Thus

$$\phi(rg) = \phi(r)\phi(g) = rf \in \phi((rA)^{\star T}) \Rightarrow rg \in (rA)^{\star T} \Rightarrow g \in ((rA)^{\star T} : r) = A^{\star T}$$

$$\Rightarrow f = \phi(g) \in \phi(A^{\star T}) = B^{\star S}.$$

□

Lemma 3.2.5. *Let $I = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$, $1 \leq i_1 \leq \cdots \leq i_v \leq t$. and $\star$ be a standard closure operation on R_t. If $I^\star$ contains a regular element $x_1^{s_1} + a_2 x_2^{s_2} + \cdots + a_t x_t^{s_t}$ with $a_\alpha \in k^\times$ for all α such that $2 \leq \alpha \leq v$ and $s_{i_u} < m_{i_u}$ for all u such that $1 \leq u \leq v$ then $I^\star = R_t$.*

Proof. Consider

$$(((x_1^{s_1} + a_2x_2^{s_2} + \cdots + a_tx_t^{s_t})(x_{i_1}^{m_{i_1}-s_{i_1}}, \ldots, x_{i_v}^{m_{i_v}-s_{i_v}}))^\star : x_1^{s_1} + a_2x_2^{s_2} + \cdots + a_tx_t^{s_t})$$

$$= (I^\star : x_1^{s_1} + a_2x_2^{s_2} + \cdots + a_tx_t^{s_t}) = R_t$$

Since $\star$ is standard $(x_{i_1}^{m_{i_1}-s_{i_1}}, \ldots, x_{i_v}^{m_{i_v}-s_{i_v}})^\star = R_t$. So $R_t = (x_{i_1}^{m_{i_1}-s_{i_1}}, \ldots, x_{i_v}^{m_{i_v}-s_{i_v}})^\star \subseteq (x_1, \ldots, x_t)^\star$ implies $(x_1, \ldots, x_t)^\star = R_t$. And by Lemma 3.2.2, $I^\star = R_t$. □

Lemma 3.2.6. *Suppose $\star$ is a standard closure operation on R_t. Then $(0)^\star$ is a monomial ideal.*

Proof. Suppose $(0)^\star$ is not a monomial ideal. Suppose W is a reduced generating set for $(0)^\star$. Then W must contain $\sum_{v=1}^{n} a_{i_v}x_{i_v}^{m_v}$, $n \geq 2$, $a_v \neq 0$ for all $1 \leq v \leq n$. There are two cases.

Case: $n = t$. Consider

$$(((\sum_{v=1}^{n} a_{i_v}x_{i_v}^{m_v})(0))^\star : \sum_{v=1}^{n} a_{i_v}x_{i_v}^{m_v}) = ((0)^\star : \sum_{v=1}^{n} a_{i_v}x_{i_v}^{m_v}) = R_t$$

since $(0)^\star$ contains $\sum_{v=1}^{n} a_{i_v}x_{i_v}^{m_v}$. Since $\star$ is standard this is a contradiction.

Case: $n < t$. Consider

$$(((x_1 + \cdots + x_t)(0))^\star : x_1 + \cdots + x_t) = ((0)^\star : x_1 + \cdots + x_t).$$

Since $\star$ is standard this implies $x_{i_v}^{m_v} \in (0)^\star$ for $1 \leq v \leq n$ which is a contradiction. □

Lemma 3.2.7. *Suppose $\star$ is a standard closure operation on R_t and $(0)^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$ for some $(m_{i_1}, \ldots, m_{i_v}) \in \mathbb{N}^v$ and v is such that $1 \leq v < t$. Then $(0)^\star = (x_{i_1}, \ldots, x_{i_v})$. Additionally, if $I \subseteq (x_{i_1}, \ldots, x_{i_v})$ then $I^\star = (x_{i_1}, \ldots, x_{i_v})$.*

Proof. Since $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}) = (0)^\star \subseteq (x_{i_1}^{m_{i_1}+1}, \ldots, x_{i_v}^{m_{i_v}+1})^\star \subseteq (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$ it must be the case that $(x_{i_1}^{m_{i_1}+1}, \ldots, x_{i_v}^{m_{i_v}+1})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$. Consider

$$(((x_1+\cdots+x_t)(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}))^\star : x_1+\cdots+x_t) = ((x_{i_1}^{m_{i_1}+1}, \ldots, x_{i_v}^{m_{i_v}+1})^\star : x_1+\cdots+x_t)$$

$$= ((x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}}) : x_1 + \cdots + x_t).$$

If $m_{i_j} > 1$ for some j such that $1 \leq j \leq v$ then, since $\star$ is standard, $x_{i_j}^{m_{i_j}-1} \in (0)^\star$ which is a contradiction.

Suppose $I \subseteq (x_{i_1}, \ldots, x_{i_v})$. The inclusion $(x_{i_1}, \ldots, x_{i_v}) = (0)^\star \subseteq I^\star \subseteq (x_{i_1}, \ldots, x_{i_v})$ implies $I^\star = (x_{i_1}, \ldots, x_{i_v})$. □

Lemma 3.2.8. *Suppose $\star$ is a standard closure operation on R_t. If $(x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})^\star = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$ for all $(m_{i_1}, \ldots, m_{i_v}) \in \mathbb{N}^v$ with $v \in \{1, \ldots, t\}$ and each $m_{i_j} > 1$, $j \in \{1, \ldots, v\}$ then $(0)^\star = (0)$.*

Proof. By Lemma 3.2.6 $(0)^\star$ is monomial. Since $(0)^\star \subseteq (x_{i_1}^2, \ldots, x_{i_v}^2)^\star = (x_{i_1}^2, \ldots, x_{i_v}^2)$, $(0)^\star \neq R_t$. Suppose $(0)^\star$ is a proper nonzero monomial ideal. Without loss of generality we have $(0)^\star = (x_{i_1}^{n_{i_1}}, \ldots, x_{i_u}^{n_{i_u}})$ for some $u \leq v$ with $n_{i_j} \geq 2$ for each $j \in \{1, \ldots, u\}$. However we have a contradiction since by Lemma 3.2.7 $(0)^\star = (x_{i_1}, \ldots, x_{i_u})$. □

Lemma 3.2.9. *Let $R = \mathbb{Z}/2\mathbb{Z}[[X, Y, Z]]/(XY, YZ, XZ)$ and I be a proper monomial ideal in R. Suppose $\star$ is a standard closure operation on R. If $I^\star \neq R$ then $I^\star$ is a proper monomial ideal.*

Proof. Suppose $\star$ is a standard closure operation on R and $I^\star \neq R$. By Proposition 3.1.14 each ideal of R has a unique reduced generating set. Thus each proper ideal of R expressed in terms of its reduced generating set will have one of the following forms:

- (0), (x^m), (y^n), (z^r), $(x^m + y^n)$, $(y^n + z^r)$, $(x^m + z^r)$, $(x^m + y^n + z^r)$
- (x^m, y^n), (y^n, z^r), (x^m, z^r), $(x^m + y^n, z^r)$, $(x^m + z^r, y^n)$, $(x^m, y^n + z^r)$, $(x^m + z^r, y^n + z^r)$
- (x^m, y^n, z^r)

m, n, $r \in \mathbb{N}$. Thus without loss of generality we need to consider the following cases where $I = (x^m)$, or $I = (x^m, y^n)$, or $I = (x^m, y^n, z^r)$:

1. $(x^m)^\star = (x^i + y^j)$
2. $(x^m)^\star = (x^i + y^j, z^l)$
3. $(x^m)^\star = (x^i + y^j + z^l)$
4. $(x^m)^\star = (x^i, y^j + z^l)$
5. $(x^m)^\star = (x^i + z^l, y^j + z^l)$
6. $(x^m, y^n)^\star = (x^i + y^j)$
7. $(x^m, y^n)^\star = (x^i + y^j, z^l)$
8. $(x^m, y^n)^\star = (x^i + y^j + z^l)$
9. $(x^m, y^n)^\star = (x^i, y^j + z^l)$
10. $(x^m, y^n)^\star = (x^i + z^l, y^j + z^l)$
11. $(x^m, y^n, z^r)^\star = (x^i, y^j + z^l)$
12. $(x^m, y^n, z^r)^\star = (x^i + y^j + z^l)$
13. $(x^m, y^n, z^r)^\star = (x^i + z^l, y^j + z^l)$

Since $I \subseteq I^\star$ cases (2), (3), (5), (7), (8), (10), (12) and (13) imply $i < m$, $j < n$ and $l < r$. By Lemma 3.2.5 $I^\star = R$ which is a contradiction.

Case (1): $(x^m)^\star = (x^i + y^j)$. Thus $(x^i + y^j) = (x^m)^\star \subseteq (x^v)^\star \subseteq (x^i + y^j)^\star \subseteq (x^i + y^j) \Rightarrow (x^v)^\star = (x^i + y^j)$ for all v such that $i < v \leq m$. Consider

$$(((x^{m-i} + y^j + z)(x^i))^\star : x^{m-i} + y^j + z) = ((x^m)^\star : x^{m-i} + y^j + z)$$

$$= ((x^i + y^j) : x^{m-i} + y^j + z).$$

Since $\star$ is standard this implies $(x^i)^\star = (x, y)$ when $m \geq i$ or $(x^i)^\star = (x^{2i-m+1}, y)$ when $m < 2i$. Suppose $(x^i)^\star = (x, y)$. Since $\star$ is weakly prime we have

$$(x + y + z)(x^i)^\star \subseteq ((x + y + z)(x^i))^\star \Rightarrow (x^2, y^2) \subseteq (x^{i+1})^\star = (x^i + y^j) \Rightarrow i = j = 1.$$

So $(x^2)^\star = (x+y)$ and $(x) = (x,y)$. Since $(x^3)^\star \subseteq (x^2)^\star = (x+y)$ we have four possibilities for $(x^3)^\star$:$(x^3)^\star = (x^3)$, $(x^3)^\star = (x^3, y^l)$ with $l > 1$, $(x^3)^\star = (x^2 + y^l)$ with $l > 1$, or $(x^3)^\star = (x+y)$. Consider $(((x^2+y+z)(x))^\star : x^2+y+z) =$ $((x^3)^\star : x^2+y+z)$. Since $\star$ is standard $(x^3)^\star = (x^3)$ implies $(x)^\star = (x)$ and $(x^3)^\star = (x^2+y^l)$ implies $(x)^\star = (x, y^l)$ with $l > 1$. These are both contradictions. Now consider $(((x+y+z)(x^2))^\star : x+y+z) = ((x^3)^\star : x+y+z)$. So $(x^3)^\star = (x^3, y^l)$ implies $(x^2)^\star = (x^2, y)$ or $(x^2)^\star = (x^2, y^{l-1})$ and $(x^3)^\star = (x+y)$ implies $(x^2)^\star = (x,y)$. Again contradictions.

Now suppose $(x^i)^\star = (x^{2i-m+1}, y)$. We have

$$(x+y+z)(x^i)^\star \subseteq ((x+y+z)(x^i))^\star \Rightarrow (x^{2i-m+2}, y^2) \subseteq (x^{i+1})^\star = (x^i + y^j) \Rightarrow j = 1.$$

Thus $(x^{i+1})^\star = (x^i + y)$. Since $(x^{i+2})^\star \subseteq (x^{i+1})^\star = (x^i + y)$ we have four possibilities for $(x^{i+2})^\star$:$(x^{i+2})^\star = (x^{i+2})$, $(x^{i+2})^\star = (x^{i+2}, y^l)$ with $l > 1$, $(x^{i+2})^\star = (x^{i+1} + y^l)$ with $l > 1$, or $(x^{i+2})^\star = (x^i + y)$. Consider $(((x^2+y+z)(x^i))^\star : x^2+y+z) =$ $((x^{i+2})^\star : x^2+y+z)$. So $(x^{i+2})^\star = (x^{i+2})$ implies $(x^i)^\star = (x^i)$ and $(x^{i+2})^\star = (x^{i+1}+y^l)$ implies $(x^i)^\star = (x, y^l)$ or $(x^i)^\star = (x^{i-1} + y^{l-1})$. These are contradictions so we are left with $(x^{i+2})^\star = (x^{i+2}, y^l)$ or $(x^{i+2})^\star = (x^i + y)$. Now consider $(((x+y+z)(x^{i+1}))^\star : x+y+z) = ((x^{i+2})^\star : x+y+z)$. So $(x^{i+2})^\star = (x^{i+2}, y^l)$ implies $(x^{i+1})^\star = (x^{i-1}, y^{l-1})$ and $(x^{i+2})^\star = (x^i + y)$ implies $(x^{i+1})^\star = (x^i, y)$. These are also contradictions.

Case (4): $(x^m)^\star = (x^i, y^j + z^l)$. If $i < m$ then Lemma 3.2.5 implies $(x^m)^\star = R$ which is a contradiction. Thus $(x^m)^\star = (x^m, y^j + z^l)$. Since $\star$ is weakly prime

$$(x+y+z)(x^m)^\star \subseteq ((x+y+z)(x^m))^\star \Rightarrow (x^{m+1}, y^{j+1} + z^{l+1}) \subseteq (x^{m+1})^\star$$

$$\Rightarrow (x^{m+1})^\star = (x^{m+1}, y^{j+1} + z^{l+1})^\star \subseteq (x^m, y^j + z^l).$$

By this and Lemma 3.2.5 $(x^{m+1})^\star$ is one of the following $(x^{m+1}, y^{j+1} + z^{l+1})$, $(x^{m+1}, y^j + z^l)$, or $(x^{m+1}, y^{j+1}, z^{l+1})$. For each of these if we compute

$(((x+y+z)(x^m))^\star : x+y+z) = ((x^{m+1})^\star : x+y+z)$ we arrive at contradiction except in the case $(x^{m+1})^\star = (x^{m+1}, y^{j+1}+z^{l+1})$. Inductively we have $(x^{m+v})^\star = (x^{m+v}, y^{j+v}+z^{l+v})$ for all $v > 0$. Consider

$$(((x^2+y+z)(x^{m+1}, y^{j+2}+z^{l+2}))^\star : x^2+y+z) = ((x^{m+3}, y^{j+3}+z^{l+3})^\star : x^2+y+z)$$

$$= (x^{m+1}, y^{j+2}+z^{l+2}).$$

This implies $(x^{m+1}, y^{j+2}+z^{l+2})^\star = (x^{m+1}, y^{j+2}+z^{l+2})$. However we have a contradiction since

$$(x^{m+1}, y^{j+1}+z^{l+1}) = (x^{m+1})^\star \subseteq (x^{m+1}, y^{j+2}+z^{l+2})^\star = (x^{m+1}, y^{j+2}+z^{l+2}).$$

Case (6): $(x^m, y^n)^\star = (x^i + y^j)$. Since $\star$ is weakly prime we have the following implication:

$$(x+y+z)(x^m, y^n)^\star \subseteq ((x+y+z)(x^m, y^n)^\star \Rightarrow (x^{i+1}+y^{j+1}) \subseteq (x^{m+1}, y^{n+1})^\star$$

$$\Rightarrow (x^{m+1}, y^{n+1})^\star = (x^{i+1}+y^{j+1})^\star \subseteq (x^i + y^j).$$

Thus $(x^{m+1}, y^{n+1})^\star$ is one of the following $(x^{i+1}+y^{j+1})$, (x^{i+1}, y^{j+1}) or (x^i+y^j). If we compute $(((x+y+z)(x^m, y^n))^\star : x+y+z) = ((x^{m+1}y^{n+1})^\star : x+y+z)$ we arrive at a contradiction except in the case $(x^{m+1}, y^{n+1})^\star = (x^{i+1}+y^{j+1})$. Now $(x^i+y^j) = (x^m, y^n)^\star \subseteq (x^{i+1}+y^{j+1})^\star = (x^{i+1}+y^{j+1})$. This is a contradiction since $(x^{i+1}+y^{j+1})$ does not contain (x^i+y^j).

Case (9): $(x^m, y^n)^\star = (x^i, y^j+z^l)$. Since $j < n$, if $i < m$ by Lemma 3.2.5 $(x^m, y^n)^\star = R$ which is a contradiction. Thus $(x^m, y^n)^\star = (x^m, y^j+z^l)$. Since $\star$ is weakly prime

$$(x+y+z)(x^m, y^n)^\star \subseteq ((x+y+z)(x^m, y^n))^\star \Rightarrow (x^{m+1}, y^{j+1}+z^{l+1}) \subseteq (x^{m+1}, y^{n+1})^\star$$

$$\Rightarrow (x^{m+1}, y^{n+1})^\star = (x^{m+1}, y^{j+1}+z^{l+1})^\star \subseteq (x^i, y^j+z^l).$$

By this and Lemma 3.2.5 $(x^{m+1}, y^{n+1})^\star$ is one of the following $(x^{m+1}, y^{j+1} + z^{l+1})$, $(x^{m+1}, y^j + z^l)$, or $(x^{m+1}, y^{j+1}, z^{l+1})$. For each of these if we compute $(((x + y + z)(x^m, y^n))^\star : x + y + z) = ((x^{m+1}, y^{n+1})^\star : x + y + z)$ we arrive at contradiction except in the case $(x^{m+1}, y^{n+1})^\star = (x^{m+1}, y^{j+1} + z^{l+1})$. Inductively we have $(x^{m+v}, y^{n+v})^\star = (x^{m+v}, y^{j+v} + z^{l+v})$ for all $v > 0$. Consider

$$(((x^2+y^2+z)(x^{m+1}, y^{j+1}+z^{l+2}))^\star : x^2+y^2+z) = ((x^{m+3}, y^{j+3}+z^{l+3})^\star : x^2+y^2+z)$$
$$= (x^{m+1}, y^{j+1} + z^{l+2}).$$

This implies $(x^{m+1}, y^{j+1} + z^{l+2})^\star = (x^{m+1}, y^{j+1} + z^{l+2})$. However we have a contradiction since

$$(x^{m+1}, y^{j+1} + z^{l+1}) = (x^{m+1})^\star \subseteq (x^{m+1}, y^{j+1} + z^{l+2})^\star = (x^{m+1}, y^{j+1} + z^{l+2}).$$

Case (11): $(x^m, y^n, z^r)^\star = (x^i, y^j + z^l)$. This case is nearly identical to case (9).

□

Lemma 3.2.10. *Let* $I = (x_{i_1}^{m_{i_1}}, \ldots, x_{i_v}^{m_{i_v}})$, $1 \leq v \leq t$. *Suppose* $\star$ *is a standard closure operation on* R_t *and* $|k| > 2$. *If* $I^\star \neq R_t$ *then* $I^\star$ *is a proper monomial ideal.*

Proof. Suppose $I^\star \neq R_t$. Now suppose without loss of generality that $I = (x_1^{m_1}, \ldots, x_v^{m_v})$, $v \leq t$, $I^\star = (x_1^{r_1}, \ldots, x_u^{r_u}, \{f_l\}_{l=1}^d)$, $u \leq v$, $\{x_1^{r_1}, \ldots, x_u^{r_u}, \{f_l\}_{l=1}^d\}$ is a reduced generating set of $I^\star$ and the f_l are non-monomial polynomials in the k-vector space $\langle x_{u+1}^{r_{u+1}}, \ldots, x_w^{r_w} \rangle$ with each $r_j < m_j$ for j such that $u + 1 \leq j \leq v$. Suppose by way of contradiction there exists $f_l \neq 0$, say f_1. Since $\star$ is weakly prime we have

$$(x_1 + \cdots + x_t)I^\star \subseteq ((x_1 + \cdots + x_t)I)^\star$$
$$\Rightarrow (x_1^{r_1+1}, \ldots, x_u^{r_u+1}, \{(\sum_{k=1}^{t} x_k)f_l\}) \subseteq (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star$$
$$\Rightarrow (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star = (x_1^{r_1+1}, \ldots, x_u^{r_u+1}, \{(\sum_{k=1}^{t} x_k)f_l\})^\star \subseteq I^\star.$$

This implies $(\sum_{k=1}^{t} x_k)f_l \in (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star$ for each l. Again without loss of generality suppose $f_1 = \sum_{j=u+1}^{w} a_j x_j^{r_j}$ with $a_{u+1} = 1$. Let $\beta \in k^\times$ and $\beta \neq a_{u+1}$. Since $\star$ is standard we have

$$I^\star = (((x_1+\cdots+x_u+\beta x_{u+1}+x_{u+2}+\cdots+x_t)I)^\star : x_1+\cdots+x_u+\beta x_{u+1}+x_{u+2}+\cdots+x_t)$$

$$= ((x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star : x_1 + \cdots + x_u + \beta x_{u+1} + x_{u+2} + \cdots + x_t).$$

Since $(\sum_{k=1}^{t} x_k)f_1 \in (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star$ we get $\frac{1}{\beta}x_{u+1}^{r_{u+1}} + \sum_{j=u+2}^{w} a_j x_j^{r_j} \in I^\star$. This implies $(1 - \frac{1}{\beta})x_{u+1}^{r_{u+1}} = f_1 - \left(\frac{1}{\beta}x_{u+1}^{r_{u+1}} + \sum_{j=u+2}^{w} a_j x_j^{r_j} \in I^\star\right)$. Which then implies $x_{u+1}^{r_{u+1}} \in I^\star$ which is a contradiction. □

Lemma 3.2.11. *Suppose $\star$ is a standard closure operation on R_t. Further suppose $|k| > 2$ or that $t = 3$ and $k = \mathbb{Z}/2\mathbb{Z}$. If $(x_1^{m_1}, \ldots, x_t^{m_t})^\star = (x_1^{n_1}, \ldots, x_t^{n_t})$ for some $(m_1, \ldots, m_t) \in \mathbb{N}^t$ and $(n_1, \ldots, n_t) \in \mathbb{N}^t$ then $(x_1^{r_1}, \ldots, x_t^{r_t})^\star = (x_1^{s_1}, \ldots, x_t^{s_t})$ for all $(r_1, \ldots, r_t) \in \mathbb{N}^t$ such that for some $j \in \{1, \ldots, t\}$, $s_{i_j} = r_{i_j}$ for all r_{i_j} and for all other j, $s_{i_j} = 1$ for all r_{i_j}.*

Proof. Let $\mathfrak{m} = (x_1, \ldots, x_t)$. If $\mathfrak{m}^\star = R_t$ by Lemma 3.2.3 $(x_1^{m_1}, \ldots, x_t^{m_t})^\star = R_t$ which is a contradiction. Thus $\mathfrak{m}^\star = \mathfrak{m}$. Consider $I = (x_1^2, \ldots, x_t^2)$. Since $I^\star \subset \mathfrak{m}^\star = \mathfrak{m}$, $I^\star \neq R_t$. So by Lemma 3.2.5 $I^\star$ does not contain a regular element $x_1^{s_1} + a_2 x_2^{s_2} + \cdots + a_t x_t^{s_t}$ with $a_\alpha \in k^\times$ for all α such that $2 \leq \alpha \leq t$ and $s_l < 2$ for all l such that $1 \leq l \leq t$. Thus for some l such that $1 \leq l \leq t$, $x_l^2 \in I^\star$ and $x_l \notin I^\star$. So by Lemma 3.2.9 or Lemma 3.2.10 $I^\star$ is a proper monomial ideal. Thus $I^\star$ is generated by degree 1 and degree 2 monomials.

Suppose $I^\star = I$. By Lemma 3.2.1 $(x_1^{m_1}, \ldots, x_t^{m_t})^\star = (x_1^{m_1}, \ldots, x_t^{m_t})$ for all m_j, $1 \leq j \leq t$.

Now we will consider the case when $I^\star$ is generated by degree 2 monomials and at least one degree 1 monomial. Without loss of generality let

$I^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$.

$$\Rightarrow (x_1^2, x_2, \ldots, x_t)^\star = (x_1, \ldots, x_t) \text{ or } (x_1^2, x_2, \ldots, x_t).$$

Suppose $(x_1^2, x_2, \ldots, x_t)^\star = (x_1, \ldots, x_t)$. Since $\star$ is weakly prime

$$(x_1 + \cdots + x_t)(x_1^2, x_2, \ldots, x_t)^\star \subseteq ((x_1 + \cdots + x_t)(x_1^2, x_2, \ldots, x_t))^\star$$

$$\Rightarrow (x_1^2, \ldots, x_t^2) \subseteq (x_1^3, x_2^2, \ldots, x_t^2)^\star$$

$$\Rightarrow (x_1^3, x_2^2, \ldots, x_t^2)^\star = (x_1^2, \ldots, x_t^2)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$$

and

$$(x_1 + \cdots + x_t)(x_1^2, \ldots, x_t^2)^\star \subseteq ((x_1 + \cdots + x_t)(x_1^2, \ldots, x_t^2))^\star$$

$$\Rightarrow (x_1^3, \ldots, x_v^3, x_{v+1}^2, \ldots, x_t^2) \subseteq (x_1^3, \ldots, x_t^3)^\star$$

$$\Rightarrow (x_1^3, \ldots, x_t^3)^\star = (x_1^3, \ldots, x_v^3, x_{v+1}^2, \ldots, x_t^2)^\star \subseteq (x_1^3, x_2^2, \ldots, x_t^2)^\star$$

$$= (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$$

$\Rightarrow (x_1^3, \ldots, x_t^3)^\star = (x_1^{n_1}, x_2^{n_2}, \ldots, x_v^{n_v}, x_{v+1}^2, \ldots, x_t^2)$ with $n_j = 2$ or 3 for j such that $1 \leq j \leq v$ with at least one $n_j = 3$, or $(x_1^3, \ldots, x_t^3)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$. By Lemma 3.2.5 that latter case implies $(x_1^3, \ldots, x_t^3)^\star = R_t$ which is a contradiction. So suppose the former. Again without loss of generality let $(x_1^3, \ldots, x_t^3)^\star = (x_1^3, \ldots, x_u^3, x_{u+1}^2, \ldots, x_t^2)$ with $u \leq v$.
Consider

$$(((x_1 + x_2^2 + \cdots + x_t^2)(x_1^2, x_2, \ldots, x_t))^\star : x_1 + x_2^2 + \cdots + x_t^2)$$

$$= ((x_1^3, \ldots, x_t^3)^\star : x_1 + x_2^2 + \cdots + x_t^2)$$

$$= ((x_1^3, \ldots, x_u^3, x_{u+1}^2, \ldots, x_t^2) : x_1 + x_2^2 + \cdots + x_t^2) = (x_1^2, x_2, \ldots, x_t).$$

Since $\star$ is standard this implies $(x_1^2, x_2, \ldots, x_t)^\star = (x_1^2, x_2, \ldots, x_t)$ which is a contradiction of the assumption $(x_1^2, x_2, \ldots, x_t)^\star = (x_1, \ldots, x_t)$. So we are left with

$(x_1^2, x_2, \ldots, x_t)^\star = (x_1^2, x_2, \ldots, x_t)$.
Similarly for some $u \leq v$,

$$(x_1, \ldots, x_{u-1}, x_u^2, x_{u+1}, \ldots, \ldots, x_v, \ldots, x_t)^\star$$

$$= (x_1, \ldots, x_{u-1}, x_u^2, x_{u+1}, \ldots, \ldots, x_v \ldots, x_t).$$

With $(x_1^2, \ldots, x_t^2)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$ as the base case suppose

$$(x_1^2, \ldots, x_v^2, x_{v+1}^k, \ldots, x_t^k)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t) \text{ for } k \leq n.$$

Thus

$$\mathfrak{m} = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t) + (x_1, \ldots, x_v, x_{v+1}^k, \ldots, x_t^k)$$

$$= (x_1^2, \ldots, x_v^2, x_{v+1}^k, \ldots, x_t^k)^\star + (x_1, \ldots, x_v, x_{v+1}^k, \ldots, x_t^k)$$

$$\subseteq (x_1, \ldots, x_v, x_{v+1}^k, \ldots, x_t^k)^\star \subseteq \mathfrak{m}$$

$$\Rightarrow (x_1, \ldots, x_v, x_{v+1}^k, \ldots, x_t^k)^\star = \mathfrak{m}$$

Since $\star$ is weakly prime

$$(x_1 + \cdots + x_t)(x_1, \ldots, x_v, x_{v+1}^n, \ldots, x_t^n)^\star \subseteq ((x_1 + \cdots + x_t)(x_1, \ldots, x_v, x_{v+1}^n, \ldots, x_t^n))^\star$$

$$\Rightarrow (x_1^2, \ldots, x_t^2)^\star \subseteq (x_1^2, \ldots, x_v^2, x_{v+1}^{n+1}, \ldots, x_t^{n+1})^\star$$

$$\Rightarrow (x_1^2, \ldots, x_v^2, x_{v+1}^{n+1}, \ldots, x_t^{n+1})^\star = (x_1^2, \ldots, x_t^2)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$$

$$\Rightarrow (x_1^2, \ldots, x_v^2, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t) \text{ for all } n.$$

Since

$$(x_1^2, \ldots, x_v^2, x_{v+1}^n, \ldots, x_t^n)^\star$$

$$\subseteq (x_1^{r_1}, \ldots, x_v^{r_v}, x_{v+1}^n, \ldots, x_t^n)^\star \text{ (for } r_j = 1 \text{ or } 2, \text{ for } j = 1, \ldots, v)$$

$$\subseteq (x_1, \ldots, x_{u-1}, x_u^2, x_{u+1}, \ldots, \ldots, x_v, \ldots, x_t)^\star$$

$$= (x_1, \ldots, x_{u-1}, x_u^2, x_{u+1}, \ldots, \ldots, x_v \ldots, x_t)$$

for all $u \leq v$ we have $(x_1^{r_1}, \ldots, x_v^{r_v}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{r_1}, \ldots, x_v^{r_v}, x_{v+1}, \ldots, x_t)$ with $r_j = 1$ or 2 for j such that $1 \leq j \leq v$ for all n.

With $(x_1^2, \ldots, x_v^2, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^2, \ldots, x_v^2, x_{v+1}, \ldots, x_t)$ for all n as the base case suppose $(x_1^k, \ldots, x_v^k, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^k, \ldots, x_v^k, x_{v+1}, \ldots, x_t)$ for all $k \leq m$, for all n. Since $\star$ is weakly prime

$$(x_1 + \cdots + x_t)(x_1^m, \ldots, x_v^m, x_{v+1}^{n-1}, \ldots, x_t^{n-1})^\star$$

$$\subseteq ((x_1 + \cdots + x_t)(x_1^m, \ldots, x_v^m, x_{v+1}^{n-1}, \ldots, x_t^{n-1}))^\star$$

$$\Rightarrow (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_t^2) \subseteq (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star$$

$$\Rightarrow (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_t^2)^\star$$

$$\subseteq (x_1^m, \ldots, x_v^m, x_{v+1}^2, \ldots, x_t^2)^\star = (x_1^k, \ldots, x_v^k, x_{v+1}, \ldots, x_t)$$

$$\Rightarrow (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{s_1}, \ldots, x_v^{s_v}, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t})$$

with $s_j = m$ or $m+1$ for $j = 1, \ldots, v$ and $s_j = 1$ or 2 for $j = v+1, \ldots, t$.

If all $s_j = m$ for j such that $1 \leq j \leq v$ then $(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^m, \ldots, x_v^m, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t})$. However $(x_1^m, \ldots, x_v^m, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t})$ is $\star$ closed only if $s_j = 1$ for all $v+1 \leq j \leq t$ since

$$(x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t) = (x_1^m, \ldots, x_v^m, x_{v+1}^2, \ldots, x_t^2)^\star$$

$$\subseteq (x_1^m, \ldots, x_v^m, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t}).$$

Thus $(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)$.

By Lemma 3.2.5 $(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = R_t$ which is a contradiction. So for at least one j, $1 \leq j \leq v$, $s_j = m+1$. Now suppose $s_l = m$ for some $l \neq j$,$1 \leq j \leq v$. Consider

$$(((x_1 + \cdots + x_t)(x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t))^\star : x_1 + \cdots + x_t)$$

$$= ((x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_t^2)^\star : x_1 + \cdots + x_t)$$

$$= ((x_1^{s_1}, \ldots, x_v^{s_v}, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t}) : x_1 + \cdots + x_t).$$

Since $\star$ is standard $((x_1^{s_1}, \ldots, x_v^{s_v}, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t}) : x_1 + \cdots + x_t)$

$$= (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)^\star.$$

This implies $x_l^{m_l - 1} \in (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)^\star$ which is contradiction since $(x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)$.

Thus $(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^{s_{v+1}}, \ldots, x_t^{s_t})$. Without loss of generality suppose

$$(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1} \ldots, x_t).$$

Since $\star$ is weakly prime

$$(x_1 + \cdots + x_t)(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1} \ldots, x_t)^\star$$

$$\subseteq ((x_1 + \cdots + x_t)(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1} \ldots, x_t))^\star$$

$$\Rightarrow (x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_w^3, x_{w+1}^2 \ldots, x_t^2) \subseteq (x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star$$

$$\Rightarrow (x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star = (x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_w^3, x_{w+1}^2 \ldots, x_t^2)^\star$$

Consider

$$(((x_1^2 + \cdots + x_v^2 + x_{v+1} + \cdots + x_t)(x_1^m, \ldots, x_v^m, x_{v+1}^2, \ldots, x_t^2))^\star$$

$$: x_1^2 + \cdots + x_v^2 + x_{v+1} + \cdots + x_t)$$

$$= ((x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star : x_1^2 + \cdots + x_v^2 + x_{v+1} + \cdots + x_t)$$

Since $\star$ is standard if $(x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star$ contains x_j^{m+1}, for some j such that $1 \leq j \leq v$ then $(x_1^m, \ldots, x_v^m, x_{v+1}^2, \ldots, x_t^2)^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)$ contains x_j^{m-1} which is a contradiction. If $(x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star$ contains x_j^3 but not x_j^2 for some j such that $v+1 \leq j \leq w$ then $(x_1^m, \ldots, x_v^m, x_{v+1}^2, \ldots, x_t^2)^\star =$

$(x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t)$ contains x_j^2 but does not contain x_j which is a contradiction. Thus

$$(x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star = (x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^2, \ldots, x_w^2, x_{w+1}^{s_{w+1}} \ldots, x_t^{s_t})$$

with $s_j = 1$ or 2 for j such that $w+1 \leq j \leq t$.

Consider

$$(((x_1 + \cdots + x_w + x_{w+1}^2 + \cdots + x_t^2)(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1}, \ldots, x_t))^\star$$

$$: x_1 + \cdots + x_w + x_{w+1}^2 + \cdots + x_t^2)$$

$$= ((x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^3, \ldots, x_t^3)^\star : x_1 + \cdots + x_w + x_{w+1}^2 + \cdots + x_t^2)$$

$$= ((x_1^{m+2}, \ldots, x_v^{m+2}, x_{v+1}^2, \ldots, x_w^2, x_{w+1}^{s_{w+1}} \ldots, x_t^{s_t}) : x_1 + \cdots + x_w + x_{w+1}^2 + \cdots + x_t^2)$$

$$= (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}, \ldots, x_t)$$

$$\Rightarrow (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1}, \ldots, x_t)^\star = (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}, \ldots, x_t)$$

which is a contradiction since $(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_w^2, x_{w+1}, \ldots, x_t)$ is $\star$ closed. This implies

$$(x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}^2, \ldots, x_t^2)^\star$$

$$= (x_1^{m+1}, \ldots, x_v^{m+1}, x_{v+1}, \ldots, x_t).$$

Thus

$$(x_1^m, \ldots, x_v^m, x_{v+1}^n, \ldots, x_t^n)^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t) \text{ for all } m, n \in \mathbb{N}.$$

Let $N = max\{n_{v+1}, \ldots, n_t\}$ and $n = min\{n_{v+1}, \ldots, n_t\}$, $n_j \in \mathbb{N}$. We have

$$(x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t) = (x_1^m, \ldots, x_v^m, x_{v+1}^N, \ldots, x_t^N)^\star$$

$$\subseteq (x_1^m, \ldots, x_v^m, x_{v+1}^{n_{v+1}}, \ldots, x_t^{n_t})^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t).$$

$$\Rightarrow (x_1^m, \ldots, x_v^m, x_{v+1}^{n_{v+1}}, \ldots, x_t^{n_t})^\star = (x_1^m, \ldots, x_v^m, x_{v+1}, \ldots, x_t) \text{ for all } n_j \in \mathbb{N} \text{ for all } m.$$

Let $M > max\{m_1, \ldots, m_v\}$, $m_j \in \mathbb{N}$. Consider

$$(((x_1^{M-m_1} + \cdots + x_v^{M-m_v} + x_{v+1} + \cdots + x_t)(x_1^{m_1}, \ldots, x_v^{m_v}, x_{v+1}^{n_{v+1}}, \ldots, x_t^{n_t}))^\star :$$

$$x_1^{M-m_1} + \cdots + x_v^{M-m_v} + x_{v+1} + \cdots + x_t)$$

$$= ((x_1^M, \ldots, x_v^M, x_{v+1}^{n_{v+1}+1}, \ldots, x_t^{n_t+1})^\star : x_1^{M-m_1} + \cdots + x_v^{M-m_v} + x_{v+1} + \cdots + x_t)$$

$$= ((x_1^M, \ldots, x_v^M, x_{v+1}, \ldots, x_t)^\star : x_1^{M-m_1} + \cdots + x_v^{M-m_v} + x_{v+1} + \cdots + x_t)$$

$$= (x_1^{m_1}, \ldots, x_v^{m_v}, x_{v+1}, \ldots, x_t)$$

$$\Rightarrow (x_1^{m_1}, \ldots, x_v^{m_v}, x_{v+1}^{n_{v+1}}, \ldots, x_t^{n_t})^\star = (x_1^{m_1}, \ldots, x_v^{m_v}, x_{v+1}, \ldots, x_t)$$

for all $m_j, n_l \in \mathbb{N}$. □

Lemma 3.2.12. *Suppose $\star$ is a standard closure operation on R_t. Further suppose $|k| > 2$ or that $t = 3$ and $k = \mathbb{Z}/2\mathbb{Z}$. For some $(m_1, \ldots, m_v) \in \mathbb{N}^v$ let $I = (x_{i_1}^{m_1}, \ldots, x_{i_v}^{m_v})$, $1 \leq v \leq t$. If $I^\star \neq R_t$ then for all $(r_1, \ldots, r_v) \in \mathbb{N}^v$, $(x_{i_1}^{r_1}, \ldots, x_{i_v}^{r_v})^\star = (x_{i_1}^{s_1}, \ldots, x_{i_w}^{s_w})$, for some w such that $v \leq w \leq t$ with $s_j = 1$ for all r_j or $s_j = r_j$ for all j such that $1 \leq j \leq v$ and $s_j = 1$ for all j such that $v + 1 \leq j \leq w$.*

Proof. Notice $(x_{i_1}^{m_1+1}, \ldots, x_{i_v}^{m_v+1})^\star \subseteq I^\star \neq R_t$. So if any $m_n = 1$ we replace I with $(x_{i_1}^{m_1+1}, \ldots, x_{i_v}^{m_v+1})$. Thus we can suppose for all n such that $1 \leq n \leq v$ that $m_n > 1$.

By Lemma 3.2.9 or Lemma 3.2.10 $I^\star$ is a proper monomial ideal. So $I^\star = (x_{i_1}^{s_1}, \ldots, x_{i_w}^{s_w})$, with $v \leq w \leq t$ and $1 \leq s_j \leq m_j$ for all j such that $1 \leq j \leq v$. So without loss of generality suppose $I = (x_1^{m_1}, \ldots, x_v^{m_v})$, for some v such that $1 \leq v \leq t$ and $I^\star = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}^{s_{u+1}}, \ldots, x_w^{s_w})$, for some u such that $0 \leq u \leq v \leq w \leq t$ with $s_j < m_j$ for all j such that $u + 1 \leq j \leq v$.

Let $J = (x_1^{m_1}, \ldots, x_w^{m_w})$, $m_j > s_j$ for all j such that $v + 1 \leq j \leq w$. Since $I \subset J \subset I^\star$ we have $J^\star = I^\star$. If $s_j > 1$ for all $u + 1 \leq j \leq w$, since $(I^\star)^\star = I^\star$, by Lemma 3.2.1 $(x_1^{r_1}, \ldots, x_w^{r_w})^\star = (x_1^{r_1}, \ldots, x_w^{r_w})$ for all $(r_1, \ldots, r_w) \in \mathbb{N}^w$. Thus

$J^\star = (x_1^{m_1}, \ldots, x_w^{m_w})$. Since $J^\star = I^\star$ either $u = v = w$ and the lemma is proven or we have a contradiction. So suppose $s_j = 1$ for at least one j such that $u+1 \leq j \leq w$.

Now without loss of generality suppose

$$J^\star = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}^{s_{u+1}}, \ldots, x_y^{s_y}, x_{y+1}, \ldots, x_w),$$

for some y such that $u+1 \leq y \leq w$ and $s_j > 1$ for all j such that $u+1 \leq j \leq y$. Let $H = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}^{s_{u+1}+1}, \ldots, x_y^{s_y+1}, x_{y+1}^2, \ldots, x_w^2)$. Since $J \subset H \subset J^\star$, $H^\star = J^\star$. Let $H_{-1} = (x_1^{m_1-1}, \ldots, x_u^{m_u-1}, x_{u+1}^{s_{u+1}}, \ldots, x_y^{s_y}, x_{y+1}, \ldots, x_w)$. Consider

$$(((x_1 + \cdots + x_t)H_{-1})^\star : x_1 + \cdots + x_t) = (H^\star : x_1 + \cdots + x_t) = (J^\star : x_1 + \cdots + x_t)$$

$$= (x_1^{m_1-1}, \ldots, x_u^{m_u-1}, x_{u+1}^{s_{u+1}-1}, \ldots, x_y^{s_y-1}, x_{y+1}, \ldots, x_w).$$

Since $\star$ is standard this implies

$$(H_{-1})^\star = (x_1^{m_1-1}, \ldots, x_u^{m_u-1}, x_{u+1}^{s_{u+1}-1}, \ldots, x_y^{s_y-1}, x_{y+1}, \ldots, x_w).$$

Consider

$$(((x_1 + \cdots + x_u + x_{u+1}^2 + \cdots + x_y^2 + x_{y+1} + \cdots + x_w)(H_{-1})^\star)^\star$$

$$: x_1 + \cdots + x_u + x_{u+1}^2 + \cdots + x_y^2 + x_{y+1} + \cdots + x_w)$$

$$= (H^\star : x_1 + \cdots + x_u + x_{u+1}^2 + \cdots + x_y^2 + x_{y+1} + \cdots + x_w)$$

$$= (x_1^{m_1-1}, \ldots, x_u^{m_u-1}, x_{u+1}^{\gamma_{u+1}}, \ldots, x_y^{\gamma_y}, x_{y+1}, \ldots, x_w)$$

where $\gamma_j = s_j - 2$ if $s_j \geq 3$ or $\gamma_j = 1$ for all j such that $u+1 \leq j \leq y$. Since $\star$ is standard $(H_{-1})^\star = ((H_{-1})^\star)^\star = (x_1^{m_1-1}, \ldots, x_u^{m_u-1}, x_{u+1}^{\gamma_{u+1}}, \ldots, x_y^{\gamma_y}, x_{y+1}, \ldots, x_w)$. So if $s_j \geq 3$ for any j such that $u+1 \leq j \leq y$ we have a contradiction. So $s_j = 2$ for all j such that $u+1 \leq j \leq y$. Thus

$$J^\star = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}^2, \ldots, x_y^2, x_{y+1}, \ldots, x_w)$$

and

$$H = (x_1^{m_1}, \dots, x_u^{m_u}, x_{u+1}^3, \dots, x_y^3, x_{y+1}^2, \dots, x_w^2).$$

Let $K = (x_1^{m_1+1}, \dots, x_u^{m_u+1}, x_{u+1}^3, \dots, x_y^3, x_{y+1}^2, \dots, x_w^2)$. Since $K^\star \subseteq J^\star$ we have $K^\star = (x_1^{n_1}, \dots, x_u^{n_u}, x_{u+1}^{\alpha_{u+1}}, \dots, x_y^{\alpha_y}, x_{y+1}^{\beta_{y+1}}, \dots, x_w^{\beta_w})$ with $m_j \leq n_j \leq m_j + 1$ for all j such that $1 \leq j \leq u$, $2 \leq \alpha_j \leq 3$ for all j such that $u+1 \leq j \leq y$ and $1 \leq \beta_j \leq 2$ for all j such that $y+1 \leq j \leq w$. Consider

$$(((x_1 + \dots + x_t)J^\star)^\star : x_1 + \dots + x_t) = (K^\star : x_1 + \dots + x_t)$$

$$= (x_1^{n_1-1}, \dots, x_u^{n_u-1}, x_{u+1}^{\alpha_{u+1}-1}, \dots, x_y^{\alpha_y-1}, x_{y+1}, \dots, x_w) = (J^\star)^\star = J^\star.$$

Since $\star$ is standard. This implies $n_j = m_j + 1$ for all j such that $1 \leq j \leq u$ and $\alpha_j = 3$ for all j such that $u+1 \leq j \leq w$. Thus

$$K^\star = (x_1^{m_1+1}, \dots, x_u^{m_u+1}, x_{u+1}^3, \dots, x_y^3, x_{y+1}^{\beta_{y+1}}, \dots, x_w^{\beta_w}).$$

Let $L = (x_1^{m_1+2}, \dots, x_u^{m_u+2}, x_{u+1}^4, \dots, x_y^4, x_{y+1}^3, \dots, x_w^3)$. Since $L^\star \subseteq K^\star$ we have $L^\star = (x_1^{p_1}, \dots, x_u^{p_u}, x_{u+1}^{\nu_{u+1}}, \dots, x_y^{\nu_y}, x_{y+1}^{\rho_{y+1}}, \dots, x_w^{\rho_w})$ with $m_j + 1 \leq p_j \leq m_j + 2$ for all j such that $1 \leq j \leq u$, $3 \leq \nu_j \leq 4$ for all j such that $u+1 \leq j \leq y$ and $\beta_j \leq \rho_j \leq \beta_j + 1$ for all j such that $y+1 \leq j \leq w$. Consider

$$(((x_1 + \dots + x_t)K)^\star : x_1 + \dots + x_t) = (L^\star : x_1 + \dots + x_t)$$

$$= (x_1^{p_1-1}, \dots, x_u^{p_u-1}, x_{u+1}^{\nu_{u+1}-1}, \dots, x_y^{\nu_y-1}, x_{y+1}^{\phi_{y+1}}, \dots, x_w^{\phi_w}) = K^\star$$

since $\star$ is standard (with $1 \leq \phi_j \leq 2$ for all j such that $y+1 \leq j \leq w$). Thus $p_j = m_j + 2$ for all j such that $1 \leq j \leq u$ and $\nu_j = 4$ for all j such that $u+1 \leq j \leq y$. So $L^\star = (x_1^{m_1+2}, \dots, x_u^{m_u+2}, x_{u+1}^4, \dots, x_y^4, x_{y+1}^{\rho_{y+1}}, \dots, x_w^{\rho_w})$. Consider

$$(((x_1^2 + \dots + x_t^2 + x_{u+1} + \dots + x_t)H)^\star : x_1^2 + \dots + x_t^2 + x_{u+1} + \dots + x_t)$$

$$= (L^\star : x_1^2 + \dots + x_t^2 + x_{u+1} + \dots + x_t)$$

$$= (x_1^{m_1}, \dots, x_u^{m_u}, x_{u+1}^3, \dots, x_y^3, x_{y+1}^{\eta_{y+1}}, \dots, x_w^{\eta_w})$$

with $1 \leq \eta_j \leq 3$ for all j such that $y+1 \leq j \leq w$. Since $\star$ is standard this implies $H^\star = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}^3, \ldots, x_y^3, x_{y+1}^{\eta_{y+1}}, \ldots, x_w^{\eta_w})$ with $\eta_j = 1$ or 2 for all j such that $y+1 \leq j \leq w$.. This is a contradiction since $H^\star = J^\star$. So it must be $s_j = 1$ for all $u+1 \leq j \leq w$.

Thus $I^\star = (x_1^{m_1}, \ldots, x_u^{m_u}, x_{u+1}, \ldots, x_w)$.
Now we must show $(x_1^{r_1}, \ldots, x_v^{r_v})^\star = (x_1^{r_1}, \ldots, x_u^{r_u}, x_{u+1}, \ldots, x_w)$ for all $(r_1, \ldots, r_v) \in \mathbb{N}^v$.
Since $\star$ is weakly prime we have

$$(x_1 + \cdots + x_t)(x_1^{m_1}, \ldots, x_v^{m_v})^\star \subseteq ((x_1 + \cdots + x_t)(x_1^{m_1}, \ldots, x_v^{m_v}))^\star$$

$$\Rightarrow (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}^2 \ldots, x_w^2) \subseteq (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star$$

$$\Rightarrow (x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star = (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}^2 \ldots, x_w^2)^\star$$

Let $J_{+1} = (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}^2 \ldots, x_w^2)$. Since $J_{+1}^\star \subseteq I^\star$,
$J_{+1}^\star = (x_1^{q_1}, \ldots, x_u^{q_u}, x_{u+1}^{\zeta_{u+1}}, \ldots, x_w^{\zeta_w})$ with $m_j \leq q_j \leq m_j+1$ for all j such that $1 \leq j \leq u$ and $1 \leq \zeta_j \leq 2$ for all j such that $u+1 \leq j \leq w$. Now we have

$$(((x_1 + \cdots + x_t)I)^\star : x_1 + \cdots + x_t) = ((x_1^{m_1+1}, \ldots, x_v^{m_v+1})^\star : x_1 + \cdots + x_t)$$

$$= ((x_1^{q_1}, \ldots, x_u^{q_u}, x_{u+1}^{\zeta_{u+1}}, \ldots, x_w^{\zeta_w}) : x_1 + \cdots + x_t) = (x_1^{q_1-1}, \ldots, x_u^{q_u-1}, x_{u+1} \ldots, x_w) = I^\star$$

since $\star$ is standard.
This implies $q_j = m_j + 1$ for all j such that $1 \leq j \leq u$. Thus
$J_{+1}^\star = (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}^{\zeta_{u+1}}, \ldots, x_w^{\zeta_w})$. Now suppose $\zeta_j = 2$ for all j such that $u+1 \leq j \leq w$. Then by Lemma 3.2.1 $(x_1^{r_1}, \ldots, x_w^{r_w})^\star = (x_1^{r_1}, \ldots, x_w^{r_w})$ for all $(r_1, \ldots, r_w) \in \mathbb{N}^w$. However $J^\star$ contradicts this. So $\zeta_j = 1$ for some j such that $u+1 \leq j \leq w$. So without loss of generality suppose
$J_{+1}^\star = (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}^2, \ldots, x_y^2, x_{y+1}, \ldots, x_w)$. Now with J_{+1} in the role of J from the previous argument it must be the case that
$J_{+1}^\star = (x_1^{m_1+1}, \ldots, x_u^{m_u+1}, x_{u+1}, \ldots, x_w)$. Inductively we have $(x_1^{m_1+n}, \ldots, x_v^{m_v+n})^\star =$

$(x_1^{m_1+n}, \ldots, x_u^{m_u+n}, x_{u+1} \ldots, x_w)$ for any $n \in \mathbb{N}$.

Consider $(x_1^{r_1}, \ldots, x_v^{r_v})$ where $(r_1, \ldots, r_v) \in \mathbb{N}^v$. Let $N = max\{r_1, \ldots, r_v\}$. Consider

$$((x_1^{m_1+N-r_1} + \cdots + x_1^{m_v+N-r_v} + x_{v+1} + \cdots + x_t)(x_1^{r_1}, \ldots, x_v^{r_v}))^\star$$

$$: x_1^{m_1+N-r_1} + \cdots + x_1^{m_v+N-r_v} + x_{v+1} + \cdots + x_t)$$

$$= ((x_1^{m_1+N}, \ldots, x_v^{m_v+N})^\star : x_1^{m_1+N-r_1} + \cdots + x_1^{m_v+N-r_v} + x_{v+1} + \cdots + x_t)$$

$$= ((x_1^{m_1+N}, \ldots, x_u^{m_u+N}, x_{u+1} \ldots, x_w) : x_1^{m_1+N-r_1} + \cdots + x_1^{m_v+N-r_v} + x_{v+1} + \cdots + x_t)$$

$$= (x_1^{r_1}, \ldots, x_u^{r_u}, x_{u+1}, \ldots, x_w).$$

Since $\star$ is standard this completes the proof.

□

3.3 Counting Standard Closures on R_2

The author and J. Vassilev showed that the ring $R_2 = k[[x, y]]/(xy)$ has 24 standard closure operations [MV]. I will include a slightly diiferent proof of this fact here which utililzes several of the lemmas and propositions from the previous sections. The following claim is applicable for both the R_2 and R_3 cases.

Claim 3.3.1. *Suppose c is a closure on R_2 or R_3 with $i_j \in \{1, 2\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$.*

1. *If $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^c = (x_{i_1}^{r_{i_1}}, x_{i_2})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ then $(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^c = (x_{i_1}^{r_{i_1}}, x_{i_2})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$.*
2. *If $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^c = (x_{i_1}, x_{i_2})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ then $(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^c = (x_{i_1}, x_{i_2})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$.*

Suppose c is a closure on R_3 with $i_j \in \{1, 2, 3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$.

3. *If* $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^c = (x_{i_1}^{r_{i_1}}, x_{i_2}, x_{i_3})$ *for all* $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ *then* $(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^c = (x_{i_1}^{r_{i_1}}, x_{i_2}, x_{i_3})$ *for all* $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ *and* $a \in k^\times$.

Proof. (1) Without loss of generality let $x_{i_1} = x$, $x_{i_2} = y$, $r_{i_1} = m$ and $r_{i_2} = n$. So we assume $(x^m, y^n)^c = (x^m, y)$. Let $a \in k^\times$. We have the following inclusion.

$$(x^{m+1}, y) = (x^{m+1}, y^{n+1})^c \subseteq (x^m + ay^n)^c \subseteq (x^m, y^n)^c = (x^m, y)$$

This implies $(x^m + ay^n)^c = (x^m, y)$.

The proofs of (2) and (3) are similar to (1). □

Theorem 3.3.2. *There are 24 standard closure operations on* R_2.

1. $\star_1$:$I^{\star_1} = R_2$ $\forall$ *ideals* I
2. $\star_{1.1}$:$(0)^{\star_{1.1}} = (x^m)^{\star_{1.1}} = (x)$ $\forall$ $m \in \mathbb{N}$, $I^{\star_{1.1}} = R_2$ $\forall$ *other ideals* I
3. $\star_{1.2}$:$(0)^{\star_{1.2}} = (y^n)^{\star_{1.2}} = (y)$ $\forall$ $n \in \mathbb{N}$, $I^{\star_{1.2}} = R_2$ $\forall$ *other ideals* I
4. $\star_{1.3}$:$(0)^{\star_{1.3}} = (0)$, $I^{\star_{1.3}} = R_2$ $\forall$ *other ideals* I
5. $\star_{1.4}$:$(0)^{\star_{1.4}} = (0)$, $(y^n)^{\star_{1.4}} = (y)$ $\forall$ $n \in \mathbb{N}$, $I^{\star_{1.4}} = R_2$ $\forall$ *other ideals* I
6. $\star_{1.5}$:$(0)^{\star_{1.5}} = (0)$, $(y^n)^{\star_{1.5}} = (y^n)$ $\forall$ $n \in \mathbb{N}$, $I^{\star_{1.5}} = R_2$ $\forall$ *other ideals* I
7. $\star_{1.6}$:$(0)^{\star_{1.6}} = (0)$, $(x^m)^{\star_{1.6}} = (x)$ $\forall$ $m \in \mathbb{N}$, $I^{\star_{1.6}} = R_2$ $\forall$ *other ideals* I
8. $\star_{1.7}$:$(0)^{\star_{1.7}} = (0)$, $(x^m)^{\star_{1.7}} = (x)$, $(y^n)^{\star_{1.7}} = (y)$ $\forall$ $m, n \in \mathbb{N}$, $I^{\star_{1.7}} = R_2$ $\forall$ *other ideals* I
9. $\star_{1.8}$:$(0)^{\star_{1.8}} = (0)$, $(x^m)^{\star_{1.8}} = (x)$, $(y^n)^{\star_{1.8}} = (y^n)$ $\forall$ $m, n \in \mathbb{N}$, $I^{\star_{1.8}} = R_2$ $\forall$ *other ideals* I
10. $\star_{1.9}$:$(0)^{\star_{1.9}} = (0)$, $(x^m)^{\star_{1.9}} = (x^m)$ $\forall$ $m \in \mathbb{N}$, $I^{\star_{1.9}} = R_2$ $\forall$ *other ideals* I
11. $\star_{1.10}$:$(0)^{\star_{1.10}} = (0)$, $(x^m)^{\star_{1.10}} = (x^m)$, $(y^n)^{\star_{1.10}} = (y)$ $\forall$ $m, n \in \mathbb{N}$, $I^{\star_{1.10}} = R_2$ $\forall$ *other ideals* I
12. $\star_{1.11}$:$(0)^{\star_{1.11}} = (0)$, $(x^m)^{\star_{1.11}} = (x^m)$, $(y^n)^{\star_{1.11}} = (y^n)$ $\forall$ $m, n \in \mathbb{N}$, $I^{\star_{1.11}} = R_2$ $\forall$ *other ideals* I

13. $\star_2$:$(0)^{\star_2} = (x^m)^{\star_2} = (x)$, $(y^n)^{\star_2} = (x^m + ay^n)^{\star_2} = (x^m, y^n)^{\star_2} = (x, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
14. $\star_{2.1}$:$(0)^{\star_{2.1}} = (0)$, $(x^m)^{\star_{2.1}} = (x)$, $(y^n)^{\star_{2.1}} = (x^m + ay^n)^{\star_{2.1}} = (x^m, y^n)^{\star_{2.1}} = (x, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
15. $\star_{2.2}$:$(0)^{\star_{2.2}} = (0)$, $(x^m)^{\star_{2.2}} = (x)$, $(y^n)^{\star_{2.2}} = (y^n)$, $(x^m + ay^n)^{\star_{2.2}} = (x^m, y^n)^{\star_{2.2}} = (x, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
16. $\star_{2.3}$:$(0)^{\star_{2.3}} = (0)$, $(x^m)^{\star_{2.3}} = (x^m)$, $(y^n)^{\star_{2.3}} = (x^m + ay^n)^{\star_{2.3}} = (x^m, y^n)^{\star_{2.3}} = (x, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
17. $\star_{2.4}$:$(0)^{\star_{2.4}} = (0)$, $(x^m)^{\star_{2.4}} = (x^m)$, $(y^n)^{\star_{2.4}} = (y^n)$, $(x^m + ay^n)^{\star_{2.4}} = (x^m, y^n)^{\star_{2.4}} = (x, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
18. $\star_{2.5}$:$(0)^{\star_{2.5}} = (y^n)^{\star_{2.5}} = (y)$, $(x^m)^{\star_{2.5}} = (x^m + ay^n)^{\star_{2.5}} = (x^m, y^n)^{\star_{2.5}} = (x^m, y)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
19. $\star_{2.6}$:$(0)^{\star_{2.6}} = (0)$, $(x^m)^{\star_{2.6}} = (x^m + ay^n)^{\star_{2.6}} = (x^m, y^n)^{\star_{2.6}} = (x^m, y)$, $(y^n)^{\star_{2.6}} = (y)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
20. $\star_{2.7}$:$(0)^{\star_{2.7}} = (0)$, $(x^m)^{\star_{2.7}} = (x^m)$, $(y^n)^{\star_{2.7}} = (y)$, $(x^m + ay^n)^{\star_{2.7}} = (x^m, y^n)^{\star_{2.7}} = (x^m, y)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
21. $\star_{2.8}$:$(0)^{\star_{2.8}} = (0)$, $(x^m)^{\star_{2.8}} = (x^m + ay^n)^{\star_{2.8}} = (x^m, y^n)^{\star_{2.8}} = (x^m, y)$, $(y^n)^{\star_{2.8}} = (y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
22. $\star_{2.9}$:$(0)^{\star_{2.9}} = (0)$, $(x^m)^{\star_{2.9}} = (x^m)$, $(y^n)^{\star_{2.9}} = (y^n)$, $(x^m + ay^n)^{\star_{2.9}} = (x^m, y^n)^{\star_{2.9}} = (x^m, y)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
23. $\star_{2.10}$:$(0)^{\star_{2.10}} = (0)$, $(x^m)^{\star_{2.10}} = (x^m)$, $(y^n)^{\star_{2.10}} = (y^n)$, $(x^m + ay^n)^{\star_{2.10}} = (x^m, y^n)^{\star_{2.10}} = (x^m, y^n)$ $\forall$ $m, n \in \mathbb{N}$ *and* $a \in k^\times$
24. $\star_{2.11}$:$I^{\star_{2.11}} = I$ $\forall$ *ideals* I.

In the proof of the theorem (and for the proof of Theorem 3.4.4) there are many cases, subcases, sub-subcases and so forth. To keep track of all of these cases we use the following convention: The number denotes the case, the first letter denotes the subcase, the second letter denotes the sub-subcase etc. For example (1dabb) is the second sub-sub-sub-subcase of the second sub-sub-subcase of the first sub-subcase of the fourth subcase of the first case.

Proof. Suppose $\star$ is a standard closure operation on R_2. By Proposition 3.1.14 each ideal of R_2 has a unique reduced generating set. Thus each proper ideal of R_2 expressed in terms of its reduced generating set will have one of the following forms: (0), (x^m), (y^n), $(x^m + ay^n)$, and (x^m, y^n), m, $n \in \mathbb{N}$ and $a \in k^{\times}$.

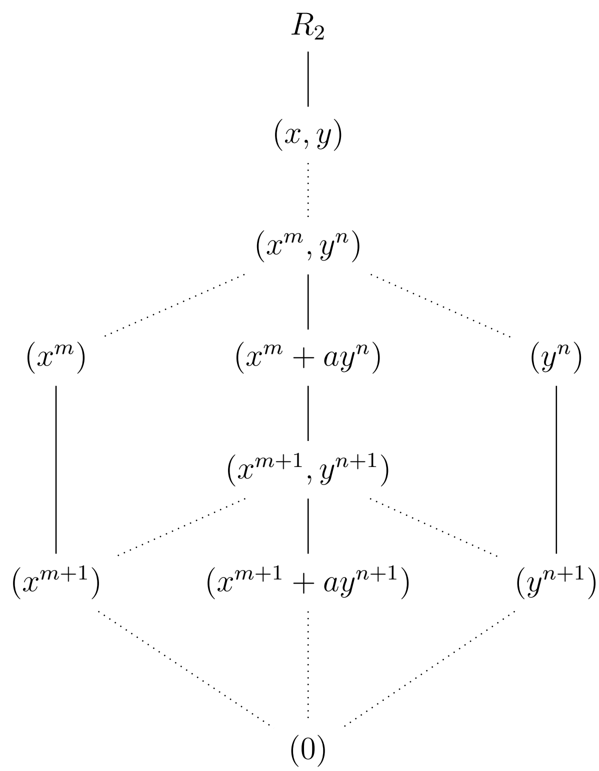

Figure 3.1: Ideal lattice of R_2.
A dash indicates inclusion. Also a solid dash between an ideal I and J, with I below J, indicates $J/I \cong k$.

Every non unit regular element in R has the form $x^n + ay^m$, $m, n \in \mathbb{N}$ and $a \in k^{\times}$. By computing $(((x^n + ay^m)I)^{\star} : x^n + ay^m)$ for $I = (0), I = (x^n), I = (y^n), I =$

$(x^m + ay^n)$, and $I = (x^m, y^n)$ one can verify $(((x^n + ay^m)I)^\star : x^n + ay^m) = I^\star$ to show each closure from the above list is in fact standard.

Now we will show that these are in fact all the standard closures on R. Since (x, y) is the maximal ideal either $(x, y)^\star = R_2$ or (x, y) is $\star$ closed.

(1) Suppose $(x, y)^\star = R_2$. By Lemma 3.2.3 $(x^m + ay^n)^\star = (x^m, y^n)^\star = R_2$. Since $(x^m + ay^n)$ and (x^m, y^n) are not $\star$ closed there are only four remaining possibilities for $(0)^\star$:$(0)^\star = R_2$, $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$, $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$ or $(0)^\star = (0)$.

(1a) Suppose $(0)^\star = R_2$. Since $R_2 = (0)^\star \subseteq I$ for all ideals I we have $I^\star = R_2$ for all I. This is $\star_1$.

(1b) Suppose $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. Since $(x) = (0)^\star \subseteq (y^n)^\star$ we have $R_2 = (x, y^n)^\star \subseteq (y^n)^\star$. Thus $(y^n)^\star = R_2$ for all $n \in \mathbb{N}$. The inclusion $(x) = (0)^\star \subseteq (x^m)^\star \subseteq (x)$ implies $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. This is $\star_{1.1}$.

(1c) Suppose $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$. Similar to case (1b) we get $\star_{1.2}$.

(1d) Suppose $(0)^\star = (0)$. If $(x^i)^\star$ contains a power of y for some $i \in \mathbb{N}$ then by Lemma 3.2.3 $(x^i)^\star = R_2$. So there are only two possibilities for $(x^i)^\star$:$(x^i)^\star = R_2$ for some $i \in \mathbb{N}$ or $(x^i)^\star = (x^j)$ for some $i, j \in \mathbb{N}$.

(1da) Suppose $(x^i)^\star = R_2$ for some $i \in \mathbb{N}$ then by Lemma 3.2.3 $(x^m)^\star = R_2$ for all $m \in \mathbb{N}$. If $(y^j)^\star$ contains a power of y for some $j \in \mathbb{N}$ then by Lemma 3.2.3 $(y^j)^\star = R_2$. So there are only two possibilities for $(y^j)^\star$:$(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ or $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$.

(1daa) Suppose $(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ then by Lemma 3.2.3 $(y^n)^\star = R_2$ for all $n \in \mathbb{N}$.This is $\star_{1.3}$.

(1dab) Suppose $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$. By Lemma 3.2.12 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(1daba) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. This is $\star_{1.4}$.

(1dabb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{1.5}$.

(1db) Suppose $(x^i)^\star = (x^j)$ for some $i, j \in \mathbb{N}$. By Lemma 3.2.12 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(1dba) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Similar to case (1da) $(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ or $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$.

(1dbaa) Suppose $(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ then by Lemma 3.2.3 $(y^n)^\star = R_2$ for all $n \in \mathbb{N}$. This is $\star_{1.6}$.

(1dbab) Suppose $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$. By Lemma 3.2.12 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(1dbaba) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. This is $\star_{1.7}$.

(1dbabb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{1.8}$.

(1dbb) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similar to case (1da) $(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ or $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$.

(1dbba) Suppose $(y^j)^\star = R_2$ for some $j \in \mathbb{N}$ then by Lemma 3.2.3 $(y^n)^\star = R_2$ for all $n \in \mathbb{N}$. This is $\star_{1.9}$.

(1dbbb) Suppose $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$. By Lemma 3.2.12 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(1dbbba) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. This is $\star_{1.10}$.

(1dbbbb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{1.11}$.

(2) Suppose $(x, y)^\star = (x, y)$. By Lemma 3.2.11 $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in$

$\mathbb{N}$, $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

(2a) Suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Since $(x^m)^\star \subseteq \cap_{n=1}^\infty (x^m, y^n)^\star = \cap_{n=1}^\infty (x, y^n) = (x)$ by Lemma 3.2.12 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(2aa) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Since $(y^2)^\star \subsetneq (x, y^2)^\star = (x, y^2)$ by Lemma 3.2.12 $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(2aaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Since $(0)^\star \subseteq (x)$, $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$ or $(0)^\star = (0)$.

(2aaaa) Suppose $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. This is $\star_2$.

(2aaab) Suppose $(0)^\star = (0)$. This is $\star_{2.1}$.

(2aab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{2.2}$.

(2ab) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (2aa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(2aba) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. This is $\star_{2.3}$.

(2abb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{2.4}$.

(2b) Suppose $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$. This case is similar to case (2b) with the roles of x and y reversed. We get $\star_{2.5}$ through $\star_{2.9}$.

(2c) Suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. The inclusion $(x^m)^\star \subseteq \cap_{n=1}^\infty (x^m, y^n)^\star = \cap_{n=1}^\infty (x^m, y^n) = (x^m)$ implies $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similarly $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Since for each $i, j \in \mathbb{N}$ and $b \in k^\times$, $(x^i + by^j)^\star \subseteq (x^i, y^j)^\star = (x^i, y^j)$, for each $i, j \in \mathbb{N}$ and $b \in k^\times$, $(x^i + by^j)^\star = (x^i, y^j)$ or $(x^i + by^j)^\star = (x^i + by^j)$.

(2ca) Suppose $(x^i + by^j)^\star = (x^i, y^j)$ for some $i, j \in \mathbb{N}$ and $b \in k^\times$. Let $a \in k^\times$ and $v \in \mathbb{N}$. Since $\star$ is weakly prime

$$(x^v + \frac{a}{b}y^v)(x^i + by^j)^\star \subseteq ((x^v + \frac{a}{b}y^v)(x^i + by^j))^\star \Rightarrow (x^{i+v}, y^{j+v}) \subseteq (x^{i+v} + by^{j+v})^\star$$

$$\Rightarrow (x^{i+v} + by^{j+v})^\star = (x^{i+v}, y^{j+v})^\star = (x^{i+v}, y^{j+v}).$$

Now let $m, n \in \mathbb{N}$ and $M = \max\{m, n\}$. Consider

$$(((x^{i+M-m} + y^{j+M-n})(x^m + ay^n))^\star : x^{i+M-m} + y^{j+M-n})$$

$$= ((x^{i+M} + ay^{j+M})^\star : x^{i+M-m} + y^{j+M-n})$$

$$= ((x^{i+M}, y^{j+M}) : x^{i+M-m} + y^{j+M-n}) = (x^m, y^n).$$

Since $\star$ is standard $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{2.10}$.

(2cb) Suppose $(x^i + by^j)^\star = (x^i + by^j)$ for some $i, j \in \mathbb{N}$ and $b \in k^\times$. Now if for some $r, s \in \mathbb{N}$ and $c \in k^\times$, $(x^r + cy^s)^\star = (x^r, y^s)$ then case (2ca) implies $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ which is a contradiction. Thus $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{2.11}$ the identity.

□

Corollary 3.3.3.

$$|S_f(R_2)| = 24$$

Proof. By Theorem 3.3.2 and Lemma 1.2.8. □

3.4 Counting Standard Closures on R_3

The ring R_2 is isomorphic to the rings $k[[X, Y]]/(XY)$, $k[[X, Z]]/(XZ)$ and $k[[Y, Z]]/(YZ)$. By Lemma 3.2.4 each of the standard closure operations on these rings induce a standard closure operation on R_3. This results in 62 distinct standard closure operations on R_3. However, we will see that the total number of standard closure operations on R_3 is far higher than this.

In order to classify and count the standard closures on R_3 we need to better understand the closures of non-monomial ideals. The following lemma shows, similar to the monomial case, that the closure of a single non-monomial ideal induces closures on other non-monomial ideals.

Lemma 3.4.1. *Suppose $\star$ is a standard closure operation on R_3 with $i_j \in \{1,2,3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$.*

1. *If $(x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}})^\star = (x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}})$ for some $(m_{i_1}, m_{i_2}) \in \mathbb{N}^2$ and $b \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$.*
2. *If $(x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}})^\star = (x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}})$ for some $(m_{i_1}, m_{i_2}) \in \mathbb{N}^2$, $(n_{i_1}, n_{i_2}) \in \mathbb{N}^2$ and $b \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{s_{i_1}}, x_{i_2}^{s_{i_2}})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$ such that for each $j \in \{1,2\}$, $s_{i_j} = 1$ for all r_{i_j} or $s_{i_j} = r_{i_j}$ for all r_{i_j}.*
3. *If $(x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}})^\star = (x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}})$ for some $(m_{i_1}, m_{i_2}, m_{i_3}) \in \mathbb{N}^3$ and $b \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$.*
4. *If $(x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}})^\star = (x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}})$ for some $(m_{i_1}, m_{i_2}) \in \mathbb{N}^2$, $(n_{i_1}, n_{i_2}, n_{i_3}) \in \mathbb{N}^3$ and $b \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{s_{i_1}}, x_{i_2}^{s_{i_2}}, x_{i_3})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$ such that for some $j \in \{1,2\}$, $s_{i_j} = r_{i_j}$ for all r_{i_j} and for all other j, $s_{i_j} = 1$ for all r_{i_j}.*
5. *If $(x_{i_1}^{m_{i_1}} + cx_{i_2}^{m_{i_2}} + dx_{i_3}^{m_{i_3}})^\star = (x_{i_1}^{m_{i_1}} + cx_{i_2}^{m_{i_2}} + dx_{i_3}^{m_{i_3}})$ for some $(m_{i_1}, m_{i_2}, m_{i_3}) \in \mathbb{N}^3$ and $c, d \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}} + bx_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}} + bx_{i_3}^{r_{i_3}})$ for all $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^3$ and $a, b \in k^\times$.*
6. *If $(x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}}, x_{i_3}^{m_{i_3}})^\star = (x_{i_1}^{m_{i_1}} + bx_{i_2}^{m_{i_2}}, x_{i_3}^{n_{i_3}})$ for some $(m_{i_1}, m_{i_2}, m_{i_3}) \in \mathbb{N}^3$, $n_{i_3} \in \mathbb{N}$ and $b \in k^\times$ then $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3})$ for all $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^2$ and $a \in k^\times$ or $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3}^{r_{i_3}})$ for all $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^2$ and $a \in k^\times$.*

7. *If* $(x_{i_1}^{m_{i_1}} + cx_{i_2}^{m_{i_2}} + dx_{i_3}^{m_{i_3}})^\star = (x_{i_1}^{m_{i_1}} + cx_{i_2}^{m_{i_2}}, x_{i_3}^{n_{i_3}})$ *for some* $(m_{i_1}, m_{i_2}, m_{i_3}) \in \mathbb{N}^3$, $n_{i_3} \in \mathbb{N}$ *and* $c, d \in k^\times$ *then* $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}} + bx_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}}, x_{i_3}^{s_{i_3}})$ *for all* $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^3$ *and* $a, b \in k^\times$ *such that* $s_{i_3} = 1$ *for all* r_{i_3} *or* $s_{i_3} = r_{i_3}$.
8. *If* $(x_{i_1}^{m_{i_1}} + cx_{i_2}^{m_{i_2}} + dx_{i_3}^{m_{i_3}})^\star = (x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}})$ *for some* $(m_{i_1}, m_{i_2}, m_{i_3}) \in \mathbb{N}^3$, $(n_{i_1}, n_{i_2}, n_{i_3}) \in \mathbb{N}^3$ *and* $c, d \in k^\times$ *then* $(x_{i_1}^{r_{i_1}} + ax_{i_2}^{r_{i_2}} + bx_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{s_{i_1}}, x_{i_2}^{s_{i_2}}, x_{i_3}^{s_{i_3}})$ *for all* $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^3$ *and* $a, b \in k^\times$ *such that for some* $j \in \{1, 2, 3\}$, $s_{i_j} = r_{i_j}$ *for all* r_{i_j} *and for all other* j, $s_{i_j} = 1$ *for all* r_{i_j}.

Proof. Without loss of generality let $x_{i_1} = x$, $x_{i_2} = y$ and $x_{i_3} = z$.

(1) $(x^{m_1} + by^{m_2})^\star = (x^{m_1} + by^{m_2})$ for some $m_1, m_2 \in \mathbb{N}$ and $b \in k^\times$. Let $a \in k^\times$. So $(x^{m_1+1} + ay^{m_2+1})^\star \subseteq (x^{m_1} + by^{m_2})^\star = (x^{m_1} + by^{m_2})$. Thus $(x^{m_1+1} + ay^{m_2+1})^\star$ is $(x^{m_1+1} + ay^{m_2+1})$, (x^{m_1+1}, y^{m_2+1}), or $(x^{m_1} + by^{m_2})$. If we compute

$$(((x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2}))^\star : x + \frac{a}{b}y + z) = ((x^{m_1+1} + ay^{m_2+1})^\star : x + \frac{a}{b}y + z)$$

the latter two possibilities result in a contradiction since $\star$ is standard. Thus $(x^{m_1+1} + ay^{m_2+1})^\star = (x^{m_1+1} + ay^{m_2+1})$. Inductively we have $(x^{m_1+v} + ay^{m_2+v})^\star = (x^{m_1+v} + ay^{m_2+v})$ for all $v \in \mathbb{N}$. Consider $(x^m + ay^n)$, $m, n \in \mathbb{N}$. Let $M = max\{m, n\}$. We have

$$(((x^{m_1+M-m} + y^{m_2+M-n} + z)(x^m + ay^n))^\star : x^{m_1+M-m} + y^{m_2+M-n} + z)$$
$$= ((x^{m_1+M} + ay^{m_2+M})^\star : x^{m_1+M-m} + y^{m_2+M-n} + z)$$
$$= ((x^{m_1+M} + ay^{m_2+M}) : x^{m_1+M-m} + y^{m_2+M-n} + z)$$
$$= (x^m + ay^n)$$

Since $\star$ is standard $(x^m + ay^n)^\star = (x^m + ay^n)$. Hence $(x^m + ay^n)^\star = (x^m + ay^n)$ for all m, n and a.

(2) Suppose $(x^{m_1} + by^{m_2})^\star = (x^{n_1}, y^{n_2})$ for some $m_1, m_2, n_1, n_2 \in \mathbb{N}$ and $b \in k^\times$. The inclusion $(x^{n_1}, y^{n_2}) = (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star \subseteq (x^{n_1}, y^{n_2})$ implies

$(x^{m_1}, y^{m_2})^\star = (x^{n_1}, y^{n_2})$. By Lemma 3.2.12 $(x^m, y^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

Case: $(x^m, y^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$. This implies $(x^{m_1} + by^{m_2})^\star = (x, y)$. Let $a \in k^\times$. Since $\star$ is weakly prime

$$(x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2})^\star \subseteq ((x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2}))^\star$$

$$\Rightarrow (x^2, y^2) \subseteq (x^{m_1+1} + ay^{m_2+1})^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1})^\star = (x^2, y^2)^\star = (x, y).$$

By induction $(x^{m_1+v} + ay^{m_2+v})^\star = (x, y)$ for all $v \in \mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence $(x^m + ay^n)^\star = (x, y)$ for all m, n and a.

The cases $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ are similar.

(3) Suppose $(x^{m_1} + by^{m_2})^\star = (x^{m_1} + by^{m_2}, z^{m_3})$ for some $m_1, m_2, m_3 \in \mathbb{N}$ and $b \in k^\times$. If $(x, y)^\star = R_3$ then by Lemma 3.2.3 $(x^{m_1} + by^{m_2})^\star = R_3$ which is a contradiction so $(x, y)^\star \neq R_3$. This implies $(x^m, y^n)^\star \neq R_3$ for all $m, n \in \mathbb{N}$. By Lemma 3.2.12 $(x^m, y^n)^\star$ is either (x^m, y^n) for all $m, n \in \mathbb{N}$, (x^m, y) for all $m, n \in \mathbb{N}$, (x, y^n) for all $m, n \in \mathbb{N}$, (x, y) for all $m, n \in \mathbb{N}$, (x^m, y^n, z) for all $m, n \in \mathbb{N}$, (x^m, y, z) for all $m, n \in \mathbb{N}$, or (x, y^n, z) for all $m, n \in \mathbb{N}$. Suppose for all $m, n \in \mathbb{N}$ $(x^m, y^n)^\star$ does not contain z. This implies $(x^{m_1}, y^{m_2})^\star$ does not contain z^{m_3}.
So $(x^{m_1} + by^{m_2}, z^{m_3}) = (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star$ implies $z^{m_3} \in (x^{m_1}, y^{m_2})^\star$ a contradiction. Hence $(x^m, y^n)^\star$ is either (x^m, y^n, z) for all $m, n \in \mathbb{N}$, (x^m, y, z) for all $m, n \in \mathbb{N}$, or (x, y^n, z) for all $m, n \in \mathbb{N}$.

Case: $(x^m, y^n)^\star = (x^m, y^n, z)$ for all $m, n \in \mathbb{N}$. The inclusion $(x^{m_1+1}, y^{m_2+1}, z) = (x^{m_1+1}, y^{m_2+1})^\star \subseteq (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star = (x^{m_1}, y^{m_2}, z) \Rightarrow (x^{m_1} + by^{m_2})^\star =$

$(x^{m_1} + by^{m_2}, z)$. Since $\star$ is weakly prime

$$(x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2})^\star \subseteq ((x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2}))^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1}, z^2) \subseteq (x^{m_1+1} + ay^{m_2+1})^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z^2)^\star \subseteq (x^{m_1} + by^{m_2}, z).$$

This implies $(x^{m_1+1} + ay^{m_2+1})^\star$ is either $(x^{m_1+1} + ay^{m_2+1})$, (x^{m_1+1}, y^{m_2+1}), $(x^{m_1} + by^{m_2})$, $(x^{m_1+1} + ay^{m_2+1}, z^l)$ or $(x^{m_1+1}, y^{m_2+1}, z^l)$ with $l = 1$ or $l = 2$. Consider

$$(((x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2}))^\star : x + \frac{a}{b}y + z) = ((x^{m_1+1} + ay^{m_2+1})^\star : x + \frac{a}{b}y + z).$$

Since $\star$ is standard the above yields a contradiction unless $(x^{m_1+1} + ay^{m_2+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z^l)$. The inclusion

$$(x^{m_1+2}, y^{m_2+2}, z) = (x^{m_1+2}, y^{m_2+2})^\star \subseteq (x^{m_1+1} + ay^{m_2+1})^\star$$

$$\subseteq (x^{m_1+1}, y^{m_2+1})^\star = (x^{m_1+1}, y^{m_2+1}, z)$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z).$$

Inductively we have $(x^{m_1+v} + ay^{m_2+v})^\star = (x^{m_1+v} + ay^{m_2+v}, z)$ for all $v \in \mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence $(x^m + ay^n)^\star = (x^m + ay^n, z)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

Case: $(x^m, y^n)^\star = (x^m, y, z)$ for all $m, n \in \mathbb{N}$. The inclusion $(x^{m+1}, y, z) \subseteq (x^{m_1+1}, y^{m_2+1})^\star \subseteq (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star = (x^{m_1}, y, z)$ implies $(x^{m_1} + by^{m_2})^\star = (x^{m_1}, y, z)$ which is a contradiction.

Case: $(x^m, y^n)^\star = (x, y^n, z)$ for all $m, n \in \mathbb{N}$. This gives a contradiction as in the previous case.

(4) Suppose $(x^{m_1} + by^{m_2})^\star = (x^{n_1}, y^{n_2}, z^{n_3})$ for some $m_1, m_2, n_1, n_2, n_3 \in \mathbb{N}$ and $b \in k^\times$. The inclusion $(x^{n_1}, y^{n_2}, z^{n_3}) = (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star \subseteq (x^{n_1}, y^{n_2}, z^{n_3})$ implies $(x^{m_1}, y^{m_2})^\star = (x^{n_1}, y^{n_2}, z^{n_3})$. By Lemma 3.2.12 $(x^m, y^n)^\star = (x, y^n, z)$ for all

$m, n \in \mathbb{N}$, $(x^m, y^n, z)^\star = (x^m, y, z)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n, z)$ for all $m, n \in \mathbb{N}$.

Case: $(x^m, y^n)^\star = (x, y^n, z)$ for all $m, n \in \mathbb{N}$. The inclusion $(x, y^{m_2+1}, z) = (x^{m_1+1}, y^{m_2+1})^\star \subseteq (x^{m_1} + by^{m_2})^\star \subseteq (x^{m_1}, y^{m_2})^\star = (x, y^{m_2+1}, z)$ implies $(x^{m_1} + by^{m_2})^\star = (x, y^{m_2}, z)$. Let $a \in k^\times$. Also the inclusion $(x, y^n, z) = (x^m, y^n)^\star = (x^m, y^n, z^r)^\star \subseteq (x, y^n, z)$ implies $(x^m, y^n, z^r)^\star = (x, y^n, z)$ for all $m, n, r \in \mathbb{N}$. Since $\star$ is weakly prime

$$(x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2})^\star \subseteq ((x + \frac{a}{b}y + z)(x^{m_1} + by^{m_2}))^\star$$

$$\Rightarrow (x^2, y^{m_2+1}, z^2) \subseteq (x^{m_1+1} + ay^{m_2+1})^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1})^\star = (x^2, y^{m_2+1}, z^2)^\star = (x, y^{m_2+1}, z).$$

By induction $(x^{m_1+v} + ay^{m_2+v})^\star = (x, y^{m_2+v}, z)$ for all $v \in \mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence $(x^m + ay^n)^\star = (x, y^n, z)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

The cases $(x^m, y^n)^\star = (x^m, y, z)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x^m, y^n, z)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ are similar.

(5) Suppose $(x^{m_1} + cy^{m_2} + dz^{m_3})^\star = (x^{m_1} + cy^{m_2} + dz^{m_3})$ for some m_1, m_2, $m_3 \in \mathbb{N}$ and $c, d \in k^\times$. For $a, b \in k^\times$ we have $(x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star \subseteq (x^{m_1} + cy^{m_2} + dz^{m_3})^\star = (x^{m_1} + cy^{m_2} + dz^{m_3})$. This implies $(x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star$ is either $(x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})$, $(x^{m_1+1} + ay^{m_2+1}, z^{m_3+1})$, $(x^{m_1+1}, y^{m_2+1} + \frac{b}{a}z^{m_3+1})$, $(x^{m_1+1} + \alpha z^{m_3+1}, y^{m_2+1} + \beta z^{m_3+1})$, or $(x^{m_1+1}, y^{m_2+1}, z^{m_3+1})$. If we compute

$$(((x + \frac{a}{c}y + \frac{b}{d}z)(x^{m_1} + cy^{m_2} + dz^{m_3}))^\star : x + \frac{a}{c}y + \frac{b}{d}z)$$

$$= ((x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star : x + \frac{a}{c}y + \frac{b}{d}z)$$

since $\star$ is standard we get a contradiction in each case except when $(x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star = (x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})$. Inductively we have $(x^{m_1+v} + ay^{m_2+v} + bz^{m_3+v})^\star = (x^{m_1+v} + ay^{m_2+v} + bz^{m_3+v})$ for all $v \geq 0$. And as in (1) we obtain the result for all exponents. Hence $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(6) Suppose $(x^{m_1} + by^{m_2}, z^{m_3})^\star = (x^{m_1} + by^{m_2}, z^{n_3})$ for some m_1, m_2, m_3, $n_3 \in \mathbb{N}$ and $b \in k^\times$. If $(x, y, z)^\star = R_3$ then by Lemma 3.2.3 $(x^{m_1}+by^{m_2}, z^{m_3})^\star = R_3$, a contradiction. Thus $(x, y, z)^\star = (x, y, z)$. By Lemma 3.2.11 $(x^{r_1}, y^{r_2}, z^{r_3})^\star = (x_1^{s_1}, y^{s_2}, z^{s_3})$ for all $(r_1, r_2, r_3) \in \mathbb{N}^3$ with $s_j = r_j$ for all $r_j \in \mathbb{N}$ for some $j \in \{1, 2, 3\}$ and $s_j = 1$ for all other $j \in \{1, 2, 3\}$. Since

$$(x^{m_1+1}, y^{m_2+1}, z^{m_3})^\star \subseteq (x^{m_1} + by^{m_2}, z^{m_3})^\star = (x^{m_1} + by^{m_2}, z^{n_3})$$

we have either $(x^m, y^n, z^r)^\star = (x^m, y^n, z)$ for all $m, n, r \in \mathbb{N}$ or $(x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$.

Case: $(x^m, y^n, z^r)^\star = (x^m, y^n, z)$ for all $m, n, r \in \mathbb{N}$. Since

$$(x^{m_1+1}, y^{m_2+1}, z) = (x^{m_1+1}, y^{m_2+1}, z^{m_3})^\star \subseteq (x^{m_1} + by^{m_2}, z^{n_3})$$

we must have $n_3 = 1$. The inclusion

$$(x^{m_1+2}, y^{m_2+2}, z) = (x^{m_1+2}, y^{m_2+2}, z^{m_3+1})^\star \subseteq (x^{m_1+1} + ay^{m_2+1}, z^{m_3+1})^\star$$

$$\subseteq (x^{m_1+1}, y^{m_2+1}, z^{m_3+1})^\star = (x^{m_1+1}, y^{m_2+1}, z)$$

implies either $(x^{m_1+1} + ay^{m_2+1}, z^{m_3+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z)$
or $(x^{m_1+1}, y^{m_2+1}, z^{m_3+1})^\star = (x^{m_1+1}, y^{m_2+1}, z)$. Suppose the latter. Consider

$$(((x+\frac{a}{b}y+z)(x^{m_1}+by^{m_2}, z^{m_3}))^\star : x+\frac{a}{b}y+z) = ((x^{m_1+1}+ay^{m_2+1}, z^{m_3+1})^\star) : x+\frac{a}{b}y+z)$$

$$= ((x^{m_1+1}, y^{m_2+1}, z) : x + \frac{a}{b}y + z) = (x^{m_1}, y^{m_2}, z).$$

Since $\star$ is standard this is a contradiction. Thus $(x^{m_1+1} + ay^{m_2+1}, z^{m_3+1})^\star$
$= (x^{m_1+1} + ay^{m_2+1}, z)$. Inductively $(x^{m_1+v} + ay^{m_2+v}, z^{m_3+v})^\star = (x^{m_1+v} + ay^{m_2+v}, z)$ for all $v \in \mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence
$(x^m + ay^n, z^r)^\star = (x^m + ay^n, z)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$.

Case: $(x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$. Similar to the previous case we get $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$.

(7) Suppose $(x^{m_1} + cy^{m_2} + dz^{m_3})^\star = (x^{m_1} + cy^{m_2}, z^{n_3})$ for some $(m_1, m_2, m_3) \in \mathbb{N}^3$, $n_3 \in \mathbb{N}$ and $c, d \in k^\times$. Since $(x^{m_1} + cy^{m_2}, z^{n_3})$ must be $\star$ closed by (6) we have either $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$.

Case: $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Since $\star$ is weakly prime for $a, b \in k^\times$ we have

$$(x + \frac{a}{c}y + \frac{b}{d}z)(x^{m_1} + cy^{m_2} + dz^{m_3})^\star \subseteq ((x + \frac{a}{c}y + \frac{b}{d}z)(x^{m_1} + cy^{m_2} + dz^{m_3}))^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1}, z^{n_3+1}) \subseteq (x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star$$

$$\Rightarrow (x^{m_1+1} + ay^{m_2+1} + bz^{m_3+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z^{n_3+1})^\star = (x^{m_1+1} + ay^{m_2+1}, z).$$

Inductively $(x^{m_1+v} + ay^{m_2+v} + bz^{m_3+v})^\star = (x^{m_1+v} + ay^{m_2+v}, z)$ for all $v \in \mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence $(x^m + ay^n + bz^r)^\star = (x^m + ay^n, z)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

Case: $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to the previous case we get $(x^m + ay^n + bz^r)^\star = (x^m + ay^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(8) Suppose $(x^{m_1} + cy^{m_2} + dz^{m_3})^\star = (x^{n_1}, y^{n_2}, z^{n_3})$ for some $m_1, m_2, m_3, n_1, n_2, n_3 \in \mathbb{N}$ and $c, d \in k^\times$. The inclusion $(x^{n_1}, y^{n_2}, z^{n_3}) = (x^{m_1} + cy^{m_2} + dz^{m_3})^\star \subseteq (x^{m_1}, y^{m_2}, z^{m_3})^\star \subseteq (x^{n_1}, y^{n_2}, z^{n_3})$ implies $(x^{m_1}, y^{m_2}, z^{m_3})^\star = (x^{n_1}, y^{n_2}, z^{n_3})$. So by Lemma 3.2.11 for all $r_1, r_2, r_3 \in \mathbb{N}$ $(x^{r_1}, y^{r_2}, z^{r_3})^\star = (x^{s_1}, y^{s_2}, z^{s_3})$for all $(r_1, r_2, r_3) \in \mathbb{N}^3$ with $s_j = r_j$ for all $r_j \in \mathbb{N}$ for some $j \in \{1, 2, 3\}$ and $s_j = 1$ for all other $j \in \{1, 2, 3\}$. There are seven cases.

Case: $(x^m, y^n, z^r)^\star = (x, y, z^r)$ for all $m, n \in \mathbb{N}$. This implies

$(x^{m_1}+cy^{m_2}+dz^{m_3})^\star=(x,y,z^{m_3})$. Let $a,b\in k^\times$. Since $\star$ is weakly prime

$$(x+\frac{a}{c}y+\frac{b}{d}z)(x^{m_1}+cy^{m_2}+dz^{m_3})^\star\subseteq((x+\frac{a}{c}y+\frac{b}{d}z)(x^{m_1}+cy^{m_2}+dz^{m_3}))^\star$$
$$\Rightarrow(x^2,y^2,z^{m_3+1})\subseteq(x^{m_1+1}+ay^{m_2+1}+bz^{m_3+1})^\star$$
$$\Rightarrow(x^{m_1+1}+ay^{m_2+1}+bz^{m_3+1})^\star=(x^2,y^2,z^{m_3+1})^\star=(x,y,z^{m_3+1}).$$

By induction $(x^{m_1+v}+ay^{m_2+v}+bz^{m_3+v})^\star=(x,y,z^{m_3+v})$ for all $v\in\mathbb{N}$. And as in (1) we obtain the result for all exponents. Hence $(x^m+ay^n+bz^r)^\star=(x,y,z^r)$ for all $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$.

The other six cases are similar.

□

In the case of R_2 the number of standard closures was not dependent on the field k. However this is not so for the R_3 case. The following two lemmas show that there are more possible standard closures if $k=\mathbb{Z}/2\mathbb{Z}$ or $k=\mathbb{Z}/3\mathbb{Z}$ than if $|k|\geq 4$.

Lemma 3.4.2. *Suppose $\star$ is a standard closure operation on R_3.*

1. *If $k=\mathbb{Z}/2\mathbb{Z}$ or $|k|\geq 4$ then $(x^{m_1}+ay^{m_2}+bz^{m_3})^\star\neq(x^{m_1}+\alpha z^{m_3},y^{m_2}+\beta z^{m_3})$ for all $(m_1,m_2,m_3)\in\mathbb{N}^3$ and $a,b,\alpha,\beta\in k^\times$.*
2. *If $k=\mathbb{Z}/3\mathbb{Z}$ and $(x^{m_1}+ay^{m_2}+bz^{m_3})^\star=(x^{m_1}+\alpha z^{m_3},y^{m_2}+\beta z^{m_3})$ for some $(m_1,m_2,m_3)\in\mathbb{N}^3$ and $a,b,\alpha,\beta\in k^\times$ then $(x^m+y^n+z^r)^\star=(x^m+2z^r,y^n+2z^r)$, $(x^m+y^n+2z^r)^\star=(x^m+z^r,y^n+z^r)$, $(x^m+2y^n+z^r)^\star=(x^m+2z^r,y^n+z^r)$ and $(x^m+2y^n+2z^r)^\star=(x^m+z^r,y^n+2z^r)$ for all $m,n,r\in\mathbb{N}$.*

Proof. First suppose $(x^{m_1}+ay^{m_2}+bz^{m_3})^\star=(x^{m_1}+\alpha z^{m_3},y^{m_2}+\beta z^{m_3})$ for some $(m_1,m_2,m_3)\in\mathbb{N}$ and $a,b,\alpha,\beta\in k^\times$. Since $(x^{m_1}+\alpha z^{m_3},y^{m_2}+\beta z^{m_3})$ must contain $x^{m_1}+ay^{m_2}+bz^{m_3}$ we have $(x^{m_1}+ay^{m_2}+bz^{m_3})^\star=(x^{m_1}+(b-a\beta)z^{m_3},y^{m_2}+\beta z^{m_3})$. This also implies $b-a\beta\neq 0$. Since $\star$ is weakly prime we have

$$(x+\frac{c}{a}y+\frac{d}{b}z)(x^{m_1}+ay^{m_2}+bz^{m_3})^\star\subseteq((x+\frac{c}{a}y+\frac{d}{b}z)(x^{m_1}+ay^{m_2}+bz^{m_3}))^\star$$

$$\Rightarrow (x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, \frac{c}{a}y^{m_2+1} + \frac{d\beta}{b}z^{m_3+1}) \subseteq (x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star$$

$$\Rightarrow (x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, y^{m_2+1} + \frac{ad\beta}{bc}z^{m_3+1}) \subseteq (x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star$$

$$\Rightarrow (x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = (x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, y^{m_2+1} + \frac{ad\beta}{bc}z^{m_3+1})^\star$$

$$\subseteq (x^{m_1} + (b-a\beta)z^{m_3}, y^{m_2} + \beta z^{m_3})$$

(since $x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1} = x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1} + c(y^{m_2+1} + \frac{ad\beta}{bc}z^{m_3+1})$). This implies one of the following.

$$(x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = (x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, y^{m_2+1} + \frac{ad\beta}{bc}z^{m_3+1}) \quad (3.1)$$

$$(x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = (x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, y^{m_2} + \beta z^{m_3}) \quad (3.2)$$

$$(x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = (x^{m_1} + (b-a\beta)z^{m_3}, y^{m_2+1} + \frac{ad\beta}{bc}z^{m_3+1}) \quad (3.3)$$

$$(x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = (x^{m_1} + (b-a\beta)z^{m_3}, y^{m_2} + \beta z^{m_3}) \quad (3.4)$$

Equation (4) implies $x^{m_1} + ay^{m_2} + bz^{m_3} \in (x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star$. By Lemma 3.2.5 $(x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star = R_3$ which is a contradiction. Now suppose equation (2). Consider

$$(((x + \frac{c}{a}y + \frac{d}{b}z)(x^{m_1} + ay^{m_2} + bz^{m_3}))^\star : x + \frac{c}{a}y + \frac{d}{b}z)$$

$$= ((x^{m_1+1} + cy^{m_2+1} + dz^{m_3+1})^\star : x + \frac{c}{a}y + \frac{d}{b}z)$$

$$= ((x^{m_1+1} + \frac{d(b-a\beta)}{b}z^{m_3+1}, y^{m_2} + \beta z^{m_3}) : x + \frac{c}{a}y + \frac{d}{b}z)$$

$$= \begin{cases} (x^{m_1} + \frac{d(b-a\beta)}{b} z^{m_3}, \frac{c}{a} y^{m_2-1} + \frac{d\beta}{b} z^{m_3-1}) \text{ if } m_2 > 1, m_3 > 1 \\ (x^{m_1} + \frac{d(b-a\beta)}{b} z^{m_3}, y, z^{m_3}) \text{ if } m_2 = 1, m_3 > 1 \\ (x^{m_1} + \frac{d(b-a\beta)}{b} z^{m_3}, y^{m_3}, z) \text{ if } m_2 > 1, m_3 = 1 \\ (x^{m_1} + \frac{d(b-a\beta)}{b} z^{m_3}, y, z) \text{ if } m_2 = m_3 = 1 \end{cases}$$

Since $\star$ is standard all of these are contradictions. Equation (3) gives a similar contradiction. So we are left with equation (1). Inductively $(x^{m_1+v} + cy^{m_2+v} + dz^{m_3+v})^\star = (x^{m_1+v} + \frac{d(b-a\beta)}{b} z^{m_3+v}, y^{m_2+v} + \frac{ad\beta}{bc} z^{m_3+v})$. And as in Lemma 3.4.1 (1) we obtain the result for all exponents. Hence $(x^m + cy^n + dz^r)^\star = (x^m + \frac{d(b-a\beta)}{b} z^r, y^n + \frac{ad\beta}{bc} z^r)$ for all $m, n, r \in \mathbb{N}$, $c, d \in k^\times$ for some fixed $a, b, \beta \in k^\times$.

(1) If $k = \mathbb{Z}/2\mathbb{Z}$ then there does not exist $\alpha, \beta \in k^\times$ such that $x^{m_1} + y^{m_2} + z^{m_3} \in (x^{m_1} + \alpha z^{m_3}, y^{m_2} + \beta z^{m_3})$ for any $(m_1, m_2, m_3) \in \mathbb{N}^3$. Thus $(x^{m_1} + y^{m_2} + z^{m_3})^\star \neq (x^{m_1} + \alpha z^{m_3}, y^{m_2} + \beta z^{m_3})$ for all $(m_1, m_2, m_3) \in \mathbb{N}^3$ and $\alpha, \beta \in k^\times$.

So suppose $|k| \geq 4$ and to the contrary that we have $(x^{m_1} + ay^{m_2} + bz^{m_3})^\star = (x^{m_1} + \alpha z^{m_3}, y^{m_2} + \beta z^{m_3})$ for some $(m_1, m_2, m_3) \in \mathbb{N}^3$ and $a, b, \alpha, \beta \in k^\times$. Thus, as shown above, $(x^{r_1} + cy^{r_2} + dz^{r_3})^\star = (x^{r_1} + \frac{d(b-a\beta)}{b} z^{r_3}, y^{r_2} + \frac{ad\beta}{bc} z^{r_3})$ for all $(r_1, r_2, r_3) \in \mathbb{N}^3$ and $c, d \in k^\times$. Consider $(x + \frac{a\beta}{b} y + dz)$. Since $|k| \geq 4$ we can choose d such that $d \neq 0$, $d \neq 1$, and $d \neq \frac{a\beta}{b}$. Thus $(x + \frac{a\beta}{b} y + dz)^\star = (x + \frac{d(b-a\beta)}{b} z, y + dz)$. Notice $(x + \frac{a\beta}{b} y + dz) \subseteq (x + (d - \frac{a\beta}{b})z, y + z)$. Since $(x + \frac{ad\beta}{b} y + \frac{bd - a\beta}{b - a\beta} z)^\star = (x + (d - \frac{a\beta}{b})z, y + z)$, $(x + (d - \frac{a\beta}{b})z, y + z)$ is $\star$ closed. The order preservation property implies that $(x + \frac{a\beta}{b} y + dz)^\star \subseteq (x + (d - \frac{a\beta}{b})z, y + z)$. Thus $(x + \frac{d(b-a\beta)}{b} z, y + dz) \subseteq (x + (d - \frac{a\beta}{b})z, y + z)$. This implies $y + dz \in (x + (d - \frac{a\beta}{b})z, y + z)$. And since $d \neq 1$, $z \in (x + (d - \frac{a\beta}{b})z, y + z)$ which is a contradiction. Thus $(x^{m_1} + ay^{m_2} + bz^{m_3})^\star \neq (x^{m_1} + \alpha z^{m_3}, y^{m_2} + \beta z^{m_3})$ for all $(m_1, m_2, m_3) \in \mathbb{N}^3$ and $a, b, \alpha, \beta \in k^\times$.

(2) Suppose $k = \mathbb{Z}/3\mathbb{Z}$ and $(x^{m_1} + ay^{m_2} + bz^{m_3})^\star = (x^{m_1} + \alpha z^{m_3}, y^{m_2} + \beta z^{m_3})$ for some $(m_1, m_2, m_3) \in \mathbb{N}^3$ and $a, b, \alpha, \beta \in k^\times$. Thus, as shown above,

$(x^{r_1} + cy^{r_2} + dz^{r_3})^\star = (x^{r_1} + \frac{d(b-a\beta)}{b} z^{r_3}, y^{r_2} + \frac{ad\beta}{bc} z^{r_3})$ for all $(r_1, r_2, r_3) \in \mathbb{N}^3$ and $c, d \in k^\times$. For a, b and β such that $b - a\beta \neq 0$ we have the following possibilities:

$$\beta = 1, a = 1, b = 2; \beta = 1, a = 2, b = 1; \beta = 2, a = 1, b = 1 \text{ or } \beta = 2, a = 2, b = 2.$$

For each of the above choices of a, b and β we compute the closures of $(x^m + y^n + z^r)$, $(x^m + y^n + 2z^r)$, $(x^m + 2y^n + z^r)$ and $(x^m + 2y^n + 2z^r)$. In each case we get $(x^m + y^n + z^r)^\star = (x^m + 2z^r, y^n + 2z^r)$, $(x^m + y^n + 2z^r)^\star = (x^m + z^r, y^n + z^r)$, $(x^m + 2y^n + z^r)^\star = (x^m + 2z^r, y^n + z^r)$ and $(x^m + 2y^n + 2z^r)^\star = (x^m + z^r, y^n + 2z^r)$ for all $m, n, r \in \mathbb{N}$.

□

Lemma 3.4.3. *Suppose $\star$ is a standard closure operation on R_3 with $i_j \in \{1, 2, 3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$. If $(x_{i_1}^m + ax_{i_2}^n + bx_{i_3}^r)^\star = (x_{i_1}^m, x_{i_2}^n + \frac{b}{a} x_{i_3}^r)$ and $(x_{i_1}^m + ax_{i_3}^r, x_{i_2}^n + bx_{i_3}^r)^\star = (x_{i_1}^m + ax_{i_3}^r, x_{i_2}^n + bx_{i_3}^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ then $|k| = 2$.*

Proof. Without loss of generality let $x_{i_1} = x$, $x_{i_2} = y$ and $x_{i_3} = z$. Suppose to the contrary that $|k| \geq 3$. Thus there exists $\beta \in k^\times$ such that $\beta \neq 1$. So we have the inclusion $(x + y + z) \subseteq (x + (1-\beta)z, y + \beta z)$. However $(x + y + z)^\star = (x, y + z)$ and $(x + (1-\beta)z, y + \beta z)^\star = (x + (1-\beta)z, y + \beta z)$. By order perservation we must have $(x + y + z)^\star \subseteq (x + (1-\beta)z, y + \beta z)^\star$ which implies $(x, y + z) \subseteq (x + (1-\beta)z, y + \beta z)$. Thus $z \in (x + (1-\beta)z, y + \beta z)$ which implies $(x + (1-\beta)z, y + \beta z) = (x, y, z)$ a contradiction. □

Theorem 3.4.4. *If $|k| \geq 4$ then there are 1522 standard closure operations on R_3. If $k = \mathbb{Z}/3\mathbb{Z}$ then there are 1523 standard closure operations on R_3. If $k = \mathbb{Z}/2\mathbb{Z}$ then there are 1525 standard closure operations on R_3.*

The standard closure operations from Theorem 3.4.4 are listed in appendices A, B and C.

Proof. Suppose $\star$ is a standard closure operation on R_3. By Proposition 3.1.14 each ideal of R_3 has a unique reduced generating set. Thus each proper ideal of R_3 expressed in terms of its reduced generating set will have one of the following forms:

- (0), (x^m), (y^n), (z^r), $(x^m + ay^n)$, $(y^n + az^r)$, $(x^m + az^r)$, $(x^m + ay^n + bz^r)$
- (x^m, y^n), (y^n, z^r), (x^m, z^r), $(x^m + ay^n, z^r)$, $(x^m + az^r, y^n)$, $(x^m, y^n + az^r)$, $(x^m + az^r, y^n + bz^r)$
- (x^m, y^n, z^r)

m, n, $r \in \mathbb{N}$ and $a, b \in k^\times$.

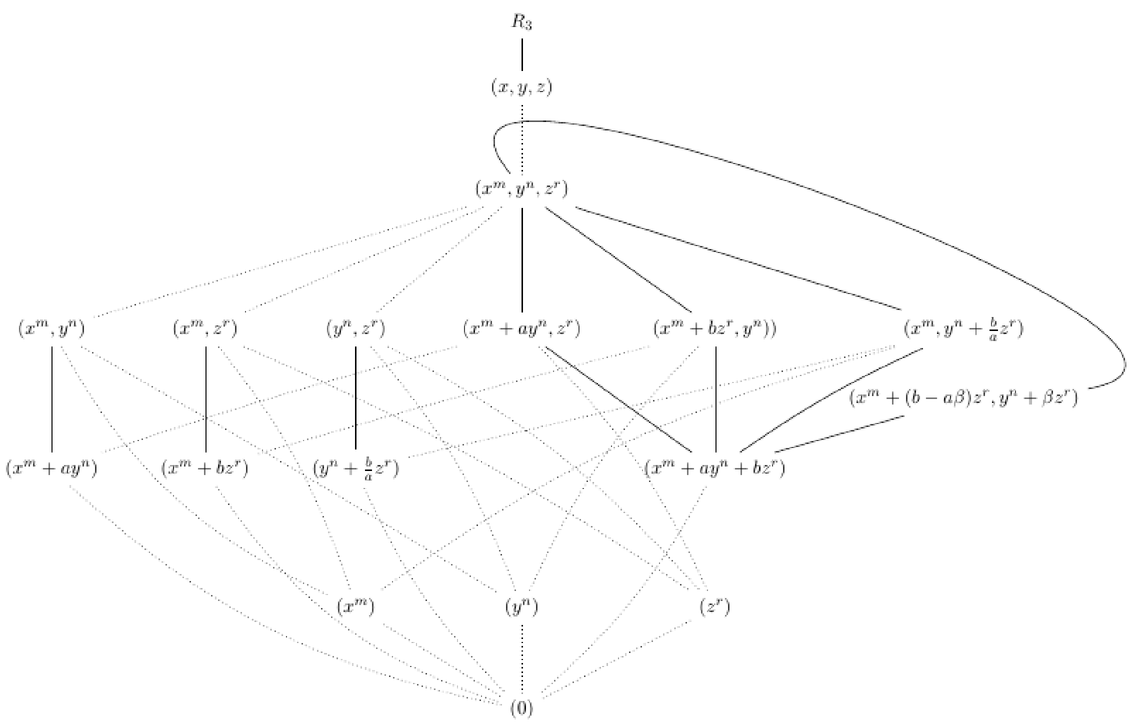

Figure 3.2: Ideal lattice of R_3.
A dash indicates inclusion. Also a solid dash between an ideal I and J, with I below J, indicates $J/I \cong k$.

For the first seven cases we will assume $(x, y, z)^\star = R_3$. By Lemma 3.2.3 we have the following:

$(x^m + ay^n + bz^r)^\star = (x^m, y^n)^\star = (x^m, z^r)^\star = (x^m + ay^n, z^r)^\star = (x^m + az^r, y^n)^\star = (x^m, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)^\star = R_3$
for all m, n, $r \in \mathbb{N}$ and $a, b \in k^\times$.

(1) Suppose $(x)^\star = R_3$, $(y)^\star = R_3$, and $(z)^\star = R_3$. By Lemma 3.2.3 $(x^m)^\star = (y^n)^\star = (z^r)^\star = (x^m + ay^n)^\star = (y^n + az^r)^\star = (x^m + az^r)^\star = (x^m, y^n)^\star = (y^n, z^r)^\star = (x^m, z^r)^\star = R_3$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma $(0)^\star$ is monomial. If $(0)^\star$ is a proper nonzero monomial ideal then it contains a power of x , y or z. Thus by Lemma 3.2.3 $(0)^\star = R_3$, a contradiction. Thus $(0)^\star = R_3$ or $(0)^\star = (0)$. Thus we have $\star_1$ and $\star_{1.1}$.

(2) Here we will consider all cases where the closure of two of the ideals, (x), (y), or (z), are R_3 and the closure of the remaining ideal is not R_3.

(2a) Suppose $(x)^\star = (y)^\star = R_3$ and $(z)^\star \neq R_3$. By By Lemma 3.2.3 $(x^m)^\star = (y^n)^\star = (x^m+ay^n)^\star = (y^n+az^r)^\star = (x^m+az^r)^\star = (x^m, y^n)^\star = (y^n, z^r)^\star = (x^m, z^r)^\star = R_3$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. If $(z)^\star$ contains a power of x or y then by Lemma 3.2.3 $(z)^\star = R_3$. Thus $(z)^\star \subseteq (z)$. By Lemma 3.2.12 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(2aa) Suppose $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is proper nonzero monomial ideal or $(0)^\star = (0)$.

(2aaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (z)$, $(0)^\star = (z^l)$ for some $l \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (z)$. This is $\star_2$.

(2aab) Suppose $(0)^\star = (0)$. This is $\star_{2.1}$.

(2ab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{2.2}$.

(2b) Suppose $(x)^\star = (z)^\star = R_3$ and $(y)^\star \neq R_3$. Similar to (2a) we get $\star_{2.3}$, $\star_{2.4}$ and $\star_{2.5}$.

(2b) Suppose $(y)^\star = (z)^\star = R_3$ and $(x)^\star \neq R_3$. Similar to (2a) we get $\star_{2.6}$, $\star_{2.7}$ and $\star_{2.8}$.

(3) Here we will consider all cases where the closure of one of the ideals, (x), (y), or (z) is R_3, and the closure of the remaining two ideals are not R_3.

The following claims will be useful throughout the remainder of the proof.

Claim 3.4.5. *Suppose $\star$ is a standard closure on R_3 with $i_j \in \{1,2,3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$ and that $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ then either*

1. *$(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$ or*
2. *$(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})$ for all $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ and $a \in k^\times$*

Proof. Without loss of generality let $x_{i_1} = x$, $x_{i_2} = y$, $r_{i_1} = m$ and $r_{i_2} = n$. Assume $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Let $a \in k^\times$. The inclusion $(x^m + ay^n)^\star \subseteq (x^m, y^n)^\star = (x^m, y^n)$ implies for each $i, j \in \mathbb{N}$ and $b \in k^\times$, $(x^i + by^j)^\star = (x^i, y^j)$ or $(x^i + by^j)^\star = (x^i + by^j)$.

Suppose $(x^i + by^j)^\star = (x^i, y^j)$ for some $i, j \in \mathbb{N}$ and $b \in k^\times$. By Lemma 3.4.1 either $(x^m + ay^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$, $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$, $(x^m + ay^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m + ay^n)^\star \subseteq (x^m, y^n)^\star = (x^m, y^n)$ leads to a contradiction except in the case $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

Suppose $(x^i + by^j)^\star = (x^i + by^j)$ for some $i, j \in \mathbb{N}$ and $b \in k^\times$. By Lemma 3.4.1 $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. This concludes the proof of the claim. □

Claim 3.4.6. *Suppose $\star$ is a standard closure on R_3 with $i_j \in \{1,2,3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$.*

1. *If* $(x_{i_1}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}, x_{i_2})$ *for all* $r_{i_2} \in \mathbb{N}$ *and* $(x_{i_2}^{r_{i_2}}, x_{i_3})^\star = (x_{i_2}, x_{i_3})$ *for all* $r_{i_2} \in \mathbb{N}$ *then* $(x_{i_2}^{r_{i_2}})^\star = (x_{i_2})$ *for all* $r_{i_2} \in \mathbb{N}$ *or* $(x_{i_2}^{r_{i_2}})^\star = (x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$.
2. *If* $(x_{i_1}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}, x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$ *and* $(x_{i_2}^{r_{i_2}}, x_{i_3})^\star = (x_{i_2}, x_{i_3})$ *for all* $r_{i_2} \in \mathbb{N}$ *then* $(x_{i_2}^{r_{i_2}})^\star = (x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$.
3. *If* $(x_{i_1}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}, x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$ *and* $(x_{i_2}^{r_{i_2}}, x_{i_3})^\star = (x_{i_2}^{r_{i_2}}, x_{i_3})$ *for all* $r_{i_2} \in \mathbb{N}$ *then* $(x_{i_2}^{r_{i_2}})^\star = (x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$.
4. *If* $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}, x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$ *then* $(x_{i_1}^{r_{i_1}})^\star = (x_{i_1})$ *for all* $r_{i_1} \in \mathbb{N}$ *or* $(x_{i_1}^{r_{i_1}})^\star = (x_{i_1}^{r_{i_1}})$ *for all* $r_{i_1} \in \mathbb{N}$.
5. *If* $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}})$ *for all* $r_{i_2} \in \mathbb{N}$ *then* $(x_{i_1}^{r_{i_1}})^\star = (x_{i_1}^{r_{i_1}})$ *for all* $r_{i_1} \in \mathbb{N}$.

Proof. Without loss of generality let $x_{i_1} = x$, $x_{i_2} = y$, $x_{i_3} = z$ and $r_{i_2} = n$.

(1) Assume $(x, y^n)^\star = (x, y^n)$ and $(y^n, z)^\star = (y, z)$ $n \in \mathbb{N}$. The inclusion $(y^j)^\star \subseteq (x, y^j)^\star \cap (y^j, z)^\star = (x, y) \cap (y, z) = (y)$ implies $(y^j)^\star = (y^i)$ for some $i, j \in \mathbb{N}$. By Lemma 3.2.12 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(2) Assume $(x, y^n)^\star = (x, y)$ and $(y^n, z)^\star = (y, z)$ $n \in \mathbb{N}$. The inclusion $(y^n)^\star \subseteq (x, y^n)^\star \cap (y^n, z)^\star = (x, y^n) \cap (y, z) = (y^n)$ implies $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(3) Proof similar to (2). (4) Assume $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. Since $(x^m)^\star \subseteq \cap_{n=1}^{\infty} (x^m, y^n)^\star = \cap_{n=1}^{\infty} (x, y^n) = (x)$ by Lemma 3.2.12 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(5) Assume $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Since $(x^m)^\star \subseteq \cap_{n=1}^{\infty} (x^m, y^n)^\star = \cap_{n=1}^{\infty} (x^m, y^n) = (x^m)$ thus $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

This concludes the proof of the claim. □

(3a) Suppose $(x)^\star = R_3$, $(y)^\star \neq R_3$, and $(z)^\star \neq R_3$. By Lemma 3.2.3 $(x^m)^\star = (x^m + ay^n)^\star = (x^m + az^r)^\star = (x^m, y^n)^\star = (x^m, z^r)^\star = R_3$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. If $(y, z)^\star$ contains a power of x then by Lemma 3.2.3 $(y, z)^\star = R_3$. Thus $(y, z)^\star = R_3$ or $(y, z)^\star \subseteq (y, z)$.

(3aa) Suppose $(y,z)^\star = R_3$. If $(y)^\star$ contains a power of x or z then by Lemma 3.2.3 $(y)^\star = R_3$, a contradiction. Thus $(y)^\star \subseteq (y)$. By Lemma 3.2.12 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similarly $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Since $(0)^\star \subseteq (y) \cap (z) = (0)$, $(0)^\star = (0)$. Thus we have the four closures $\star_3$ through $\star_{3.3}$.

(3ab) Suppose $(y,z)^\star \subseteq (y,z)$. By Lemma 3.2.12 $(y^n,z^r)^\star = (y,z)$ for all $n,r \in \mathbb{N}$, $(y^n,z^r)^\star = (y,z^r)$ for all $n,r \in \mathbb{N}$, $(y^n,z^r)^\star = (y^n,z)$ for all $n,r \in \mathbb{N}$ or $(y^n,z^r)^\star = (y^n,z^r)$ for all $n,r \in \mathbb{N}$

(3aba) Suppose $(y^n,z^r)^\star = (y,z)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y,z)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Since $(y)^\star \subseteq (y,z)$ by Lemma 3.2.12 $(y^n)^\star = (y,z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y^n,z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Since (y^n,z) is not $\star$ closed for all $n \in \mathbb{N}$ only three possibilities remain.

(3abaa) Suppose $(y^n)^\star = (y,z)$ for all $n \in \mathbb{N}$. Since $(z)^\star \subseteq (y,z)$ by Lemma 3.2.12 $(z^r)^\star = (y,z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y,z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Since (y,z^r) is not $\star$ closed for all $r \in \mathbb{N}$ only three possibilities remain.

(3abaaa) Suppose $(z^r)^\star = (y,z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(3abaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (y,z)$, $(0)^\star = (y^j,z^l)$ for some $j,l \in \mathbb{N}$, $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$ or $(0)^\star = (z^l)$ for some $l \in \mathbb{N}$.

(3abaaaaa) Suppose $(0)^\star = (y^j,z^l)$ for some $j,l \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (y,z)$. This is $\star_{3.4}$.

(3abaaaab) Suppose $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (y)$.

This is $\star_{3.5}$.

(3abaaaac) Suppose $(0)^\star = (z^l)$ for some $l \in \mathbb{N}$. Similar to the previous case $(0)^\star = (z)$. This is $\star_{3.6}$.

(3abaaab) Suppose $(0)^\star = (0)$. This is $\star_{3.7}$.

(3abaab) Suppose $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(3abaaba) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to case (3abaaaab) $(0)^\star = (z)$. This is $\star_{3.8}$.

(3abaabb) Suppose $(0)^\star = (0)$. This is $\star_{3.9}$.

(3abaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{3.10}$.

(3abab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Similar to case (3abaa) we have $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(3ababa) Suppose $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(3ababaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to case (3abaaaab) $(0)^\star = (y)$. This is $\star_{3.11}$.

(3ababab) Suppose $(0)^\star = (0)$. This is $\star_{3.12}$.

(3ababb) Suppose $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$. Since $(0)^\star \subseteq (y) \cap (z) = (0)$, $(0)^\star = (0)$. This is $\star_{3.13}$.

(3ababc) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{3.14}$.

(3abac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (3abaa) we have $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{3.15}$, $\star_{3.16}$ and $\star_{3.17}$.

(3abb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(3abba) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Since $(z^2)^\star \subseteq (y, z^2)$ by Lemma 3.2.12 $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(3abbaa) Suppose $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(3abbaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subsetneq (y)$, $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (y)$. This is $\star_{3.18}$.

(3abbaab) Suppose $(0)^\star = (0)$. This is $\star_{3.19}$.

(3abbab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{3.20}$.

(3abbb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (3abba) $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{3.21}$ and $\star_{3.22}$.

(3abc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. Similar to case (3abb) we get $\star_{3.23}$ through $\star_{3.26}$.

(3abd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{3.27}$ and $\star_{3.28}$.

(3b) Suppose $(x)^\star \neq R_3$, $(y)^\star = R_3$, and $(z)^\star \neq R_3$. Similar to case (3a) we get

$\star_{3.29}$ through $\star_{3.57}$.

(3c) Suppose $(x)^\star \neq R_3$, $(y)^\star \neq R_3$, and $(z)^\star = R_3$. Similar to case (3a) we get $\star_{3.58}$ through $\star_{3.86}$.

For parts (4), (5) and (6) We will cover those cases where $(x)^\star \neq R_3$, $(y)^\star \neq R_3$, and $(z)^\star \neq R_3$.

(4) Suppose $(x,y)^\star = (x,z)^\star = (y,z)^\star = R_3$. By Lemma 3.2.3 $(x^m + ay^n)^\star = (x^m + az^r)^\star = (y^n + az^r)^\star = (x^m, y^n)^\star = (x^m, z^r)^\star = (y^n, z^r)^\star = R_3$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. If $(x)^\star$ contains a power of y or z then $(x)^\star = R_3$, a contradiction. Thus $(x)^\star \subseteq (x)$. By Lemma 3.2.12 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similarly $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Since $(0)^\star \subseteq (x) \cap (y) = (0)$, $(0)^\star = (0)$. Thus we have the closures $\star_4$ through $\star_{4.7}$.

(5) Here we will consider those cases where the closures of two of the ideals, (x,y), (x,z) or (y,z), are R_3 and the closure of the remaining ideal is not R_3.

(5a) Suppose $(x,y)^\star = (x,z)^\star = R_3$ and $(y,z)^\star \neq R_3$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.3 $(x^m + ay^n)^\star = (x^m + az^r)^\star = (x^m, y^n)^\star = (x^m, z^r)^\star = R_3$. Similar to case (4) $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. And since $(0)^\star \subseteq (x) \cap (y,z) = (0)$, $(0)^\star = (0)$.

(5aa) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. If $(y,z)^\star$ contains a power of x then by Lemma 3.2.3 $(y,z)^\star = R_3$, a contradiction. Thus $(y,z)^\star \subseteq (y,z)$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y,z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(5aaa) Suppose $(y^n, z^r)^\star = (y,z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y,z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(y)^\star \subseteq (y,z)$ by Lemma 3.2.12 $(y^n)^\star = (y,z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$

for all $n \in \mathbb{N}$. However since (y^n, z) is not $\star$ closed for all $n \in \mathbb{N}$ we are left with just the three possibilities.

(5aaaa) Suppose $(y^n)^\star = (y, z)$ for all $n \in \mathbb{N}$. Since $(z)^\star \subseteq (y, z)$ by Lemma 3.2.12 $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since (y, z^r) is not $\star$ closed for all $r \in \mathbb{N}$ we are left with just the three possibilities: $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_5$, $\star_{5.1}$ and $\star_{5.2}$.

(5aaab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Similar to (5aaaa) $(z^r)^\star = (y, z)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{5.3}$, $\star_{5.4}$ and $\star_{5.5}$.

(5aaac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (5aaab) we have $\star_{5.6}$, $\star_{5.7}$ and $\star_{5.8}$.

(5aab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(5aaba) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Since $(z^2)^\star \subseteq (y, z^2)$ by Lemma 3.2.12 $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{5.9}$ and $\star_{5.10}$.

(5aabb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case(5aaba) we get This is $\star_{5.11}$ and This is $\star_{5.12}$.

(5aac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. This is similar to case (5aab) with the roles of y and z exchanged. Thus we have $\star_{5.13}$ through $\star_{5.16}$.

(5aad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(5aada) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{5.17}$.

(5aadb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{5.18}$.

(5ab) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. This case is similar to case (5ac). Thus we get $\star_{5.19}$ through $\star_{5.37}$.

(5b) Suppose $(x, y)^\star = (y, z)^\star = R_3$ and $(x, z)^\star \neq R_3$. Similar to case (5a) we get $\star_{5.38}$ through $\star_{5.75}$.

(5c) Suppose $(x, z)^\star = (y, z)^\star = R_3$ and $(x, y)^\star \neq R_3$. Similar to case (5a) we get $\star_{5.76}$ through $\star_{5.113}$.

(6) Here we will consider those cases where the closure of one of the ideals, (x, y), (x, z) or (y, z), is R_3 and the closure of the remaining ideals is not R_3. (6a) Suppose $(x, y)^\star = R_3$, $(x, z)^\star \neq R_3$ and $(y, z)^\star \neq R_3$. By Lemma 3.2.3 $(x^m + ay^n)^\star = (x^m, y^n)^\star = R_3$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. If $(x, z)^\star$ contains a power of y then by Lemma 3.2.3 $(x, z)^\star = R_3$, a contradiction. Thus $(x, z)^\star \subseteq (x, z)$. By Lemma 3.2.12 $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(6aa) Suppose $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6a) $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(6aaa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(x)^\star \subseteq (x, z)$ by Lemma 3.2.12 $(x^m)^\star = (x, z)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x^m, z)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. However since (x^m, z) is not $\star$-closed for all $m \in \mathbb{N}$ we are left with just three cases.

(6aaaa) Suppose $(x^m)^\star = (x, z)$ for all $m \in \mathbb{N}$. Since $(y)^\star \subseteq (y, z)$ Lemma 3.2.12 $(y^n)^\star = (y, z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Again (y^n, z) is not $\star$-closed for all $n \in \mathbb{N}$ we are left with just three cases.

(6aaaaa) Suppose $(y^n)^\star = (y, z)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. (6aaaaaa) Suppose $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(6aaaaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (z)$, $(0)^\star = (z^l)$ for some $l \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (z)$. This is $\star_6$.

(6aaaaaab) Suppose $(0)^\star = (0)$. This is $\star_{6.1}$.

(6aaaaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{6.2}$.

(6aaaab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Since $(0)^\star \subseteq (y) \cap (x, z) = (0)$, $(0)^\star = (0)$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Her we have $\star_{6.3}$ and $\star_{6.4}$.

(6aaaac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (6aaaab) we get $\star_{6.5}$ through $\star_{6.6}$.

(6aaab) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Since $(0)^\star \subseteq (x) \cap (y, z) = (0)$, $(0)^\star = (0)$. Similar to case (6aaaa) $(y^n)^\star = (y, z)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(6aaaba) Suppose $(y^n)^\star = (y, z)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Hence we have $\star_{6.7}$ and $\star_{6.8}$.

(6aaabb) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{6.9}$ and $\star_{6.10}$.

(6aaabc) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Similar to case (6aaabb) we have $\star_{6.11}$ through $\star_{6.12}$.

(6aaac) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similar to (6aaab) we get $\star_{6.13}$ through $\star_{6.18}$.

(6aab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. And by Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (6aaa) $(x^m)^\star = (x, z)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(6aaba) Suppose $(x^m)^\star = (x, z)$ for all $m \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Here we have $\star_{6.19}$ and $\star_{6.20}$.

(6aabb) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Similar to (6aaba) $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. So we have $\star_{6.21}$ and $\star_{6.22}$.

(6aabc) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similar to case (6aabb) we get $\star_{6.23}$ and $\star_{6.24}$.

(6aac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. Similar to case (6aab) with the roles of y and z exchanged we get $\star_{6.25}$ and $\star_{6.30}$.

(6aad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(6aada) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to (6aaa) $(x^m)^\star = (x, z)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. So we have $\star_{6.31}$, $\star_{6.32}$ and $\star_{6.33}$.

(6aadb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$. Similar to case (6aada) we get $\star_{6.34}$ through $\star_{6.36}$.

(6ab) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6aa) $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(6aba) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. This is the same as as case (6aab) with the roles of x and y exchanged. We get $\star_{6.37}$ through $\star_{6.42}$.

(6abb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{6.43}$ through $\star_{6.46}$.

(6abc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. And again by Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(6abca) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Since $(y^2)^\star \subseteq (y^2, z)$, by Lemma 3.2.12 $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. (6abcaa) Suppose $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$. This is $\star_{6.47}$.

(6abcab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{6.48}$.

(6abcb) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similar to case (6abca) we get $\star_{6.49}$ or $\star_{6.50}$.

(6abd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$

for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(6abda) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $n \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $n \in \mathbb{N}$. Thus we have $\star_{6.51}$ and $\star_{6.52}$.

(6abdb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6abda) we have $\star_{6.53}$ and $\star_{6.54}$.

(6ac) Suppose $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6aa) $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(6aca) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. Similar to case (6aac) with the roles of x and y exchanged we get $\star_{6.55}$ and $\star_{6.60}$.

(6acb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (6abc) we get $\star_{6.61}$ and $\star_{6.64}$.

(6acc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(x^2)^\star \subsetneq (x^2, z)$ by Lemma 3.2.12 $(x^m)^\star = (x^m, z)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(6acca) Suppose $(x^m)^\star = (x^m, z)$ for all $m \in \mathbb{N}$. Since $(y^2)^\star \subsetneq (y^2, z)$ by Lemma 3.2.12 $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(6accaa) Suppose $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(6accaaa) Suppose $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(6accaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subsetneq (z)$,

$(0)^\star = (z^l)$ for some $l \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (z)$. This is $\star_{6.65}$.

(6accaaab) Suppose $(0)^\star = (0)$. This is $\star_{6.66}$.

(6accaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{6.67}$.

(6accab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Here we have $\star_{6.68}$ and $\star_{6.69}$.

(6accb) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (6acca) $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(6accba) Suppose $(y^n)^\star = (y^n, z)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Here we have $\star_{6.70}$ and $\star_{6.71}$.

(6accbb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Hence we have $\star_{6.72}$ and $\star_{6.73}$.

(6acd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(6acda) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6acc) $(x^m)^\star = (x^m, z)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. So we have $\star_{6.74}$ and $\star_{6.75}$.

(6acdb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (6acda) we get $\star_{6.76}$ and $\star_{6.77}$.

(6ad) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. (6ada) Now suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. Similar to case (6aad) we get $\star_{6.78}$ and $\star_{6.83}$.

(6adb) Now suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (6abd) we get $\star_{6.84}$ and $\star_{6.87}$.

(6adc) Now suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. Similar to case (6acd) we get $\star_{6.88}$ and $\star_{6.91}$.

(6add) Now suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{6.92}$ through $\star_{6.95}$.

(6b) Suppose $(x, y)^\star \neq R_3$, $(x, z)^\star = R_3$ and $(y, z)^\star \neq R_3$. Similar to case (6a) we get $\star_{6.96}$ through $\star_{6.191}$.

(6c) Suppose $(x, y)^\star \neq R_3$, $(x, z)^\star \neq R_3$ and $(y, z)^\star = R_3$. Similar to case (6a) we get $\star_{6.192}$ through $\star_{6.287}$.

(7) Suppose $(x, y)^\star \neq R_3$, $(x, z)^\star \neq R_3$ and $(y, z)^\star \neq R_3$. If $(x, y)^\star$ contains a power of x then $(x, y)^\star = R_3$, a contradiction. Thus $(x, y)^\star \subseteq (x, y)$. Similarly $(x, z)^\star \subseteq (x, z)$ and $(y, z)^\star \subseteq (y, z)$. By Lemma 3.2.12 $(x^m, y^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$, $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

(7a) Suppose $(x^m, y^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x, y)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(7aa) Suppose $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all

$n,r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n,r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n,r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n,r \in \mathbb{N}$. Since $(0)^\star \subseteq (x,y) \cap (x,z) \cap (y,z) = (0)$, $(0)^\star = (0)$.

(7aaa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. This leaves only the colosure of the principal monmial ideals to be determined. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_7$ through $\star_{7.7}$.

(7aab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.8}$ through $\star_{7.11}$.

(7aac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7aab) with the roles of y and z exchanged. We get $\star_{7.12}$ and $\star_{7.15}$.

(7aad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n,r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$.

(7aada) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Thus we have $\star_{7.16}$ and $\star_{7.17}$.

(7aadb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7aada) we have $\star_{7.18}$ and $\star_{7.19}$.

(7ab) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m,r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star =$

(x, z^r) for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7aa) $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (7aa) $(0)^\star = (0)$.

(7aba) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. Similar to case (7aab) with the roles of x and y exchanged we have $\star_{7.20}$ through $\star_{7.23}$.

(7abb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.24}$ through $\star_{7.27}$.

(7abc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.28}$ through $\star_{7.29}$.

(7abd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(7abda) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Thus we have $\star_{7.30}$ and $\star_{7.31}$.

(7abdb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7abda) we have $\star_{7.32}$ and $\star_{7.33}$.

(7ac) Suppose $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.4.6 $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (7aa) $(0)^\star = (0)$.

(7aca) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.34}$ through $\star_{7.37}$.

(7acb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (7abc) we have $\star_{7.38}$ and $\star_{7.39}$.

(7acc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.40}$ and $\star_{7.41}$.

(7acd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{7.42}$ and $\star_{7.43}$.

(7ad) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.4.6 $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(7ada) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(7adaa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This gives us $\star_{7.44}$ and $\star_{7.45}$.

(7adab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star =$

(y, z^r) for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This gives us $\star_{7.46}$ and $\star_{7.47}$.

(7adac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. This is $\star_{7.48}$.

(7adad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{7.49}$ and $\star_{7.50}$.

(7adb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7ada) we have $\star_{7.51}$ through $\star_{7.57}$.

(7b) Suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. By Lemma 3.2.12 $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(7ba) Suppose $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$.

(7baa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.58}$ through $\star_{7.61}$.

(7bab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.62}$ and

$\star_{7.63}$.

(7bac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$ and $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.64}$ through $\star_{7.67}$.

(7bad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(7bada) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Thus we have $\star_{7.68}$ and $\star_{7.69}$.

(7badb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7bada) we have $\star_{7.70}$ and $\star_{7.71}$.

(7bb) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$.

(7bba) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Thus we have $\star_{7.72}$ and $\star_{7.73}$.

(7bbb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (7bba) we have $\star_{7.74}$ and $\star_{7.75}$.

(7bbc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. Simlar to case (7bba) we have $\star_{7.76}$ and $\star_{7.77}$.

(7bbd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$.

(7bbda) Suppose $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Thus we have $\star_{7.78}$ and $\star_{7.79}$.

(7bbdb) Suppose $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7bbda) we have $\star_{7.80}$ and $\star_{7.81}$.

(7bc) Suppose $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$.

(7bca) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{7.82}$ and $\star_{7.83}$.

(7bcb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{7.84}$.

(7bcc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{7.85}$ and $\star_{7.86}$.

(7bcd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{7.87}$ and $\star_{7.88}$.

(7bd) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(7bda) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$, $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$.

(7bdaa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{7.89}$.

(7bdab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{7.90}$.

(7bdac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{7.91}$.

(7bdad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{7.92}$ and $\star_{7.93}$.

(7bdb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7bda) we get $\star_{7.94}$ through $\star_{7.98}$.

(7c) Suppose $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$. Similar to case (7b) with the roles of x and y exchanged we get $\star_{7.99}$ through $\star_{7.139}$.

(7d) Suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. By Claim 3.4.6 $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$ and $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

(7da) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, z)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(7daa) Suppose $(x^m, z^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(7daaa) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.140}$ and $\star_{7.141}$.

(7daab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{7.142}$.

(7daac) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{7.143}$ and $\star_{7.144}$.

(7daad) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{7.145}$ and $\star_{7.146}$.

(7dab) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.12 $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(7daba) Suppose $(y^n, z^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{7.147}$.

(7dabb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (7daba) we have $\star_{7.148}$.

(7dabc) Suppose $(y^n, z^r)^\star = (y^n, z)$ for all $n, r \in \mathbb{N}$. Similar to case (7daba) we have $\star_{7.149}$.

(7dabd) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{7.150}$ and $\star_{7.151}$.

(7db) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (7da) we have $\star_{7.152}$ through $\star_{7.163}$.

Since we have now covered all cases such that $(x, y, z)^\star = R_3$ we will now suppose for the remainder of the proof that $(x, y, z)^\star = (x, y, z)$. By Lemma 3.2.11 we have the following cases:

(8a) $(x^m, y^n, z^r)^\star = (x, y, z^r)\, \forall m, n, r \in \mathbb{N}$

(8b) $(x^m, y^n, z^r)^\star = (x, y^n, z)\, \forall m, n, r \in \mathbb{N}$

(8c) $(x^m, y^n, z^r)^\star = (x^m, y, z)\, \forall m, n, r \in \mathbb{N}$

(9a) $(x^m, y^n, z^r)^\star = (x, y^n, z^r)\, \forall m, n, r \in \mathbb{N}$

(9b) $(x^m, y^n, z^r)^\star = (x^m, y, z^r)\, \forall m, n, r \in \mathbb{N}$

(9c) $(x^m, y^n, z^r)^\star = (x^m, y^n, z)\, \forall m, n, r \in \mathbb{N}$

(10) $(x^m, y^n, z^r)^\star = (x^m, y^n, z^r)\, \forall m, n, r \in \mathbb{N}$

(8a) Suppose $(x^m, y^n, z^r)^\star = (x, y, z^r)\, \forall m, n, r \in \mathbb{N}$. The inclusion

$$(x, y, z^{r+1}) = (x^{m+1}, y^{n+1}, z^{r+1})^\star \subseteq (x^m + ay^n + bz^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x, y, z^r)$$

implies $(x^m + ay^n + bz^r)^\star = (x, y, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similarly $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n + az^r)^\star = (x^m + ay^n, z^r)^\star = (x^m + az^r, y^n)^\star = (x, y, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Since $(x^m, y^n)^\star \subseteq \cap_{r=1}^{\infty}(x^m, y^n, z^r)^\star =$

$\cap_{r=1}^{\infty}(x,y,z^r) = (x,y)$ by Lemma 3.2.12 $(x^m,y^n)^\star = (x,y)$ for all $m,n \in \mathbb{N}$, $(x^m,y^n)^\star = (x,y^n)$ for all $m,n \in \mathbb{N}$, $(x^m,y^n)^\star = (x^m,y)$ for all $m,n \in \mathbb{N}$ or $(x^m,y^n)^\star = (x^m,y^n)$ for all $m,n \in \mathbb{N}$.

(8aa) Suppose $(x^m,y^n)^\star = (x,y)$ for all $m,n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x,y)$ for all $m,n \in \mathbb{N}$ and $a \in k^\times$. Since $(x)^\star \subseteq (x,y)$ by Lemma 3.2.12 we have $(x^m)^\star = (x,y)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x^m,y)$ for all $m \in \mathbb{N}$, $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. However $(x^m,y^n)^\star \neq (x^m,y)$ for all $m,n \in \mathbb{N}$ since (x^m,y) is not $\star$ closed for $m > 1$. This leaves only three cases.

(8aaa) Suppose $(x^m)^\star = (x,y)$ for all $m \in \mathbb{N}$. The inclusion $(x,y) = (x^m)^\star \subseteq (x^m,z^r)^\star \subseteq (x,y,z^r)$ implies $(x^m,z^r)^\star = (x,y,z^r)$ for all $m,r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x,y,z^r)$ for all $m,r \in \mathbb{N}$ and $a \in k^\times$. Since $(y)^\star \subseteq (x,y)$ by Lemma 3.2.12 we have $(y^n)^\star = (x,y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (x,y^n)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. However $(y^n)^\star \neq (x,y^n)$ for all $n \in \mathbb{N}$ since these ideals are not $\star$ closed.

(8aaaa) Suppose $(y^n)^\star = (x,y)$ for all $n \in \mathbb{N}$. The inclusion $(x,y) = (y^n)^\star = (y^n,z^r)^\star \subseteq (x,y,z^r)$ implies $(y^n,z^r)^\star = (x,y,z^r)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x,y,z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x,y,z^2)$ by Lemma 3.2.12 $(z^r)^\star = (x,y,z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x,z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since the ideals (x,z^r) and (y,z^r) are not $\star$ closed it must be the case $(z^r)^\star = (x,y,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8aaaaa) Suppose $(z^r)^\star = (x,y,z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(8aaaaaa) Suppose $(0)^\star$ isa proper nonzero monomial ideal. Since $(0)^\star \subseteq (x,y)$ the only $\star$ closed monomial ideal $(0)^\star$ could be is (x,y). Thus $(0)^\star = (x,y)$. This is $\star_8$.

(8aaaaab) Suppose $(0)^\star = (0)$. This is $\star_{8.1}$.

(8aaaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.2}$.

(8aaab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. The inclusion $(y) = (y^2)^\star = (y^2, z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 imply $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aaaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since (x, z^r) and (y, z^r) are not $\star$ closed for all $r \in \mathbb{N}$ it must be the case that $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8aaabaa) Suppose $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$

(8aaabaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (y)$, $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (y)$. This is $\star_{8.3}$.

(8aaabaab) Suppose $(0)^\star = (0)$. This is $\star_{8.4}$.

(8aaabab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.5}$.

(8aaabb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (y, z^2)$ by Lemma 3.2.12 $(z^r)^\star \subseteq (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star \subseteq (z^r)$ for all $r \in \mathbb{N}$.

(8aaabba) Suppose $(z^r)^\star \subseteq (y, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$

(8aaabbaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (y)$, $(0)^\star = (y^j)$ for some $j \in \mathbb{N}$. By Lemma 3.2.7, $(0)^\star = (y)$. This is $\star_{8.6}$.

(8aaabbab) Suppose $(0)^\star = (0)$. This is $\star_{8.7}$.

(8aaabbb) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.8}$.

(8aaac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. The inclusion $(y^2, z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 imply $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. However $(y^n, z^r)^\star \neq (x, y^n, z^r)$ since (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$. This leaves only the three cases.

(8aaaca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Since (x, z^r) and (y, z^r) are not $\star$ closed for all $r \in \mathbb{N}$ we are left with $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.9}$ and $\star_{8.10}$.

(8aaacb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (y, z^2)$ by Lemma 3.2.12 $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.11}$ and $\star_{8.12}$.

(8aaacc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.13}$ and $\star_{8.14}$.

(8aab) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. The inclusion $(x) = (x^m)^\star \subseteq (x^m, z^r)^\star \subseteq (x, y, z^r)$ and Lemma 3.2.12 implies $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$

or $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$.

(8aaba) Suppose $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aaa) $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8aabaa) Suppose $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$. Similar to case (8aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8aabaaa) Suppose $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(8aabaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subsetneq (x)$, $(0)^\star = (x^j)$ for some $j \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. This is $\star_{8.15}$.

(8aabaaab) Suppose $(0)^\star = (0)$. This is $\star_{8.16}$.

(8aabaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.17}$.

(8aabab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Since $(0)^\star \subsetneq (x) \cap (y) = (0)$ we have $(0)^\star = (0)$. Since $(y) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 either $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aababa) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 imply $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since (x, z^r) and (y, z^r) are not $\star$ closed for all $r \in \mathbb{N}$ we must have either $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.18}$ and $\star_{8.19}$.

(8aababb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1

$(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (y, z^2)$ and Lemma 3.2.12 imply $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.20}$ and $\star_{8.21}$.

(8aabac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 we have $(0)^\star = (0)$. Since $(y^2, z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 either $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Since (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$ we have only the three cases.

(8aabaca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aababa) $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.22}$ and $\star_{8.23}$.

(8aabacb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (8aababb) $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.24}$ and $\star_{8.25}$.

(8aabacc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 we have $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. So we have $\star_{8.26}$ and $\star_{8.27}$.

(8aabb) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 Similar to case (8aaa) $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8aabba) Suppose $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$. Similar to case (8aabaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x, z^2)$, by Lemma 3.2.12 either $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8aabbaa) Suppose $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(8aabbaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to (8aabaaaa) $(0)^\star = (x)$. This is $\star_{8.28}$.

(8aabbaab) Suppose $(0)^\star = (0)$. This is $\star_{8.29}$.

(8aabbab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.30}$.

(8aabbb) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Similar to (8aabab) $(0)^\star = (0)$ and $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aabbba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. Since $(z^2)^\star \subseteq (x, z^2)$, by Lemma 3.2.12 $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.31}$ and $\star_{8.32}$.

(8aabbbb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.33}$.

(8aabbc) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (8aabac) $(0)^\star = (0)$ and either $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8aabbca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aabbba) $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and$(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.34}$ and $\star_{8.35}$.

(8aabbcb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aabbbb) $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.36}$.

(8aabbcc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.37}$ and $\star_{8.38}$.

(8aac) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Since $(x^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, y^n, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. However (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$ so we are left with the remaining three cases.

(8aaca) Suppose $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aaa) $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8aacaa) Suppose $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$. Similar to case (8aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.39}$ and $\star_{8.40}$.

(8aacab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. The inclusion $(y) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 implies $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aacaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aababa) we have $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we get $\star_{8.41}$ and $\star_{8.42}$.

(8aacabb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (y, z^2)$ and Lemma 3.2.12 imply $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.43}$ and $\star_{8.44}$.

(8aacac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Since $(y^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. However (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$ so we are left with only the three cases.

(8aacaca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aaaa) $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.45}$ and $\star_{8.46}$.

(8aacacb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aacabb) we have $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This give us $\star_{8.47}$ and $\star_{8.48}$.

(8aacacc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.49}$ and $\star_{8.50}$.

(8aacb) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aaa) $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8aacba) Suppose $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$. Similar to case (8aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (x, z^2)$ and Lemma 3.2.12 imply $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Here we have $\star_{8.51}$ and $\star_{8.52}$.

(8aacbb) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. Similar to case (8aabab) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aacbba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aacba)

$(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.53}$ and $\star_{8.54}$.

(8aacbbb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.55}$.

(8aacbc) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aacac) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8aacbca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aacba) $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.56}$ and $\star_{8.57}$.

(8aacbcb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aacbbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.58}$.

(8aacbcc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to (8aacbbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.59}$ and $\star_{8.60}$.

(8aacc) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(8aacca) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Similar to case (8aaa) $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$, $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8aaccaa) Suppose $(y^n)^\star = (x, y)$ for all $n \in \mathbb{N}$. Similar to case (8aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.61}$.

(8aaccab) Suppose $(y^n)^\star = (y)$ for all $n \in \mathbb{N}$. The inclusion $(y) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 imply $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$.

(8aaccaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.62}$.

(8aaccabb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.63}$.

(8aaccac) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (8aabac) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8aaccaca) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.64}$.

(8aaccacb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.65}$.

(8aaccacc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. So we have $\star_{8.66}$ and $\star_{8.67}$.

(8aaccb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aacca) we get $\star_{8.68}$ through $\star_{8.74}$.

(8ab) Suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(8aba) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. The inclusion $(x) = (x^2)^\star \subseteq (x^2, z^2)^\star \subseteq (x, y, z^2)$ and Lemma 3.2.12 imply $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$

or $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$.

(8abaa) Suppose $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Since $(y^2)^\star \subseteq (x, y^2)$, by Lemma 3.2.12 $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8abaaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Since $(x, y^2) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. However the latter is a contradiction since (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x, y, z^2)$ Lemma 3.2.12 implies $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since (x, z^r) and (y, z^r) are not $\star$ closed for all $r \in \mathbb{N}$ we have $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8abaaaa) Suppose $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(8abaaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (x)$, $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. This is $\star_{8.75}$.

(8abaaaab) Suppose $(0)^\star = (0)$. This is $\star_{8.76}$.

(8abaaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.77}$.

(8abaab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.By Lemma 3.2.8 $(0)^\star = (0)$. Since $(y^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. However since (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$ we are left with only three cases.

(8abaaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1

$(y^n+az^r)^\star = (x,y,z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x,y,z^2)$, by Lemma 3.2.12 $(z^r)^\star = (x,y,z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x,z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. However since (x,z^r) and (y,z^r) are not $\star$ closed for all $r \in \mathbb{N}$ we are left with $(z^r)^\star = (x,y,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.78}$ and $\star_{8.79}$.

(8abaabb) Suppose $(y^n,z^r)^\star = (y,z^r)$ for all $n,r \in \mathbb{N}$. By Claim 3.3.1 $(y^n+az^r)^\star = (y,z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (y,z^2)$, by Lemma 3.2.12 $(z^r)^\star = (y,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.80}$ and $\star_{8.81}$.

(8abaabc) Suppose $(y^n,z^r)^\star = (y^n,z^r)$ for all $n,r \in \mathbb{N}$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n+az^r)^\star = (y^n,z^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n+az^r)^\star = (y^n+az^r)$ for all $n,r \in \mathbb{N}$ and $a \in k^\times$. So we have $\star_{8.82}$ and $\star_{8.83}$.

(8abab) Suppose $(x^m,z^r)^\star = (x,z^r)$ for all $m,r \in \mathbb{N}$. By Claim 3.3.1 $(x^m+az^r)^\star = (x,z^r)$ for all $m,r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaa) $(y^n)^\star = (x,y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8ababa) Suppose $(y^n)^\star = (x,y^n)$ for all $n \in \mathbb{N}$. Since $(x,y^2) = (y^2)^\star \subseteq (y^2,z^2)^\star \subseteq (x,y,z^2)$ by Lemma 3.2.12 $(y^n,z^r)^\star = (x,y,z^r)$ for all $n,r \in \mathbb{N}$ or $(y^n,z^r)^\star = (x,y^n,z^r)$ for all $n,r \in \mathbb{N}$. However (x,y^n,z^r) is not $\star$ closed for all $n,r \in \mathbb{N}$. Thus $(y^n,z^r)^\star = (x,y,z^r)$ for all $n,r \in \mathbb{N}$. The inclusion $(z^2)^\star \subseteq (x,z^2)$ and Lemma 3.2.12 implies $(z^r)^\star = (x,z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(8ababaa) Suppose $(z^r)^\star = (x,z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(8ababaaa) Suppose $(0)^\star$ isa proper nonzero monomial ideal. Since $(0)^\star \subseteq (x)$, $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. This is $\star_{8.84}$.

(8abababaab) Suppose $(0)^\star = (0)$. This is $\star_{8.85}$.

(8abababab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{8.86}$.

(8abababb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Since $(y^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 we have the following possible closures: $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. We can eliminate (x, y^n, z^r) since it is not $\star$ closed for all $n, r \in \mathbb{N}$.

(8abababba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (x, z^2)$ and Lemma 3.2.12 implies $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.87}$ and $\star_{8.88}$.

(8abababbb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.89}$.

(8abababbc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8abababbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{8.90}$ and $\star_{8.91}$.

(8abb) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Since $(x^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 we have the following possible closures $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, y, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. However (x^m, y, z^r) is not $\star$ closed for all $m, r \in \mathbb{N}$. This leaves only three cases.

(8abba) Suppose $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1

$(x^m + az^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8abbaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to (8abaaa) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.92}$.

(8abbab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (8abaab) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8abbaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaaba) $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.93}$ and $\star_{8.94}$.

(8abbabb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaabb) $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.95}$ and $\star_{8.96}$.

(8abbabc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8abaabc) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{8.97}$ and $\star_{8.98}$.

(8abbb) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8abbba) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to (8abaaa) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. The inclusion $(z^2)^\star \subseteq (x, z^2)$ and Lemma 3.2.12 imply $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.99}$ and

$\star_{8.100}$.

(8abbbb) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. Similar to case (8abaab) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8abbbba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abbba) $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This gives us $\star_{8.101}$ and $\star_{8.102}$.

(8abbbbb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8aabbbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.103}$.

(8abbbbc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (8aabbbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.104}$ and $\star_{8.105}$.

(8abbc) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(8abbca) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(8abbcaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Since $(y) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. However the latter is not possible since (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$. This is $\star_{8.106}$.

(8abbcab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (8aabac) $(y^n, z^r)^\star =$

(x, y, z^r) for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$, or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Thus we have $\star_{8.107}$ and $\star_{8.108}$.

(8abbcabc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{8.109}$ and $\star_{8.110}$.

(8abbcb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abbca) we get $\star_{8.111}$ through $\star_{8.115}$.

(8ac) Suppose $(x^m, y^n)^\star = (x^m, y)$ for all $m, n \in \mathbb{N}$. Similar to case (8ab) by exchanging the roles of x and y we get $\star_{8.116}$ through $\star_{8.156}$.

(8ad) Suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Claim 3.4.6 implies $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similarly $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

(8ada) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Since $(x^2, z^2)^\star \subseteq (x, y, z^2)$, by Lemma 3.2.12 we have the following possible closures: $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x^m, y, z^r)$ for all $m, r \in \mathbb{N}$, $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. However since (x^m, y, z^r) is not $\star$ closed for all $m, r \in \mathbb{N}$ we are left with only the three cases.

(8adaa) Suppose $(x^m, z^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, y, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Since $(y^2, z^2)^\star \subseteq (x, y, z^2)$ by Lemma 3.2.12 $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. However (x, y^n, z^r) is not $\star$ closed for all $n, r \in \mathbb{N}$ which leaves us with only the three cases.

(8adaaa) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. The inclusion $(z^2)^\star \subseteq$

(x, y, z^2) and Lemma 3.2.12 implies $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$, $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Since (x, z^r) and (y, z^r) are not $\star$ closed for all $r \in \mathbb{N}$ we have $(z^r)^\star = (x, y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.157}$ and $\star_{8.158}$.

(8adaab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(z^2)^\star \subseteq (y, z^2)$ and Lemma 3.2.12 implies $(z^r)^\star = (y, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{8.159}$ and $\star_{8.160}$.

(8adaac) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{8.161}$ and $\star_{8.162}$.

(8adab) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8abaab) we have $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8adaba) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8ababa) $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{8.163}$ and $\star_{8.164}$.

(8adabb) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8ababbb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. This is $\star_{8.165}$.

(8adabc) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{8.166}$ and $\star_{8.167}$.

(8adac) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(8adaca) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8ababb) $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(8adacaa) Suppose $(y^n, z^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (x, y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.168}$.

(8adacab) Suppose $(y^n, z^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.3.1 $(y^n + az^r)^\star = (y, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{8.169}$.

(8adacac) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{8.170}$ and $\star_{8.171}$.

(8adacb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8adaca) we get $\star_{8.172}$ through $\star_{8.175}$.

(8adb) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (8ada) we get $\star_{8.176}$ through $\star_{8.194}$.

(8b) Suppose $(x^m, y^n, z^r)^\star = (x, y^n, z)\, \forall m, n, r \in \mathbb{N}$. This case is similar to case (8a) with the roles of y and z exchanged. We get $\star_{8.195}$ to $\star_{8.389}$.

(8c) Suppose $(x^m, y^n, z^r)^\star = (x^m, y, z)\, \forall m, n, r \in \mathbb{N}$. This case is similar to case (8a) with the roles of x and z exchanged. We get $\star_{8.390}$ to $\star_{8.584}$.

For cases (9a), (9b) and (9c) we need the following claim.

Claim 3.4.7. *Suppose $\star$ is a standard closure on R_3 with $i_j \in \{1, 2, 3\}$ and $i_\alpha \neq i_\beta$ for $\alpha \neq \beta$ and that $(x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}}, x_{i_3}^{r_{i_3}})^\star = (x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}}, x_{i_3})$ for all $(r_{i_1}, r_{i_2}, r_{i_3}) \in \mathbb{N}^3$ then* *either*

1. $(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}}, x_{i_2}^{r_{i_2}}, x_{i_3})$ *for all* $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ *and* $a \in k^\times$ *or*
2. $(x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}})^\star = (x_{i_1}^{r_{i_1}} + a x_{i_2}^{r_{i_2}}, x_{i_3})$ *for all* $(r_{i_1}, r_{i_2}) \in \mathbb{N}^2$ *and* $a \in k^\times$

Proof. The proof is nearly identical to that of Claim 3.4.5. □

(9a) Suppose $(x^m, y^n, z^r)^\star = (x, y^n, z^r)\, \forall m, n, r \in \mathbb{N}$. The inclusion

$$(x, y^{n+1}, z^{r+1}) = (x^{m+1}, y^{n+1}, z^{r+1}) \subseteq (x^m + ay^n, z^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x, y^n, z^r)$$

implies $(x^m + ay^n, z^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similarly $(x^m + az^r, y^n)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ and $(x^m + az^r, y^n + bz^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. The inclusion

$$(x, y^{n+1}, z^{r+1}) = (x^{m+1}, y^{n+1}, z^{r+1}) \subseteq (x^m, y^n + az^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x, y^n, z^r)$$

implies that for each $i, j, l \in \mathbb{N}$ and $b \in k^\times$ either $(x^i, y^j + bz^l)^\star = (x, y^j, z^l)$ or $(x^i, y^j + bz^l)^\star = (x, y^j + bz^l)$. If we suppose the latter for some $i, j, l \in \mathbb{N}$ and $b \in k^\times$, by Lemma 3.4.1 $(x^m, y^n + az^r)^\star = (x, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Thus $(x^m, y^n + az^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m, y^n + az^r)^\star = (x, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$.

(9aa) Suppose $(x^m, y^n + az^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion

$$(x, y^{n+1}, z^{r+1}) = (x^{m+1}, y^{n+1}, z^{r+1}) \subseteq (x^m + ay^n + bz^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x, y^n, z^r)$$

and Lemma 3.4.1 imply either $(x^m + ay^n + bz^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n + bz^r)^\star = (x, y^n + \frac{b}{a} z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. However $(x, y^n + \frac{b}{a} z^r)$ is not $\star$ closed for all $m, n, r \in \mathbb{N}$. Thus $(x^m + ay^n + bz^r)^\star = (x, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$. The inclusion $(x^m)^\star \subseteq \cap_{n=1}^{\infty} \cap_{r=1}^{\infty} (x^m, y^n, z^r)^\star = \cap_{n=1}^{\infty} \cap_{r=1}^{\infty} (x, y^n, z^r) = (x)$ and Claim 3.4.6 implies $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(9aaa) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. The inclusion $(x) = (x^m)^\star \subseteq (x^m, y^n)^\star \subseteq \cap_{r=1}^{\infty}(x^m, y^n, z^r)^\star = \cap_{r=1}^{\infty}(x, y^n, z^r) = (x, y^n)$ implies $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similarly $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Since $(y^2)^\star \subseteq (x, y^2)$, by Lemma 3.2.12 $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9aaaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. The inclusion $(x, y^2) = (y^2)^\star \subseteq (y^2, z^2)^\star \subseteq (x, y^2, z^2)$ and Lemma 3.2.12 $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.7 $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. However $(x, y^n + az^r)$ is not $\star$ closed for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(z^2)^\star \subseteq (x, z^2)$, by Lemma 3.2.12 $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(9aaaaa) Suppose $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(9aaaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Since $(0)^\star \subseteq (x)$, $(0)^\star = (x^i)$ for some $i \in \mathbb{N}$. By Lemma 3.2.7 $(0)^\star = (x)$. This is $\star_9$.

(9aaaaab) Suppose $(0)^\star = (0)$. This is $\star_{9.1}$.

(9aaaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{9.2}$.

(9aaab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. The inclusion $(y^2, z^2)^\star \subseteq (x, y^2, z^2)$ and Lemma 3.2.12 imply either $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9aaaba) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$

or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(9aaabaa) Suppose $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$. Similar to case (9aaaaa) $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(9aaabaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to case (9aaaaaa) $(0)^\star = (x)$. This is $\star_{9.3}$.

(9aaabaab) Suppose $(0)^\star = (0)$. This is $\star_{9.4}$.

(9aaabab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{9.5}$.

(9aaabb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{9.6}$ and $\star_{9.7}$.

(9aab) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. The inclusion $(x^m, y^n)^\star \subseteq \cap_{r=1}^{\infty}(x^m, y^n, z^r)^\star = \cap_{r=1}^{\infty}(x, y^n, z^r) = (x, y^n)$ and Lemma 3.2.12 implies $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

(9aaba) Suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, z^r)^\star \subseteq \cap_{n=1}^{\infty}(x^m, y^n, z^r)^\star = \cap_{n=1}^{\infty}(x, y^n, z^r) = (x, z^r)$ and Lemma 3.2.12 implies $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(9aabaa) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9aabaaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$

for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{9.8}$ and $\star_{9.9}$.

(9aabaab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to (9aaab) we have either $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9aabaaba) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{9.10}$ and $\star_{9.11}$.

(9aabaabb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to (9aaabb) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us $\star_{9.12}$ and $\star_{9.13}$.

(9aabab) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Claim 3.4.6 implies $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(9aababa) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9aababaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.14}$.

(9aababab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9aabababa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.15}$.

(9aabababb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{9.16}$ and $\star_{9.17}$.

(9aababb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aababa) we get $\star_{9.18}$ through $\star_{9.21}$.

(9aabb) Suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Claim 3.4.6 implies $(y^n)^\star = (y^n)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

(9aabba) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaba) $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(9aabbaa) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9aabbaaa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{9.22}$ and $\star_{9.23}$.

(9aabbaab) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.6 $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{9.24}$ and $\star_{9.25}$.

(9aabbab) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Similar to case (9aabab) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(9aabbaba) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9aabbabaa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaaa) $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.26}$.

(9aabbabab) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. By Claim 3.4.5 $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Thus we have $\star_{9.27}$ and $\star_{9.28}$.

(9aabbabb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aabbaba) we get $\star_{9.29}$ through $\star_{9.31}$.

(9aabbb) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aabba) we get $\star_{9.32}$ through $\star_{9.41}$.

(9ab) Suppose $(x^m, y^n + az^r)^\star = (x, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion

$$\begin{aligned}(x, y^{n+1}, z^{r+1}) = (x^{m+1}, y^{n+1}, z^{r+1}) &\subseteq (x^m + ay^n + bz^r)^\star \\ &\subseteq (x^m, y^n + \frac{b}{a}z^r)^\star = (x, y^n + \frac{b}{a}z^r)\end{aligned}$$

implies $(x^m + ay^n + bz^r)^\star = (x, y^n + \frac{b}{a}z^r)$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similar to case (9aa) $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$ or $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$.

(9aba) Suppose $(x^m)^\star = (x)$ for all $m \in \mathbb{N}$. Similar to case (9aaa) we have $(x^m + ay^n)^\star = (x^m, y^n)^\star = (x, y^n)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$ and $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9abaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaa) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$, $(y^n + az^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. However the former implies

$$(x, y^n, z^r) = (y^n + az^r)^\star \subseteq (x^m, y^n + az^r)^\star = (x, y^n + az^r)$$

which is a contradiction. Thus $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. Also similar to case (9aaaa) $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(9abaaa) Suppose $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.6 $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(9abaaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to case (9aaaaa) $(0)^\star = (x)$. This is $\star_{9.42}$.

(9abaaab) Suppose $(0)^\star = (0)$. This is $\star_{9.43}$.

(9abaab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{9.44}$.

(9abab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9ababa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.

(9ababaa) Suppose $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$. Similar to case (9abaaa) $(0)^\star$ is a proper nonzero monomial ideal or $(0)^\star = (0)$.

(9ababaaa) Suppose $(0)^\star$ is a proper nonzero monomial ideal. Similar to case (9aaaaaa) $(0)^\star = (x)$. This is $\star_{9.45}$.

(9ababaab) Suppose $(0)^\star = (0)$. This is $\star_{9.46}$.

(9ababab) Suppose $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Lemma 3.2.8 $(0)^\star = (0)$. This is $\star_{9.47}$.

(9ababb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aaabb) $(0)^\star = (0)$, $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. If we suppose the former then $(y^n, z^r) = (y^n + az^r)^\star \subseteq (x, y^n + az^r)^\star = (x, y^n + az^r)$, a contradiction. Thus $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.48}$.

(9abb) Suppose $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similar to case (9aab) $(0)^\star = (0)$ and $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ or $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$.

(9abba) Suppose $(x^m, y^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$. Similar to case (9aaba) $(x^m + ay^n)^\star = (x, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ and $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(9abbaa) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, n \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9abbaaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (y^n + az^r)$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. So we have $\star_{9.49}$ and $\star_{9.50}$.

(9abbaab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to (9aaab) we have either $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9abbaaba) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. Thus we have $\star_{9.51}$ and $\star_{9.52}$.

(9abbaabb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to (9ababb) $(z^r)^\star = (z^r)$ and $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.53}$.

(9abbab) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Similar to case (9aabab) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$ and $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(9abbaba) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaa) $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$ or $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$.

(9abbabaa) Suppose $(y^n)^\star = (x, y^n)$ for all $n \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$, $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.54}$.

(9abbabab) Suppose $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9abbababa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.55}$.

(9abbababb) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9ababb) $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.56}$.

(9abbabb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9abbaba) we get $\star_{9.57}$ through $\star_{9.59}$.

(9abbb) Suppose $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Similar to case (9aabb) $(y^n)^\star = (y^n)$ for all $r \in \mathbb{N}$ and $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$.

(9abbba) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaba) $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ or $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$.

(9abbbaa) Suppose $(x^m, z^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$. By Claim 3.3.1 $(x^m + az^r)^\star = (x, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9abbbaaa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ and $(z^r)^\star = (x, z^r)$ for all $r \in \mathbb{N}$ or $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$.This gives us $\star_{9.60}$ and $\star_{9.61}$.

(9abbbaab) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9aabbaab) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. If we suppose the former then $(y^n, z^r) = (y^n + az^r)^\star \subseteq (x, y^n + az^r)^\star = (x, y^n + az^r)$ which is a contradiction. Thus $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.62}$.

(9abbbab) Suppose $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$. Similar to case (9aabab) $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. By Claim 3.4.5 $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$.

(9abbbaba) Suppose $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9aaab) $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$ or $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$.

(9abbbabaa) Suppose $(y^n, z^r)^\star = (x, y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abaa) $(y^n + az^r)^\star = (x, y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.63}$.

(9abbbabab) Suppose $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Similar to case (9abbbaab) $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{9.64}$.

(9abbbabb) Suppose $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9abbbaba) we get $\star_{9.65}$ and $\star_{9.66}$.

(9abbbb) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$. Similar to case (9abbba) we get $\star_{9.67}$ through $\star_{9.73}$.

(9b) Suppose $(x^m, y^n, z^r)^\star = (x^n, y, z^r)\,\forall m, n, r \in \mathbb{N}$. This case is similar to case (9a) with the roles of x and y exchanged. We get $\star_{9.74}$ through $\star_{9.147}$.

(9c) Suppose $(x^m, y^n, z^r)^\star = (x^n, y^n, z)\,\forall m, n, r \in \mathbb{N}$. This case is similar to case (9a) with the roles of x and z exchanged. We get $\star_{9.148}$ through $\star_{9.222}$.

(10) Suppose $(x^m, y^n, z^r)^\star = (x^m, y^n, z^r)\,\forall m, n, r \in \mathbb{N}$. The inclusion $(x^m)^\star \subseteq \bigcap_{n=1}^\infty \bigcap_{r=1}^\infty (x^m, y^n, z^r)^\star = \bigcap_{n=1}^\infty \bigcap_{r=1}^\infty (x^m, y^n, z^r) = (x^m)$ implies $(x^m)^\star = (x^m)$ for all $m \in \mathbb{N}$. Similarly we get $(y^n)^\star = (y^n)$ for all $n \in \mathbb{N}$ and $(z^r)^\star = (z^r)$ for all $r \in \mathbb{N}$. The inclusion $(x^m, y^n)^\star \subseteq \bigcap_{r=1}^\infty (x^m, y^n, z^r)^\star = \bigcap_{r=1}^\infty (x^m, y^n, z^r) = (x^m, y^n)$ implies $(x^m, y^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$. Similarly we have $(x^m, z^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $(y^n, z^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$. Now we need to determine the closures of the non-monomial ideals.

By Claim 3.4.5 $(x^m + ay^n)^\star = (x^m, y^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + ay^n)^\star = (x^m + ay^n)$ for all $m, n \in \mathbb{N}$ and $a \in k^\times$, $(x^m + az^r)^\star = (x^m, z^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m + az^r)^\star = (x^m + az^r)$ for all $m, r \in \mathbb{N}$ and $a \in k^\times$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$ or $(y^n + az^r)^\star = (y^n + az^r)$ for all $n, r \in \mathbb{N}$ and $a \in k^\times$. This gives us seven cases.

(10a) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$, $(x^m + az^r)^\star = (x^m, z^r)$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n) = (x^m + ay^n)^\star \subseteq (x^m + ay^n, z^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^n, y^n, z^r)$ implies $(x^m + ay^n, z^r)^\star = (x^n, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similarly $(x^m + az^r, y^n)^\star = (x^m, y^n + az^r)^\star$ $= (x^m + az^r, y^n + bz^r)^\star = (x^n, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Since $(x^2 + y^2 + z^2)^\star \subseteq (x^2 + y^2, z^2)^\star = (x^2, y^2, z^2)$ by Lemma 3.4.1 we have either $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(10aa) Suppose $(x^m + ay^n + bz^r)^\star = (x^n, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10}$.

(10ab) Suppose $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.1}$.

(10b) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$, $(x^m + az^r)^\star = (x^m, z^r)$ and $(y^n + az^r)^\star = (y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n) = (x^m + ay^n)^\star \subseteq (x^m + ay^n, z^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m + ay^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similarly $(x^m + az^r, y^n)^\star = (x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This leaves only ideals of the form $(x^m + ay^n + bz^r)$ and $(x^m, y^n + az^r)$ for $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(x^2, y^2 + z^2)^\star \subseteq (x^2, y^2, z^2)^\star = (x^2, y^2, z^2)$ by Lemma 3.4.1 we have either $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$ or $(x^m, y^n + az^r)^\star = (x^m, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$.

(10ba) Suppose $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (10a) $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(10baa) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.2}$.

(10bab) Suppose $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.3}$.

(10bb) Suppose $(x^m, y^n + az^r)^\star = (x^m, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Since $(x^2 + y^2 + z^2)^\star \subseteq (x^2, y^2 + z^2)^\star = (x^2, y^2 + z^2)$ by Lemma 3.4.1 we have either $(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a}z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(10bba) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a}z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.4}$.

(10bbb) Suppose $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.5}$.

(10c) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$, $(x^m + az^r)^\star = (x^m + az^r)$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (10b) we get $\star_{10.6}$ through $\star_{10.9}$.

(10d) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$, $(x^m + az^r)^\star = (x^m, z^r)$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (10b) we get $\star_{10.10}$ through $\star_{10.13}$.

(10e) Suppose $(x^m + ay^n)^\star = (x^m, y^n)$, $(x^m + az^r)^\star = (x^m + az^r)$ and $(y^n + az^r)^\star = (y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n) = (x^m + ay^n)^\star \subseteq (x^m + ay^n, z^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m + ay^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n) = (x^m - \frac{a}{b}y^n)^\star$
$\subseteq (x^m + az^r, y^n + bz^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This leaves only ideals of the form

$(x^m + ay^n + bz^r)$, $(x^m + az^r, y^n)$ and $(x^m, y^n + az^r)$ for $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Since $(x^2 + y^2 + z^2)^\star \subseteq (x^2 + y^2, z^2)^\star = (x^2, y^2, z^2)$ by Lemma 3.4.1 we have either $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$, $(x^m + ay^n + bz^r)^\star = (x^m + bz^r, y^n)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a} z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(10ea) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. The inclusion $(x^m, y^n, z^r) = (x^m + y^n + az^r)^\star \subseteq (x^m + az^r, y^n)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ imlpies $(x^m + az^r, y^n)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similarly $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.14}$.

(10eb) Suppose $(x^m + ay^n + bz^r)^\star = (x^m + bz^r, y^n)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Thus $(x^m + az^r, y^n)$ is $\star$ closed for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m + az^r, y^n) = (x^m + y^n + az^r)^\star \subseteq (x^m, y^n + az^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{10.15}$.

(10ec) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a} z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Thus $(x^m, y^n + az^r)$ is $\star$ closed for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n + az^r) = (x^m + y^n + az^r)^\star \subseteq (x^m + az^r, y^n)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m + az^r, y^n)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. This is $\star_{10.16}$.

(10f) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$, $(x^m + az^r)^\star = (x^m, z^r)$ and $(y^n + az^r)^\star = (y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (10e) we get $\star_{10.17}$ through $\star_{10.19}$.

(10g) Suppose $(x^m + ay^n)^\star = (x^m + ay^n)$, $(x^m + az^r)^\star = (x^m + az^r)$ and $(y^n + az^r)^\star = (y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. Similar to (10e) we get $\star_{10.20}$ through $\star_{10.22}$.

For case (10h) we need the following claim.

Claim 3.4.8. *Suppose $\star$ is a standard closure operation on R_3 and that $(x^m, y^n, z^r)^\star$*

$= (x^m, y^n, z^r)$ *for all* $m, n, r \in \mathbb{N}$. *If* $(x^{m_1} + cz^{m_3}, y^{m_2} + dz^{m_3})^\star = (x^{m_1}, y^{m_2}, z^{m_3})$ *for some* $(m_1, m_2, m_3) \in \mathbb{N}^3$ *and* $c, d \in k^\times$ *then* $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ *for all* $m, n, r \in \mathbb{N}$ *and* $a, b \in k^\times$.

Proof. Let $a, b \in k^\times$. Since $\star$ is weakly prime we have

$$(x + \frac{ad}{bc}y + \frac{a}{c})(x^{m_1} + cz^{m_3}, y^{m_2} + dz^{m_3})^\star \subseteq ((x + \frac{ad}{bc}y + \frac{a}{c})(x^{m_1} + cz^{m_3}, y^{m_2} + dz^{m_3}))^\star$$

$$\Rightarrow (x^{m_1+1}, y^{m_2+1}, z^{m_3+1}) \subseteq (x^{m_1+1} + az^{m_3+1}, y^{m_2+1} + bz^{m_3+1})^\star$$

$$\Rightarrow (x^{m_1+1} + az^{m_3+1}, y^{m_2+1} + bz^{m_3+1})^\star = (x^{m_1+1}, y^{m_2+1}, z^{m_3+1})^\star$$

$$= (x^{m_1+1}, y^{m_2+1}, z^{m_3+1}).$$

Inductively we have for all $v \in \mathbb{N}$,

$$(x^{m_1+v} + az^{m_3+v}, y^{m_2+v} + bz^{m_3+v})^\star = (x^{m_1+v}, y^{m_2+v}, z^{m_3+v}).$$

Then similar to Lemma 3.4.1 (1) we have the result for all exponents. Thus $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This concludes the proof of the claim. □

(10h) Suppose $(x^m+ay^n)^\star = (x^m+ay^n)$, $(x^m+az^r)^\star = (x^m+az^r)$ and $(y^n+az^r)^\star = (y^n+az^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. We need to determine the closures of ideals of the form $(x^m + ay^n + bz^r)$, $(x^m + az^r, y^n + bz^r)$, $(x^m, y^n + az^r)$, $(x^m + az^r, y^n)$ and $(x^m + ay^n, z^r)$ for $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Since $(x^2 + y^2 + z^2)^\star \subseteq (x^2, y^2, z^2)$ we have the following six cases:
by Lemma 3.4.1

$$(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)\, \forall m, n, r \in \mathbb{N}, a, b \in k^\times$$

$$(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a}z^r)\, \forall m, n, r \in \mathbb{N}, a, b \in k^\times$$

$$(x^m + ay^n + bz^r)^\star = (x^m + bz^r, y^n)\, \forall m, n, r \in \mathbb{N}, a, b \in k^\times$$

$$(x^m + ay^n + bz^r)^\star = (x^m + ay^n, z^r)\,\forall m, n, r \in \mathbb{N}, a, b \in k^\times$$

$$(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)\,\forall m, n, r \in \mathbb{N}, a, b \in k^\times$$

or by Lemma 3.4.2 $k = \mathbb{Z}/3\mathbb{Z}$ and $(x^m + y^n + z^r)^\star = (x^m + 2z^r, y^n + 2z^r)$, $(x^m + y^n + 2z^r)^\star = (x^m + z^r, y^n + z^r)$, $(x^m + 2y^n + z^r)^\star = (x^m + 2z^r, y^n + z^r)$ and $(x^m + 2y^n + 2z^r)^\star = (x^m + z^r, y^n + 2z^r)$ for all $m, n, r \in \mathbb{N}$.

(10ha) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similar to case (10ea) we have $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n + az^r)^\star$ $= (x^m + az^r, y^n)^\star = (x^m + ay^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.23}$.

(10hb) Suppose $(x^m + ay^n + bz^r)^\star = (x^m, y^n + \frac{b}{a}z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Thus $(x^m, y^n + az^r)$ is $\star$ closed for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m, y^n + az^r) = (x^m + y^n + az^r)^\star \subsetneq (x^m + az^r, y^n)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m + az^r, y^n)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a \in k^\times$. This leaves ideals of the form $(x^m + az^r, y^n + bz^r)$ for $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. For each $i, j, l \in \mathbb{N}$ and $c, d \in k^\times$, $(x^i + cz^l, y^j + dz^l)^\star \subseteq (x^i, y^j, z^l)^\star = (x^i, y^j, z^l)$. Thus for each $i, j, l \in \mathbb{N}$ and $c, d \in k^\times$ either $(x^i + cz^l, y^j + dz^l)^\star = (x^i, y^j, z^l)$ or $(x^i + cz^l, y^j + dz^l)^\star = (x^i + cz^l, y^j + dz^l)$. If $(x^i + cz^l, y^j + dz^l)^\star = (x^i, y^j, z^l)$ for some $i, j, l \in \mathbb{N}$ and $c, d \in k^\times$ by Claim 3.4.8 $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Thus either $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

(10hba) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. This is $\star_{10.24}$.

(10hbb) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. By Lemma 3.4.3 this is only a standard closure when $k = \mathbb{Z}/2\mathbb{Z}$. This is $\star_{10.24b}$.

(10hc) Suppose $(x^m+ay^n+bz^r)^\star = (x^m+bz^r, y^n)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. Thus $(x^m + az^r, y^n)$ is $\star$ closed for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m + az^r, y^n) = (x^m + y^n + az^r)^\star \subseteq (x^m, y^n + az^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. Similarly $(x^m + ay^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (10hb) $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$ or $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$.

(10hca) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. This is $\star_{10.25}$.

(10hcb) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. By Lemma 3.4.3 this is only a standard closure when $k = \mathbb{Z}/2\mathbb{Z}$. This is $\star_{10.25b}$.

(10hd) Suppose $(x^m+ay^n+bz^r)^\star = (x^m+ay^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. Thus $(x^m + ay^n, z^r)$ is $\star$ closed for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. The inclusion $(x^m + ay^n, z^r) = (x^m + y^n + az^r)^\star \subseteq (x^m, y^n + az^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. Simlarly $(x^m + az^r, y^n)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a \in k^\times$. Similar to case (10hb) $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$ or $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$.

(10hda) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. This is $\star_{10.26}$.

(10hdb) Suppose $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$. By Lemma 3.4.3 this is only a standard closure when $k = \mathbb{Z}/2\mathbb{Z}$. This is $\star_{10.26b}$.

(10he) Suppose $k = \mathbb{Z}/3\mathbb{Z}$ and $(x^m + y^n + z^r)^\star = (x^m + 2z^r, y^n + 2z^r)$, $(x^m + y^n + 2z^r)^\star = (x^m + z^r, y^n + z^r)$, $(x^m + 2y^n + z^r)^\star = (x^m + 2z^r, y^n + z^r)$

and $(x^m + 2y^n + 2z^r)^\star = (x^m + z^r, y^n + 2z^r)$ for all $m, n, r \in \mathbb{N}$. The inclusion $(x^m + 2z^r, y^n + 2z^r) = (x^m + y^n + z^r)^\star \subseteq (x^m, y^n + z^r)^\star \subseteq (x^m, y^n, z^r)^\star = (x^m, y^n, z^r)$ implies $(x^m, y^n + z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$. Similarly we have the following: $(x^m, y^n + 2z^r)^\star = (x^m + z^r, y^n)^\star = (x^m + 2z^r, y^n)^\star = (x^m + y^n, z^r)^\star = (x^m + 2y^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$. This is $\star_{\mathbb{Z}/3\mathbb{Z}}$.

(10hf) Suppose $(x^m + ay^n + bz^r)^\star = (x^m + ay^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. We need to determine the closures of ideals of the form $(x^m + az^r, y^n + bz^r)$, $(x^m + ay^n, z^r)$, $(x^m + az^r, y^n)$ and $(x^m, y^n + az^r)$ for some $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Note that ideals of one of these forms are not contained in an ideal of another one of these forms. Thus their closures are independent of each other. Since $(x^m + az^r, y^n + bz^r)^\star \subseteq (x^m, y^n, z^r)$ similar to case (10hb) either $(x^m + az^r, y^n + bz^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + az^r, y^n + bz^r)^\star = (x^m + az^r, y^n + bz^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. The inclusion $(x^m, y^n + az^r)^\star \subseteq (x^m, y^n, z^r)$ and Lemma 3.4.1 imply $(x^m, y^n + az^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m, y^n + az^r)^\star = (x^m, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similarly $(x^m + az^r, y^n)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + az^r, y^n)^\star = (x^m + az^r, y^n)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. Similarly $(x^m + ay^n, z^r)^\star = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ or $(x^m + ay^n, z^r)^\star = (x^m + ay^n, z^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$. For the ideals $(x^m + az^r, y^n + bz^r)$, $(x^m + ay^n, z^r)$, $(x^m + az^r, y^n)$ and $(x^m, y^n + az^r)$ for all $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$ we have the following case:
All four are not $\star$ closed. This is $\star_{10.27}$.

Three are not $\star$ closed. Thus we have $\star_{10.28}$ through $\star_{10.31}$.

Two are not $\star$ closed. Thus we have $\star_{10.32}$ through $\star_{10.37}$.

One is not $\star$ closed. Thus we have $\star_{10.38}$ through $\star_{10.41}$.

All four are $\star$ closed. This is $\star_{10.42}$, the identity.

$\square$

Corollary 3.4.9. *If $|k| \geq 4$ then $|S_f(R_3)| = 1522$. If $k = \mathbb{Z}/3\mathbb{Z}$ then $|S_f(R_3)| = 1523$. If $k = \mathbb{Z}/2\mathbb{Z}$ then $|S_f(R_3)| = 1525$.*

Proof. By Theorem 3.4.4 and Lemma 1.2.8. $\square$

Chapter 4

Future Research

There are many questions remaining to be answered regarding standard closure operations on rings of small dimension. We found that under certain conditions the set of standard closure operations for a 2-dimensional ring is infinite. For a 2-dimensional Noetherian domain D we know that this is always the case. We would also like to determine if this is the case for non-domains. The rings S, R_2 and R_3 provide several examples of 1-dimensional rings over fields where the set of standard closure operations is finite. However, since it has been shown that 0-dimensional rings can have infinitely many standard closures we may find 1-dimensional rings over 0-dimensional rings, that are not fields, that have infinitely many standard closures. It needs to be determined under what conditions the set of standard closures is finite for 1-dimensional rings.

We determined that the ring S has six standard closure operations. Certainly we should investigate the standard closure operations on the ring $(k[x]/(x^n))[[t]]$ for $n > 2$. Possibly we can show that the size of this set is a function of n.

Counting the set of standard closures on R_t for $t > 3$ appears to be more difficult. The number of standard closure operations on R_t seems to be increasing rapidly as

t increases. For $t = 4$ we can establish a lower bound for the number of standard closure operations.

Corollary 4.0.10. *There are at least 5977 standard closure operations on* R_4*.*

Proof. The ring R_3 is isomorpic to the rings $R_4^{\hat{4}}$, $R_4^{\hat{3}}$, $R_4^{\hat{2}}$ and $R_4^{\hat{1}}$ (as defined in Lemma 3.2.4). By Theorem 3.4.4 each of these rings has 1522 standard closure operations. By Lemma 3.2.4 each of these standard closures induces a standard closure on R_4. However some of the closures from $R_4^{\hat{4}}$, $R_4^{\hat{3}}$, $R_4^{\hat{2}}$ and $R_4^{\hat{1}}$ induce the same closure operations on R_4. Let $\pi_{4,4} : R_3 \to R_4^{\hat{4}}$ be the isomorphism that sends $x \to x_1$, $y \to x_2$ and $z \to x_3$. Let $\pi_{4,3} : R_3 \to R_4^{\hat{4}}$ be the isomorphism that sends $x \to x_1$, $y \to x_2$ and $z \to x_4$. Define $\pi_{4,2}$ and $\pi_{4,1}$ similarly.

Since $I^{\star_1} = R_3$ for all ideals I the image of this closure under the maps $\pi_{4,4}$, $\pi_{4,3}$, $\pi_{4,2}$ and $\pi_{4,1}$ will be identical. The same is true for $\star_{1.1}$. So $\star_1$ and $\star_{1.1}$ induce a total of two standard closures on R_4.

The images of $\star_2$ under the composition of ϕ (as defined in Lemma 3.2.4) with the maps $\pi_{4,4}$, $\pi_{4,2}$ and $\pi_{4,1}$ yields the same closure on R_4: $(0)^{\star_2} = (x_3^r)^{\star_2} = (x_3)$ $\forall$ $r \in \mathbb{N}$, $I^{\star_2} = R_4$ $\forall$ other I. The image of $\star_2$ under the composition of ϕ with map $\pi_{4,3}$ is $(0)^{\star_2} = (x_4^s)^{\star_2} = (x_4)$ $\forall$ $s \in \mathbb{N}$, $I^{\star_2} = R_4$ $\forall$ other I. Thus $\star_2$ induces two distinct standard closures on R_4. The same is true for $\star_{2.1}$ through $\star_{2.8}$. Thus $\star_2$ through $\star_{2.8}$ induce a total of 18 standard closures on R_4.

The images of $\star_3$ under the composition of ϕ with the maps $\pi_{4,4}$ and $\pi_{4,1}$ yields the same closure on R_4: $(0)^{\star_3} = (0)$, $(x_2^n)^{\star_3} = (x_2)$, $(x_3^r)^{\star_3} = (x_3)$ $\forall$ $n, r \in \mathbb{N}$, $I^{\star_3} = R_3$ $\forall$ other I. The image of $\star_3$ under the composition of ϕ with the map $\pi_{4,3}$ gives the following closure on R_4: $(0)^{\star_3} = (0)$, $(x_2^n)^{\star_3} = (x_2)$, $(x_4^s)^{\star_3} = (x_4)$ $\forall$ $n, s \in \mathbb{N}$, $I^{\star_3} = R_3$ $\forall$ other I. And the image of $\star_3$ under the composition of ϕ with the map $\pi_{4,2}$ gives the closure on R_4: $(0)^{\star_3} = (0)$, $(x_4^s)^{\star_3} = (x_4)$, $(x_3^r)^{\star_3} = (x_3)$ $\forall$ $r, s \in \mathbb{N}$, $I^{\star_3} = R_3$ $\forall$ other I. Thus $\star_3$ induces three distinct standard closures on R_4. The same is true for $\star_{3.1}$ through $\star_{3.86}$. Thus $\star_3$ through $\star_{3.86}$ induce a total of 261 standard closures on R_4.

The image of $\star_4$ under the composition of ϕ with the map $\pi_{4,4}$ gives the following closure on R_4: $(0)^{\star_4} = (0)$, $(x_1^m)^\star = (x_1)$, $(x_2^n)^{\star_4} = (x_2)$, $(x_3^r)^{\star_4} = (x_3)$ $\forall\ m, n, r \in \mathbb{N}$, $I^{\star_4} = R_3$ $\forall$ other I. The image of $\star_4$ under the composition of ϕ with the map $\pi_{4,3}$ gives the following closure on R_4: $(0)^{\star_4} = (0)$, $(x_1^m)^\star = (x_1)$, $(x_2^n)^{\star_4} = (x_2)$, $(x_4^s)^{\star_4} = (x_4)$ $\forall\ m, n, s \in \mathbb{N}$, $I^{\star_4} = R_3$ $\forall$ other I. The image of $\star_4$ under the composition of ϕ with the map $\pi_{4,2}$ gives the following closure on R_4: $(0)^{\star_4} = (0)$, $(x_1^m)^\star = (x_1)$, $(x_4^s)^{\star_4} = (x_4)$, $(x_3^r)^{\star_4} = (x_3)$ $\forall\ m, n, s \in \mathbb{N}$, $I^{\star_4} = R_3$ $\forall$ other I. And the image of $\star_4$ under the composition of ϕ with the map $\pi_{4,1}$ gives the following closure on R_4: $(0)^{\star_4} = (0)$, $(x_4^s)^\star = (x_4)$, $(x_2^n)^{\star_4} = (x_2)$, $(x_3^r)^{\star_4} = (x_3)$ $\forall\ n, r, s \in \mathbb{N}$, $I^{\star_4} = R_3$ $\forall$ other I. Thus $\star_4$ induces four distinct standard closures on R_4. The same is true for $\star_{4.1}$ through $\star_{10.42}$. Thus $\star_4$ through $\star_{10.42}$ induce a total of 5,696 standard closures on R_4.

Hence we have at least 5,977 standard closre operations on R_4.

□

We can certainly increase the lower bound for the number of standard closures on R_4 given in Corollary 4.0.10. If c and d are closure operations, we define $c \cap d$ to be the operation $I^{c \cap d} = I^c \cap I^d$. In [MV] we showed that if c and d are semistar operations then $c \cap d$ is a semistar operation. We can show similarly that this is the case when c and d are standard closure operations. Certainly we can "intersect" standard closure operations from Corollary 4.0.10 to find standard closures not induced by any of the standard closures on R_3. It appears the number of standard closures on R_4 is many times greater than for R_3. Determining the number of standard closures on R_t for $t > 3$ will probably require computer aided techniques.

Appendices

Appendix A

Standard closure operations from proof of Theorem 3.4.4

1. $\star_1$:$I^{\star_1} = R_3 \; \forall \; I$
2. $\star_{1.1}$:$(0)^{\star_{1.1}} = (0)$, $I^{\star_{1.1}} = R_3 \; \forall$ other I
3. $\star_2$:$(0)^{\star_2} = (z^r)^{\star_2} = (z) \; \forall \; r \in \mathbb{N}$, $I^{\star_2} = R_3 \; \forall$ other I
4. $\star_{2.1}$:$(0)^{\star_{2.1}} = (0)$, $(z^r)^{\star_{2.1}} = (z) \; \forall \; r \in \mathbb{N}$, $I^{\star_{2.1}} = R_3 \; \forall$ other I
5. $\star_{2.2}$:$(0)^{\star_{2.2}} = (0)$, $(z^r)^{\star_{2.2}} = (z^r) \; \forall \; r \in \mathbb{N}$, $I^{\star_{2.2}} = R_3 \; \forall$ other I
6. $\star_{2.3}$ through $\star_{2.8}$ see proof

12. $\star_3$:$(0)^{\star_3} = (0)$, $(y^n)^{\star_3} = (y)$, $(z^r)^{\star_3} = (z) \; \forall \; n, r \in \mathbb{N}$, $I^{\star_3} = R_3 \; \forall$ other I
13. $\star_{3.1}$:$(0)^{\star_{3.1}} = (0)$, $(y^n)^{\star_{3.1}} = (y)$, $(z^r)^{\star_{3.1}} = (z^r) \; \forall \; n, r \in \mathbb{N}$, $I^{\star_{3.1}} = R_3 \; \forall$ other I
14. $\star_{3.2}$:$(0)^{\star_{3.2}} = (0)$, $(y^n)^{\star_{3.2}} = (y^n)$, $(z^r)^{\star_{3.2}} = (z) \; \forall \; n, r \in \mathbb{N}$, $I^{\star_{3.2}} = R_3 \; \forall$ other I
15. $\star_{3.3}$:$(0)^{\star_{3.3}} = (0)$, $(y^n)^{\star_{3.3}} = (y^n)$, $(z^r)^{\star_{3.3}} = (z^r) \; \forall \; n, r \in \mathbb{N}$, $I^{\star_{3.3}} = R_3 \; \forall$ other I
16. $\star_{3.4}$:$(0)^{\star_{3.4}} = (y^n)^{\star_{3.4}} = (z^r)^{\star_{3.4}} = (y^n + az^r)^{\star_{3.4}} = (y^n, z^r)^{\star_{3.4}} = (y, z) \; \forall \; n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.4}} = R_3 \; \forall$ other I
17. $\star_{3.5}$:$(0)^{\star_{3.5}} = (y)$, $(y^n)^{\star_{3.5}} = (z^r)^{\star_{3.5}} = (y^n + az^r)^{\star_{3.5}} = (y^n, z^r)^{\star_{3.5}} = (y, z) \; \forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.5}} = R_3 \; \forall$ other I
18. $\star_{3.6}$:$(0)^{\star_{3.6}} = (z)$, $(y^n)^{\star_{3.6}} = (z^r)^{\star_{3.6}} = (y^n + az^r)^{\star_{3.6}} = (y^n, z^r)^{\star_{3.6}} = (y, z) \; \forall$

$n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.6}} = R_3$ $\forall$ other I

19. $\star_{3.7}$:$(0)^{\star_{3.7}} = (0)$, $(y^n)^{\star_{3.7}} = (z^r)^{\star_{3.7}} = (y^n + az^r)^{\star_{3.7}} = (y^n, z^r)^{\star_{3.7}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.7}} = R_3$ $\forall$ other I
20. $\star_{3.8}$:$(0)^{\star_{3.8}} = (z^r)^{\star_{3.8}} = (z)$, $(y^n)^{\star_{3.8}} = (y^n + az^r)^{\star_{3.8}} = (y^n, z^r)^{\star_{3.8}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.8}} = R_3$ $\forall$ other I
21. $\star_{3.9}$:$(0)^{\star_{3.9}} = (0)$, $(y^n)^{\star_{3.9}} = (y^n + az^r)^{\star_{3.9}} = (y^n, z^r)^{\star_{3.9}} = (y, z)$, $(z^r)^{\star_{3.9}} = (z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.9}} = R_3$ $\forall$ other I
22. $\star_{3.10}$:$(0)^{\star_{3.10}} = (0)$, $(y^n)^{\star_{3.10}} = (y^n + az^r)^{\star_{3.10}} = (y^n, z^r)^{\star_{3.10}} = (y, z)$, $(z^r)^{\star_{3.10}} = (z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.10}} = R_3$ $\forall$ other I
23. $\star_{3.11}$:$(0)^{\star_{3.11}} = (y^n)^{\star_{3.11}} = (y)$, $(z^r)^{\star_{3.11}} = (y^n + az^r)^{\star_{3.11}} = (y^n, z^r)^{\star_{3.11}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.11}} = R_3$ $\forall$ other I
24. $\star_{3.12}$:$(0)^{\star_{3.12}} = (0)$, $(y^n)^{\star_{3.12}} = (y)$, $(z^r)^{\star_{3.12}} = (y^n + az^r)^{\star_{3.12}} = (y^n, z^r)^{\star_{3.12}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.12}} = R_3$ $\forall$ other I
25. $\star_{3.13}$:$(0)^{\star_{3.13}} = (0)$, $(y^n)^{\star_{3.13}} = (y)$, $(z^r)^{\star_{3.13}} = (z)$ $(y^n + az^r)^{\star_{3.13}} = (y^n, z^r)^{\star_{3.13}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.13}} = R_3$ $\forall$ other I
26. $\star_{3.14}$:$(0)^{\star_{3.14}} = (0)$, $(y^n)^{\star_{3.14}} = (y)$, $(z^r)^{\star_{3.14}} = (z^r)$ $(y^n + az^r)^{\star_{3.14}} = (y^n, z^r)^{\star_{3.14}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.14}} = R_3$ $\forall$ other I
27. $\star_{3.15}$:$(0)^{\star_{3.15}} = (0)$, $(y^n)^{\star_{3.15}} = (y^n)$, $(z^r)^{\star_{3.15}} = (y^n + az^r)^{\star_{3.15}} = (y^n, z^r)^{\star_{3.15}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.15}} = R_3$ $\forall$ other I
28. $\star_{3.16}$:$(0)^{\star_{3.16}} = (0)$, $(y^n)^{\star_{3.16}} = (y^n)$, $(z^r)^{\star_{3.16}} = (z)$ $(y^n + az^r)^{\star_{3.16}} = (y^n, z^r)^{\star_{3.16}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.16}} = R_3$ $\forall$ other I
29. $\star_{3.17}$:$(0)^{\star_{3.17}} = (0)$, $(y^n)^{\star_{3.17}} = (y^n)$, $(z^r)^{\star_{3.17}} = (z^r)$ $(y^n + az^r)^{\star_{3.17}} = (y^n, z^r)^{\star_{3.17}} = (y, z)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.17}} = R_3$ $\forall$ other I
30. $\star_{3.18}$:$(0)^{\star_{3.18}} = (y^n)^{\star_{3.18}} = (y)$, $(z^r)^{\star_{3.18}} = (y^n + az^r)^{\star_{3.18}} = (y^n, z^r)^{\star_{3.18}} = (y, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.18}} = R_3$ $\forall$ other I
31. $\star_{3.19}$:$(0)^{\star_{3.19}} = (0)$, $(y^n)^{\star_{3.19}} = (y)$, $(z^r)^{\star_{3.19}} = (y^n + az^r)^{\star_{3.19}} = (y^n, z^r)^{\star_{3.19}} = (y, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.19}} = R_3$ $\forall$ other I
32. $\star_{3.20}$:$(0)^{\star_{3.20}} = (0)$, $(y^n)^{\star_{3.20}} = (y)$, $(y^n + az^r)^{\star_{3.20}} = (y^n, z^r)^{\star_{3.20}} = (y, z^r)$,

$(z^r)^{\star_{3.20}} = (z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.20}} = R_3$ $\forall$ other I

33. $\star_{3.21}$:$(0)^{\star_{3.21}} = (y^n)^{\star_{3.21}} = (y^n)$, $(z^r)^{\star_{3.21}} = (y^n + az^r)^{\star_{3.21}} = (y^n, z^r)^{\star_{3.21}} = (y, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.21}} = R_3$ $\forall$ other I

34. $\star_{3.22}$:$(0)^{\star_{3.22}} = (0)$, $(y^n)^{\star_{3.22}} = (y^n)$, $(z^r)^{\star_{3.22}} = (z^r)$, $(y^n + az^r)^{\star_{3.22}} = (y^n, z^r)^{\star_{3.22}} = (y, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.22}} = R_3$ $\forall$ other I

35. $\star_{3.23}$ through $\star_{3.26}$ see proof

39. $\star_{3.27}$:$(0)^{\star_{3.27}} = (0)$, $(y^n)^{\star_{3.27}} = (y^n)$, $(z^r)^{\star_{3.27}} = (z^r)$, $(y^n + az^r)^{\star_{3.27}} = (y^n, z^r)^{\star_{3.27}} = (y^n, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.27}} = R_3$ $\forall$ other I

40. $\star_{3.28}$:$(0)^{\star_{3.28}} = (0)$, $(y^n)^{\star_{3.28}} = (y^n)$, $(z^r)^{\star_{3.28}} = (z^r)$, $(y^n + az^r)^{\star_{3.28}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{3.28}} = (y^n, z^r)$ $\forall$ $n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{3.28}} = R_3$ $\forall$ other I

41. $\star_{3.29}$ through $\star_{3.57}$ see proof

70. $\star_{3.58}$ through $\star_{3.86}$ see proof

99. $\star_4$:$(0)^{\star_4} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_4} = (y)$, $(z^r)^{\star_4} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_4} = R_3$ $\forall$ other I

100. $\star_{4.1}$:$(0)^{\star_{4.1}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{4.1}} = (y)$, $(z^r)^{\star_{4.1}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.1}} = R_3$ $\forall$ other I

101. $\star_{4.2}$:$(0)^{\star_{4.2}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{4.2}} = (y^n)$, $(z^r)^{\star_{4.2}} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.2}} = R_3$ $\forall$ other I

102. $\star_{4.3}$:$(0)^{\star_{4.3}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{4.3}} = (y)$, $(z^r)^{\star_{4.3}} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.3}} = R_3$ $\forall$ other I

103. $\star_{4.4}$:$(0)^{\star_{4.4}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{4.4}} = (y^n)$, $(z^r)^{\star_{4.4}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.4}} = R_3$ $\forall$ other I

104. $\star_{4.5}$:$(0)^{\star_{4.5}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{4.5}} = (y)$, $(z^r)^{\star_{4.5}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.5}} = R_3$ $\forall$ other I

105. $\star_{4.6}$:$(0)^{\star_{4.6}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{4.6}} = (y^n)$, $(z^r)^{\star_{4.6}} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.6}} = R_3$ $\forall$ other I

106. $\star_{4.7}$:$(0)^{\star_{4.7}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{4.7}} = (y^n)$, $(z^r)^{\star_{4.7}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$, $I^{\star_{4.7}} = R_3$ $\forall$ other I

107. $\star_5$:$(0)^{\star_5} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_5} = (z^r)^{\star_5} = (y^n + az^r)^{\star_5} = (y^n, z^r)^{\star_5} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_5} = R_3$ $\forall$ other I
108. $\star_{5.1}$:$(0)^{\star_{5.1}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.1}} = (y^n + az^r)^{\star_{5.1}} = (y^n, z^r)^{\star_{5.1}} = (y, z)$, $(z^r)^{\star_{5.1}} = (z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.1}} = R_3$ $\forall$ other I
109. $\star_{5.2}$:$(0)^{\star_{5.2}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.2}} = (y^n + az^r)^{\star_{5.2}} = (y^n, z^r)^{\star_{5.2}} = (y, z)$, $(z^r)^{\star_{5.2}} = (z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.2}} = R_3$ $\forall$ other I
110. $\star_{5.3}$:$(0)^{\star_{5.3}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.3}} = (y)$, $(z^r)^{\star_{5.3}} = (y^n + az^r)^{\star_{5.3}} = (y^n, z^r)^{\star_{5.3}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.3}} = R_3$ $\forall$ other I
111. $\star_{5.4}$:$(0)^{\star_{5.4}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.4}} = (y)$, $(z^r)^{\star_{5.4}} = (z)$, $(y^n + az^r)^{\star_{5.4}} = (y^n, z^r)^{\star_{5.4}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.4}} = R_3$ $\forall$ other I
112. $\star_{5.5}$:$(0)^{\star_{5.5}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.5}} = (y)$, $(z^r)^{\star_{5.5}} = (z^r)$, $(y^n + az^r)^{\star_{5.5}} = (y^n, z^r)^{\star_{5.5}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.5}} = R_3$ $\forall$ other I
113. $\star_{5.6}$ through $\star_{5.8}$ see proof
116. $\star_{5.9}$:$(0)^{\star_{5.9}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.9}} = (y)$, $(z^r)^{\star_{5.9}} = (y^n + az^r)^{\star_{5.9}} = (y^n, z^r)^{\star_{5.9}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.9}} = R_3$ $\forall$ other I
117. $\star_{5.10}$:$(0)^{\star_{5.10}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.10}} = (y)$, $(z^r)^{\star_{5.10}} = (z^r)$, $(y^n + az^r)^{\star_{5.10}} = (y^n, z^r)^{\star_{5.10}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.10}} = R_3$ $\forall$ other I
118. $\star_{5.11}$:$(0)^{\star_{5.11}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.11}} = (y^n)$, $(z^r)^{\star_{5.11}} = (y^n + az^r)^{\star_{5.11}} = (y^n, z^r)^{\star_{5.11}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.11}} = R_3$ $\forall$ other I
119. $\star_{5.12}$:$(0)^{\star_{5.12}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.12}} = (y^n)$, $(z^r)^{\star_{5.12}} = (z^r)$, $(y^n + az^r)^{\star_{5.12}} = (y^n, z^r)^{\star_{5.12}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.12}} = R_3$ $\forall$ other I
120. $\star_{5.13}$ through $\star_{5.16}$ see proof
124. $\star_{5.17}$:$(0)^{\star_{5.17}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.17}} = (y^n)$, $(z^r)^{\star_{5.17}} = (z^r)$, $(y^n + az^r)^{\star_{5.17}} = (y^n, z^r)^{\star_{5.17}} = (y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{5.17}} = R_3$ $\forall$ other I
125. $\star_{5.18}$:$(0)^{\star_{5.18}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{5.18}} = (y^n)$, $(z^r)^{\star_{5.18}} = (z^r)$, $(y^n + az^r)^{\star_{5.18}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{5.18}} = (y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$,

$I^{\star_{5.18}} = R_3$ $\forall$ other I

126. $\star_{5.19}$ through $\star_{5.37}$ see proof

145. $\star_{5.38}$ through $\star_{5.75}$ see proof

183. $\star_{5.76}$ through $\star_{5.113}$ see proof

221. $\star_6$:$(0)^{\star_6} = (z^r)^{\star_6} = (z)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_6} = (y^n + az^r)^{\star_6} = (y^n, z^r)^{\star_6} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_6} = R_3$ $\forall$ other I

222. $\star_{6.1}$:$(0)^{\star_{6.1}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.1}} = (y^n + az^r)^{\star_{6.1}} = (y^n, z^r)^{\star_{6.1}} = (y, z)$, $(z^r)^{\star_{6.1}} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.1}} = R_3$ $\forall$ other I

223. $\star_{6.2}$:$(0)^{\star_{6.2}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.2}} = (y^n + az^r)^{\star_{6.2}} = (y^n, z^r)^{\star_{6.2}} = (y, z)$, $(z^r)^{\star_{6.2}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.2}} = R_3$ $\forall$ other I

224. $\star_{6.3}$:$(0)^{\star_{6.3}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.3}} = (y)$, $(z^r)^{\star_{6.3}} = (z)$, $(y^n + az^r)^{\star_{6.3}} = (y^n, z^r)^{\star_{6.3}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.3}} = R_3$ $\forall$ other I

225. $\star_{6.4}$:$(0)^{\star_{6.4}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.4}} = (y)$, $(z^r)^{\star_{6.4}} = (z^r)$, $(y^n + az^r)^{\star_{6.4}} = (y^n, z^r)^{\star_{6.4}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.4}} = R_3$ $\forall$ other I

226. $\star_{6.5}$:$(0)^{\star_{6.5}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.5}} = (y^n)$, $(z^r)^{\star_{6.5}} = (z)$, $(y^n + az^r)^{\star_{6.5}} = (y^n, z^r)^{\star_{6.5}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.5}} = R_3$ $\forall$ other I

227. $\star_{6.6}$:$(0)^{\star_{6.6}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.6}} = (y^n)$, $(z^r)^{\star_{6.6}} = (z^r)$, $(y^n + az^r)^{\star_{6.6}} = (y^n, z^r)^{\star_{6.6}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.6}} = R_3$ $\forall$ other I

228. $\star_{6.7}$:$(0)^{\star_{6.7}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.7}} = (y^n + az^r)^{\star_{6.7}} = (y^n, z^r)^{\star_{6.7}} = (y, z)$, $(z^r)^{\star_{6.7}} = (z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.7}} = R_3$ $\forall$ other I

229. $\star_{6.8}$:$(0)^{\star_{6.8}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.8}} = (y^n + az^r)^{\star_{6.8}} = (y^n, z^r)^{\star_{6.8}} = (y, z)$,

$(z^r)^{\star_{6.8}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.8}} = R_3$ $\forall$ other I

230. $\star_{6.9}$:$(0)^{\star_{6.9}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.9}} = (y)$, $(z^r)^{\star_{6.9}} = (z)$, $(y^n + az^r)^{\star_{6.9}} = (y^n, z^r)^{\star_{6.9}} = (y, z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.9}} = R_3$ $\forall$ other I

231. $\star_{6.10}$:$(0)^{\star_{6.10}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.10}} = (y)$, $(z^r)^{\star_{6.10}} = (z^r)$, $(y^n + az^r)^{\star_{6.10}} = (y^n, z^r)^{\star_{6.10}} = (y, z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.10}} = R_3$ $\forall$ other I

232. $\star_{6.11}$:$(0)^{\star_{6.11}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.11}} = (y^n)$, $(z^r)^{\star_{6.11}} = (z)$, $(y^n + az^r)^{\star_{6.11}} = (y^n, z^r)^{\star_{6.11}} = (y, z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.11}} = R_3$ $\forall$ other I

233. $\star_{6.12}$:$(0)^{\star_{6.12}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.12}} = (y^n)$, $(z^r)^{\star_{6.12}} = (z^r)$, $(y^n + az^r)^{\star_{6.12}} = (y^n, z^r)^{\star_{6.12}} = (y, z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.12}} = R_3$ $\forall$ other I

234. $\star_{6.13}$ through $\star_{6.18}$ see proof

240. $\star_{6.19}$:$(0)^{\star_{6.19}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.19}} = (y)$, $(z^r)^{\star_{6.19}} = (z^r)$, $(y^n + az^r)^{\star_{6.19}} = (y^n, z^r)^{\star_{6.19}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.19}} = R_3$ $\forall$ other I

241. $\star_{6.20}$:$(0)^{\star_{6.20}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.20}} = (y^n)$, $(z^r)^{\star_{6.20}} = (z^r)$, $(y^n + az^r)^{\star_{6.20}} = (y^n, z^r)^{\star_{6.20}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.20}} = R_3$ $\forall$ other I

242. $\star_{6.21}$:$(0)^{\star_{6.21}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.21}} = (y)$, $(z^r)^{\star_{6.21}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n + az^r)^{\star_{6.21}} = (y^n, z^r)^{\star_{6.21}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.21}} = R_3$ $\forall$ other I

243. $\star_{6.22}$:$(0)^{\star_{6.22}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.22}} = (y^n)$, $(z^r)^{\star_{6.22}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n + az^r)^{\star_{6.22}} = (y^n, z^r)^{\star_{6.22}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.22}} = R_3$ $\forall$ other I

244. $\star_{6.23}$:$(0)^{\star_{6.23}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.23}} = (y)$, $(z^r)^{\star_{6.23}} = (z^r)$, $(x^m + az^r)^\star =$

$(x^m, z^r)^\star = (x, z)$, $(y^n + az^r)^{\star_{6.23}} = (y^n, z^r)^{\star_{6.23}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.23}} = R_3$ $\forall$ other I

245. $\star_{6.24}$:$(0)^{\star_{6.24}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.24}} = (y^n)$, $(z^r)^{\star_{6.24}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n + az^r)^{\star_{6.24}} = (y^n, z^r)^{\star_{6.24}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.24}} = R_3$ $\forall$ other I
246. $\star_{6.25}$ through $\star_{6.30}$ see proof
252. $\star_{6.31}$:$(0)^{\star_{6.31}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$, $(y^n)^{\star_{6.31}} = (y^n)$, $(z^r)^{\star_{6.31}} = (z^r)$, $(y^n + az^r)^{\star_{6.31}} = (y^n, z^r)^{\star_{6.31}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.31}} = R_3$ $\forall$ other I
253. $\star_{6.32}$:$(0)^{\star_{6.32}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.32}} = (y^n)$, $(z^r)^{\star_{6.32}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$,$(y^n + az^r)^{\star_{6.32}} = (y^n, z^r)^{\star_{6.32}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.32}} = R_3$ $\forall$ other I
254. $\star_{6.33}$:$(0)^{\star_{6.33}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.33}} = (y^n)$, $(z^r)^{\star_{6.33}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z)$,$(y^n + az^r)^{\star_{6.33}} = (y^n, z^r)^{\star_{6.33}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.33}} = R_3$ $\forall$ other I
255. $\star_{6.34}$ through $\star_{6.36}$ see proof
258. $\star_{6.37}$ through $\star_{6.42}$ see proof
264. $\star_{6.43}$:$(0)^{\star_{6.43}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.43}} = (y)$, $(z^r)^{\star_{6.43}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$,$(y^n + az^r)^{\star_{6.43}} = (y^n, z^r)^{\star_{6.43}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.43}} = R_3$ $\forall$ other I
265. $\star_{6.44}$:$(0)^{\star_{6.44}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.44}} = (y^n)$, $(z^r)^{\star_{6.44}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$,$(y^n + az^r)^{\star_{6.44}} = (y^n, z^r)^{\star_{6.44}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.44}} = R_3$ $\forall$ other I
266. $\star_{6.45}$:$(0)^{\star_{6.45}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.45}} = (y)$, $(z^r)^{\star_{6.45}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$,$(y^n + az^r)^{\star_{6.45}} = (y^n, z^r)^{\star_{6.45}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.45}} = R_3$ $\forall$ other I
267. $\star_{6.46}$:$(0)^{\star_{6.46}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.46}} = (y^n)$, $(z^r)^{\star_{6.46}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$,$(y^n + az^r)^{\star_{6.46}} = (y^n, z^r)^{\star_{6.46}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and

$a \in k^\times$, $I^{\star_{6.46}} = R_3$ $\forall$ other I

268. $\star_{6.47}$:$(0)^{\star_{6.47}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.47}} = (y^n + az^r)^{\star_{6.47}} = (y^n, z^r)^{\star_{6.47}} = (y^n, z)$, $(z^r)^{\star_{6.47}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.47}} = R_3$ $\forall$ other I

269. $\star_{6.48}$:$(0)^{\star_{6.48}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.48}} = (y)$, $(z^r)^{\star_{6.48}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n + az^r)^{\star_{6.48}} = (y^n, z^r)^{\star_{6.48}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.48}} = R_3$ $\forall$ other I

270. $\star_{6.49}$:$(0)^{\star_{6.49}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.49}} = (y^n + az^r)^{\star_{6.49}} = (y^n, z^r)^{\star_{6.49}} = (y^n, z)$, $(z^r)^{\star_{6.49}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.49}} = R_3$ $\forall$ other I

271. $\star_{6.50}$:$(0)^{\star_{6.50}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.50}} = (y)$, $(z^r)^{\star_{6.50}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n + az^r)^{\star_{6.50}} = (y^n, z^r)^{\star_{6.50}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.50}} = R_3$ $\forall$ other I

272. $\star_{6.51}$:$(0)^{\star_{6.51}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.51}} = (y^n)$, $(z^r)^{\star_{6.51}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n + az^r)^{\star_{6.51}} = (y^n, z^r)^{\star_{6.51}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.51}} = R_3$ $\forall$ other I

273. $\star_{6.52}$:$(0)^{\star_{6.52}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.52}} = (y^n)$, $(z^r)^{\star_{6.52}} = (z^r)$, $(x^m+az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n + az^r)^{\star_{6.52}} = (y^n, z^r)^{\star_{6.52}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.52}} = R_3$ $\forall$ other I

274. $\star_{6.53}$:$(0)^{\star_{6.53}} = (0)$, $(x^m)^\star = (x)$, $(y^n)^{\star_{6.53}} = (y^n)$, $(z^r)^{\star_{6.53}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n+az^r)^{\star_{6.53}} = (y^n+az^r)$, $(y^n, z^r)^{\star_{6.53}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.53}} = R_3$ $\forall$ other I

275. $\star_{6.54}$:$(0)^{\star_{6.54}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.54}} = (y^n)$, $(z^r)^{\star_{6.54}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x, z^r)$, $(y^n + az^r)^{\star_{6.54}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{6.54}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.54}} = R_3$ $\forall$ other I

276. $\star_{6.55}$ through $\star_{6.60}$ see proof

282. $\star_{6.61}$ through $\star_{6.64}$ see proof

286. $\star_{6.65}$:$(0)^{\star_{6.65}} = (z^r)^{\star_{6.65}} = (z)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$,

$(y^n)^{\star_{6.65}} = (y^n + az^r)^{\star_{6.65}} = (y^n, z^r)^{\star_{6.65}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.65}} = R_3$ $\forall$ other I

287. $\star_{6.66}$:$(0)^{\star_{6.66}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.66}} = (y^n + az^r)^{\star_{6.66}} = (y^n, z^r)^{\star_{6.66}} = (y^n, z)$, $(z^r)^{\star_{6.66}} = (z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.66}} = R_3$ $\forall$ other I

288. $\star_{6.67}$:$(0)^{\star_{6.67}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.67}} = (y^n + az^r)^{\star_{6.67}} = (y^n, z^r)^{\star_{6.67}} = (y^n, z)$, $(z^r)^{\star_{6.67}} = (z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.67}} = R_3$ $\forall$ other I

289. $\star_{6.68}$:$(0)^{\star_{6.68}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.68}} = (y^n)$, $(z^r)^{\star_{6.68}} = (z)$ $(y^n + az^r)^{\star_{6.68}} = (y^n, z^r)^{\star_{6.68}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.68}} = R_3$ $\forall$ other I

290. $\star_{6.69}$:$(0)^{\star_{6.69}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.69}} = (y^n)$, $(z^r)^{\star_{6.69}} = (z^r)$ $(y^n + az^r)^{\star_{6.69}} = (y^n, z^r)^{\star_{6.69}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.69}} = R_3$ $\forall$ other I

291. $\star_{6.70}$:$(0)^{\star_{6.70}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.70}} = (y^n + az^r)^{\star_{6.70}} = (y^n, z^r)^{\star_{6.70}} = (y^n, z)$, $(z^r)^{\star_{6.70}} = (z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.70}} = R_3$ $\forall$ other I

292. $\star_{6.71}$:$(0)^{\star_{6.71}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.71}} = (y^n + az^r)^{\star_{6.71}} = (y^n, z^r)^{\star_{6.71}} = (y^n, z)$, $(z^r)^{\star_{6.71}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.71}} = R_3$ $\forall$ other I

293. $\star_{6.72}$:$(0)^{\star_{6.72}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.72}} = (y^n)$,$(z^r)^{\star_{6.72}} = (z)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n + az^r)^{\star_{6.72}} = (y^n, z^r)^{\star_{6.72}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.72}} = R_3$ $\forall$ other I

294. $\star_{6.73}$:$(0)^{\star_{6.73}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.73}} = (y^n)$,$(z^r)^{\star_{6.73}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n + az^r)^{\star_{6.73}} = (y^n, z^r)^{\star_{6.73}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.73}} = R_3$ $\forall$ other I

295. $\star_{6.74}$:$(0)^{\star_{6.74}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.74}} = (y^n)$,$(z^r)^{\star_{6.74}} = (z^r)$, $(y^n + az^r)^{\star_{6.74}} = (y^n, z^r)^{\star_{6.74}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and

$a \in k^\times$, $I^{\star_{6.74}} = R_3$ $\forall$ other I

296. $\star_{6.75}$:$(0)^{\star_{6.75}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.75}} = (y^n)$,$(z^r)^{\star_{6.75}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n + az^r)^{\star_{6.75}} = (y^n, z^r)^{\star_{6.75}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.75}} = R_3$ $\forall$ other I
297. $\star_{6.76}$:$(0)^{\star_{6.76}} = (0)$, $(x^m)^\star = (x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n)^{\star_{6.76}} = (y^n)$,$(z^r)^{\star_{6.76}} = (z^r)$, $(y^n + az^r)^{\star_{6.76}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{6.76}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.76}} = R_3$ $\forall$ other I
298. $\star_{6.77}$:$(0)^{\star_{6.77}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.77}} = (y^n)$,$(z^r)^{\star_{6.77}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z)$, $(y^n+az^r)^{\star_{6.77}} = (y^n+az^r)$, $(y^n, z^r)^{\star_{6.77}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.77}} = R_3$ $\forall$ other I
299. $\star_{6.78}$ through $\star_{6.83}$ see proof
305. $\star_{6.84}$ through $\star_{6.87}$ see proof
309. $\star_{6.88}$ through $\star_{6.91}$ see proof
313. $\star_{6.92}$:$(0)^{\star_{6.92}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.92}} = (y^n)$,$(z^r)^{\star_{6.92}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z^r)$, $(y^n + az^r)^{\star_{6.92}} = (y^n, z^r)^{\star_{6.92}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.92}} = R_3$ $\forall$ other I
314. $\star_{6.93}$:$(0)^{\star_{6.93}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.93}} = (y^n)$,$(z^r)^{\star_{6.93}} = (z^r)$, $(x^m + az^r)^\star = (x^m, z^r)^\star = (x^m, z^r)$, $(y^n + az^r)^{\star_{6.93}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{6.93}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.93}} = R_3$ $\forall$ other I
315. $\star_{6.94}$:$(0)^{\star_{6.94}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.94}} = (y^n)$,$(z^r)^{\star_{6.94}} = (z^r)$, $(x^m + az^r)^\star = (x^m + az^r)$, $(x^m, z^r)^\star = (x^m, z^r)$, $(y^n + az^r)^{\star_{6.94}} = (y^n, z^r)^{\star_{6.94}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.94}} = R_3$ $\forall$ other I
316. $\star_{6.95}$:$(0)^{\star_{6.95}} = (0)$, $(x^m)^\star = (x^m)$, $(y^n)^{\star_{6.95}} = (y^n)$,$(z^r)^{\star_{6.95}} = (z^r)$, $(x^m + az^r)^\star = (x^m + az^r)$, $(x^m, z^r)^\star = (x^m, z^r)$, $(y^n + az^r)^{\star_{6.95}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{6.95}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{6.95}} = R_3$ $\forall$ other I
317. $\star_{6.96}$ through $\star_{6.191}$ see proof
413. $\star_{6.192}$ through $\star_{6.287}$ see proof
509. $\star_7$:$(0)^{\star_7} = (0)$, $(x^m)^{\star_7} = (x)$, $(y^n)^{\star_7} = (y)$, $(z^r)^{\star_7} = (z)$, $(x^m + ay^n)^{\star_7} =$

$(x^m, y^n)^{\star_7} = (x, y)$, $(x^m + az^r)^{\star_7} = (x^m, z^r)^{\star_7} = (x, z)$, $(y^n + az^r)^{\star_7} = (y^n, z^r)^{\star_7} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_7} = R_3$ $\forall$ other I

510. $\star_{7.1}$:$(0)^{\star_{7.1}} = (0)$, $(x^m)^{\star_{7.1}} = (x)$, $(y^n)^{\star_{7.1}} = (y)$, $(z^r)^{\star_{7.1}} = (z^r)$, $(x^m + ay^n)^{\star_{7.1}} = (x^m, y^n)^{\star_{7.1}} = (x, y)$, $(x^m + az^r)^{\star_{7.1}} = (x^m, z^r)^{\star_{7.1}} = (x, z)$, $(y^n + az^r)^{\star_{7.1}} = (y^n, z^r)^{\star_{7.1}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.1}} = R_3$ $\forall$ other I
511. $\star_{7.2}$:$(0)^{\star_{7.2}} = (0)$, $(x^m)^{\star_{7.2}} = (x)$, $(y^n)^{\star_{7.2}} = (y^n)$, $(z^r)^{\star_{7.2}} = (z)$, $(x^m + ay^n)^{\star_{7.2}} = (x^m, y^n)^{\star_{7.2}} = (x, y)$, $(x^m + az^r)^{\star_{7.2}} = (x^m, z^r)^{\star_{7.2}} = (x, z)$, $(y^n + az^r)^{\star_{7.2}} = (y^n, z^r)^{\star_{7.2}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.2}} = R_3$ $\forall$ other I
512. $\star_{7.3}$:$(0)^{\star_{7.3}} = (0)$, $(x^m)^{\star_{7.3}} = (x^m)$, $(y^n)^{\star_{7.3}} = (y)$, $(z^r)^{\star_{7.3}} = (z)$, $(x^m + ay^n)^{\star_{7.3}} = (x^m, y^n)^{\star_{7.3}} = (x, y)$, $(x^m + az^r)^{\star_{7.3}} = (x^m, z^r)^{\star_{7.3}} = (x, z)$, $(y^n + az^r)^{\star_{7.3}} = (y^n, z^r)^{\star_{7.3}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.3}} = R_3$ $\forall$ other I
513. $\star_{7.4}$:$(0)^{\star_{7.4}} = (0)$, $(x^m)^{\star_{7.4}} = (x)$, $(y^n)^{\star_{7.4}} = (y^n)$, $(z^r)^{\star_{7.4}} = (z^r)$, $(x^m + ay^n)^{\star_{7.4}} = (x^m, y^n)^{\star_{7.4}} = (x, y)$, $(x^m + az^r)^{\star_{7.4}} = (x^m, z^r)^{\star_{7.4}} = (x, z)$, $(y^n + az^r)^{\star_{7.4}} = (y^n, z^r)^{\star_{7.4}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.4}} = R_3$ $\forall$ other I
514. $\star_{7.5}$:$(0)^{\star_{7.5}} = (0)$, $(x^m)^{\star_{7.5}} = (x^m)$, $(y^n)^{\star_{7.5}} = (y)$, $(z^r)^{\star_{7.5}} = (z^r)$, $(x^m + ay^n)^{\star_{7.5}} = (x^m, y^n)^{\star_{7.5}} = (x, y)$, $(x^m + az^r)^{\star_{7.5}} = (x^m, z^r)^{\star_{7.5}} = (x, z)$, $(y^n + az^r)^{\star_{7.5}} = (y^n, z^r)^{\star_{7.5}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.5}} = R_3$ $\forall$ other I
515. $\star_{7.6}$:$(0)^{\star_{7.6}} = (0)$, $(x^m)^{\star_{7.6}} = (x^m)$, $(y^n)^{\star_{7.6}} = (y^n)$, $(z^r)^{\star_{7.6}} = (z)$, $(x^m + ay^n)^{\star_{7.6}} = (x^m, y^n)^{\star_{7.6}} = (x, y)$, $(x^m + az^r)^{\star_{7.6}} = (x^m, z^r)^{\star_{7.6}} = (x, z)$, $(y^n + az^r)^{\star_{7.6}} = (y^n, z^r)^{\star_{7.6}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.6}} = R_3$ $\forall$ other I
516. $\star_{7.7}$:$(0)^{\star_{7.7}} = (0)$, $(x^m)^{\star_{7.7}} = (x^m)$, $(y^n)^{\star_{7.7}} = (y^n)$, $(z^r)^{\star_{7.7}} = (z^r)$, $(x^m + ay^n)^{\star_{7.7}} = (x^m, y^n)^{\star_{7.7}} = (x, y)$, $(x^m + az^r)^{\star_{7.7}} = (x^m, z^r)^{\star_{7.7}} = (x, z)$, $(y^n + az^r)^{\star_{7.7}} = (y^n, z^r)^{\star_{7.7}} = (y, z)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.7}} = R_3$ $\forall$ other I
517. $\star_{7.8}$:$(0)^{\star_{7.8}} = (0)$, $(x^m)^{\star_{7.8}} = (x)$, $(y^n)^{\star_{7.8}} = (y)$, $(z^r)^{\star_{7.8}} = (z^r)$, $(x^m + ay^n)^{\star_{7.8}} = (x^m, y^n)^{\star_{7.8}} = (x, y)$, $(x^m + az^r)^{\star_{7.8}} = (x^m, z^r)^{\star_{7.8}} = (x, z)$, $(y^n + az^r)^{\star_{7.8}} = (y^n, z^r)^{\star_{7.8}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.8}} = R_3$ $\forall$ other I
518. $\star_{7.9}$:$(0)^{\star_{7.9}} = (0)$, $(x^m)^{\star_{7.9}} = (x)$, $(y^n)^{\star_{7.9}} = (y^n)$, $(z^r)^{\star_{7.9}} = (z^r)$, $(x^m + ay^n)^{\star_{7.9}} =$

$(x^m, y^n)^{\star_{7.9}} = (x, y)$, $(x^m + az^r)^{\star_{7.9}} = (x^m, z^r)^{\star_{7.9}} = (x, z)$, $(y^n + az^r)^{\star_{7.9}} = (y^n, z^r)^{\star_{7.9}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.9}} = R_3$ $\forall$ other I

519. $\star_{7.10}$:$(0)^{\star_{7.10}} = (0)$, $(x^m)^{\star_{7.10}} = (x^m)$, $(y^n)^{\star_{7.10}} = (y)$, $(z^r)^{\star_{7.10}} = (z^r)$, $(x^m + ay^n)^{\star_{7.10}} = (x^m, y^n)^{\star_{7.10}} = (x, y)$,
$(x^m + az^r)^{\star_{7.10}} = (x^m, z^r)^{\star_{7.10}} = (x, z)$, $(y^n + az^r)^{\star_{7.10}} = (y^n, z^r)^{\star_{7.10}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.10}} = R_3$ $\forall$ other I

520. $\star_{7.11}$:$(0)^{\star_{7.11}} = (0)$, $(x^m)^{\star_{7.11}} = (x^m)$, $(y^n)^{\star_{7.11}} = (y^n)$, $(z^r)^{\star_{7.11}} = (z^r)$,
$(x^m + ay^n)^{\star_{7.11}} = (x^m, y^n)^{\star_{7.11}} = (x, y)$, $(x^m + az^r)^{\star_{7.11}} = (x^m, z^r)^{\star_{7.11}} = (x, z)$,
$(y^n + az^r)^{\star_{7.11}} = (y^n, z^r)^{\star_{7.11}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.11}} = R_3$ $\forall$ other I

521. $\star_{7.12}$ through $\star_{7.15}$ see proof

525. $\star_{7.16}$:$(0)^{\star_{7.16}} = (0)$, $(x^m)^{\star_{7.16}} = (x)$, $(y^n)^{\star_{7.16}} = (y^n)$, $(z^r)^{\star_{7.16}} = (z^r)$,
$(x^m + ay^n)^{\star_{7.16}} = (x^m, y^n)^{\star_{7.16}} = (x, y)$, $(x^m + az^r)^{\star_{7.16}} = (x^m, z^r)^{\star_{7.16}} = (x, z)$,
$(y^n + az^r)^{\star_{7.16}} = (y^n, z^r)^{\star_{7.16}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.16}} = R_3$ $\forall$ other I

526. $\star_{7.17}$:$(0)^{\star_{7.17}} = (0)$, $(x^m)^{\star_{7.17}} = (x^m)$, $(y^n)^{\star_{7.17}} = (y^n)$, $(z^r)^{\star_{7.17}} = (z^r)$,
$(x^m + ay^n)^{\star_{7.17}} = (x^m, y^n)^{\star_{7.17}} = (x, y)$, $(x^m + az^r)^{\star_{7.17}} = (x^m, z^r)^{\star_{7.17}} = (x, z)$,
$(y^n + az^r)^{\star_{7.17}} = (y^n, z^r)^{\star_{7.17}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.17}} = R_3$ $\forall$ other I

527. $\star_{7.18}$:$(0)^{\star_{7.18}} = (0)$, $(x^m)^{\star_{7.18}} = (x)$, $(y^n)^{\star_{7.18}} = (y^n)$, $(z^r)^{\star_{7.18}} = (z^r)$,
$(x^m + ay^n)^{\star_{7.18}} = (x^m, y^n)^{\star_{7.18}} = (x, y)$, $(x^m + az^r)^{\star_{7.18}} = (x^m, z^r)^{\star_{7.18}} = (x, z)$,
$(y^n + az^r)^{\star_{7.18}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.18}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.18}} = R_3$ $\forall$ other I

528. $\star_{7.19}$:$(0)^{\star_{7.19}} = (0)$, $(x^m)^{\star_{7.19}} = (x^m)$, $(y^n)^{\star_{7.19}} = (y^n)$, $(z^r)^{\star_{7.19}} = (z^r)$,
$(x^m + ay^n)^{\star_{7.19}} = (x^m, y^n)^{\star_{7.19}} = (x, y)$, $(x^m + az^r)^{\star_{7.19}} = (x^m, z^r)^{\star_{7.19}} = (x, z)$,
$(y^n + az^r)^{\star_{7.19}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.19}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.19}} = R_3$ $\forall$ other I

529. $\star_{7.20}$ through $\star_{7.23}$ see proof

533. $\star_{7.24}$:$(0)^{\star_{7.24}} = (0)$, $(x^m)^{\star_{7.24}} = (x)$, $(y^n)^{\star_{7.24}} = (y)$, $(z^r)^{\star_{7.24}} = (z^r)$, $(x^m + ay^n)^{\star_{7.24}} = (x^m, y^n)^{\star_{7.24}} = (x, y)$, $(x^m + az^r)^{\star_{7.24}} = (x^m, z^r)^{\star_{7.24}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.24}} = (y^n, z^r)^{\star_{7.24}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.24}} = R_3$ $\forall$ other I

534. $\star_{7.25}$:$(0)^{\star_{7.25}} = (0)$, $(x^m)^{\star_{7.25}} = (x)$, $(y^n)^{\star_{7.25}} = (y^n)$, $(z^r)^{\star_{7.25}} = (z^r)$, $(x^m + ay^n)^{\star_{7.25}} = (x^m, y^n)^{\star_{7.25}} = (x, y)$, $(x^m + az^r)^{\star_{7.25}} = (x^m, z^r)^{\star_{7.25}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.25}} = (y^n, z^r)^{\star_{7.25}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.25}} = R_3$ $\forall$ other I

535. $\star_{7.26}$:$(0)^{\star_{7.26}} = (0)$, $(x^m)^{\star_{7.26}} = (x^m)$, $(y^n)^{\star_{7.26}} = (y)$, $(z^r)^{\star_{7.26}} = (z^r)$, $(x^m + ay^n)^{\star_{7.26}} = (x^m, y^n)^{\star_{7.26}} = (x, y)$, $(x^m + az^r)^{\star_{7.26}} = (x^m, z^r)^{\star_{7.26}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.26}} = (y^n, z^r)^{\star_{7.26}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.26}} = R_3$ $\forall$ other I

536. $\star_{7.27}$:$(0)^{\star_{7.27}} = (0)$, $(x^m)^{\star_{7.27}} = (x^m)$, $(y^n)^{\star_{7.27}} = (y^n)$, $(z^r)^{\star_{7.27}} = (z^r)$, $(x^m + ay^n)^{\star_{7.27}} = (x^m, y^n)^{\star_{7.27}} = (x, y)$, $(x^m + az^r)^{\star_{7.27}} = (x^m, z^r)^{\star_{7.27}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.27}} = (y^n, z^r)^{\star_{7.27}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.27}} = R_3$ $\forall$ other I

537. $\star_{7.28}$:$(0)^{\star_{7.28}} = (0)$, $(x^m)^{\star_{7.28}} = (x)$, $(y^n)^{\star_{7.28}} = (y^n)$, $(z^r)^{\star_{7.28}} = (z^r)$, $(x^m + ay^n)^{\star_{7.28}} = (x^m, y^n)^{\star_{7.28}} = (x, y)$, $(x^m + az^r)^{\star_{7.28}} = (x^m, z^r)^{\star_{7.28}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.28}} = (y^n, z^r)^{\star_{7.28}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.28}} = R_3$ $\forall$ other I

538. $\star_{7.29}$:$(0)^{\star_{7.29}} = (0)$, $(x^m)^{\star_{7.29}} = (x^m)$, $(y^n)^{\star_{7.29}} = (y^n)$, $(z^r)^{\star_{7.29}} = (z^r)$, $(x^m + ay^n)^{\star_{7.29}} = (x^m, y^n)^{\star_{7.29}} = (x, y)$, $(x^m + az^r)^{\star_{7.29}} = (x^m, z^r)^{\star_{7.29}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.29}} = (y^n, z^r)^{\star_{7.29}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.29}} = R_3$ $\forall$ other I

539. $\star_{7.30}$:$(0)^{\star_{7.30}} = (0)$, $(x^m)^{\star_{7.30}} = (x)$, $(y^n)^{\star_{7.30}} = (y^n)$, $(z^r)^{\star_{7.30}} = (z^r)$, $(x^m + ay^n)^{\star_{7.30}} = (x^m, y^n)^{\star_{7.30}} = (x, y)$, $(x^m + az^r)^{\star_{7.30}} = (x^m, z^r)^{\star_{7.30}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.30}} = (y^n, z^r)^{\star_{7.30}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.30}} = R_3$ $\forall$ other I

540. $\star_{7.31}$:$(0)^{\star_{7.31}} = (0)$, $(x^m)^{\star_{7.31}} = (x^m)$, $(y^n)^{\star_{7.31}} = (y^n)$, $(z^r)^{\star_{7.31}} = (z^r)$, $(x^m + ay^n)^{\star_{7.31}} = (x^m, y^n)^{\star_{7.31}} = (x, y)$, $(x^m + az^r)^{\star_{7.31}} = (x^m, z^r)^{\star_{7.31}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.31}} = (y^n, z^r)^{\star_{7.31}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.31}} = R_3$ $\forall$ other I

541. $\star_{7.32}$:$(0)^{\star_{7.32}} = (0)$, $(x^m)^{\star_{7.32}} = (x)$, $(y^n)^{\star_{7.32}} = (y^n)$, $(z^r)^{\star_{7.32}} = (z^r)$, $(x^m + ay^n)^{\star_{7.32}} = (x^m, y^n)^{\star_{7.32}} = (x, y)$, $(x^m + az^r)^{\star_{7.32}} = (x^m, z^r)^{\star_{7.32}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.32}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.32}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.32}} = R_3$ $\forall$ other I

542. $\star_{7.33}$:$(0)^{\star_{7.33}} = (0)$, $(x^m)^{\star_{7.33}} = (x^m)$, $(y^n)^{\star_{7.33}} = (y^n)$, $(z^r)^{\star_{7.33}} = (z^r)$, $(x^m + ay^n)^{\star_{7.33}} = (x^m, y^n)^{\star_{7.33}} = (x, y)$, $(x^m + az^r)^{\star_{7.33}} = (x^m, z^r)^{\star_{7.33}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.33}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.33}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.33}} = R_3$ $\forall$ other I

543. $\star_{7.34}$:$(0)^{\star_{7.34}} = (0)$, $(x^m)^{\star_{7.34}} = (x^m)$, $(y^n)^{\star_{7.34}} = (y)$, $(z^r)^{\star_{7.34}} = (z)$, $(x^m + ay^n)^{\star_{7.34}} = (x^m, y^n)^{\star_{7.34}} = (x, y)$, $(x^m + az^r)^{\star_{7.34}} = (x^m, z^r)^{\star_{7.34}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.34}} = (y^n, z^r)^{\star_{7.34}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.34}} = R_3$ $\forall$ other I

544. $\star_{7.35}$:$(0)^{\star_{7.35}} = (0)$, $(x^m)^{\star_{7.35}} = (x^m)$, $(y^n)^{\star_{7.35}} = (y)$, $(z^r)^{\star_{7.35}} = (z^r)$, $(x^m + ay^n)^{\star_{7.35}} = (x^m, y^n)^{\star_{7.35}} = (x, y)$, $(x^m + az^r)^{\star_{7.35}} = (x^m, z^r)^{\star_{7.35}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.35}} = (y^n, z^r)^{\star_{7.35}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.35}} = R_3$ $\forall$ other I

545. $\star_{7.36}$:$(0)^{\star_{7.36}} = (0)$, $(x^m)^{\star_{7.36}} = (x^m)$, $(y^n)^{\star_{7.36}} = (y^n)$, $(z^r)^{\star_{7.36}} = (z)$, $(x^m + ay^n)^{\star_{7.36}} = (x^m, y^n)^{\star_{7.36}} = (x, y)$, $(x^m + az^r)^{\star_{7.36}} = (x^m, z^r)^{\star_{7.36}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.36}} = (y^n, z^r)^{\star_{7.36}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.36}} = R_3$ $\forall$ other I

546. $\star_{7.37}$:$(0)^{\star_{7.37}} = (0)$, $(x^m)^{\star_{7.37}} = (x^m)$, $(y^n)^{\star_{7.37}} = (y^n)$, $(z^r)^{\star_{7.37}} = (z^r)$, $(x^m + ay^n)^{\star_{7.37}} = (x^m, y^n)^{\star_{7.37}} = (x, y)$, $(x^m + az^r)^{\star_{7.37}} = (x^m, z^r)^{\star_{7.37}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.37}} = (y^n, z^r)^{\star_{7.37}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.37}} = R_3$ $\forall$ other I

547. $\star_{7.38}$ through $\star_{7.39}$ see proof

549. $\star_{7.40}$:$(0)^{\star_{7.40}} = (0)$, $(x^m)^{\star_{7.40}} = (x^m)$, $(y^n)^{\star_{7.40}} = (y^n)$, $(z^r)^{\star_{7.40}} = (z)$, $(x^m + ay^n)^{\star_{7.40}} = (x^m, y^n)^{\star_{7.40}} = (x, y)$, $(x^m + az^r)^{\star_{7.40}} = (x^m, z^r)^{\star_{7.40}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.40}} = (y^n, z^r)^{\star_{7.40}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.40}} = R_3$ $\forall$ other I

550. $\star_{7.41}$:$(0)^{\star_{7.41}} = (0)$, $(x^m)^{\star_{7.41}} = (x^m)$, $(y^n)^{\star_{7.41}} = (y^n)$, $(z^r)^{\star_{7.41}} = (z^r)$, $(x^m + ay^n)^{\star_{7.41}} = (x^m, y^n)^{\star_{7.41}} = (x, y)$, $(x^m + az^r)^{\star_{7.41}} = (x^m, z^r)^{\star_{7.41}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.41}} = (y^n, z^r)^{\star_{7.41}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.41}} = R_3$ $\forall$ other I

551. $\star_{7.42}$:$(0)^{\star_{7.42}} = (0)$, $(x^m)^{\star_{7.42}} = (x^m)$, $(y^n)^{\star_{7.42}} = (y^n)$, $(z^r)^{\star_{7.42}} = (z)$, $(x^m + ay^n)^{\star_{7.42}} = (x^m, y^n)^{\star_{7.42}} = (x, y)$, $(x^m + az^r)^{\star_{7.42}} = (x^m, z^r)^{\star_{7.42}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.42}} = (y^n, z^r)^{\star_{7.42}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.42}} = R_3$ $\forall$ other I

552. $\star_{7.43}$:$(0)^{\star_{7.43}} = (0)$, $(x^m)^{\star_{7.43}} = (x^m)$, $(y^n)^{\star_{7.43}} = (y^n)$, $(z^r)^{\star_{7.43}} = (z^r)$, $(x^m + ay^n)^{\star_{7.43}} = (x^m, y^n)^{\star_{7.43}} = (x, y)$, $(x^m + az^r)^{\star_{7.43}} = (x^m, z^r)^{\star_{7.43}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.43}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.43}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.43}} = R_3$ $\forall$ other I

553. $\star_{7.44}$:$(0)^{\star_{7.44}} = (0)$, $(x^m)^{\star_{7.44}} = (x^m)$, $(y^n)^{\star_{7.44}} = (y)$, $(z^r)^{\star_{7.44}} = (z^r)$, $(x^m + ay^n)^{\star_{7.44}} = (x^m, y^n)^{\star_{7.44}} = (x, y)$, $(x^m + az^r)^{\star_{7.44}} = (x^m, z^r)^{\star_{7.44}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.44}} = (y^n, z^r)^{\star_{7.44}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.44}} = R_3$ $\forall$ other I

554. $\star_{7.45}$:$(0)^{\star_{7.45}} = (0)$, $(x^m)^{\star_{7.45}} = (x^m)$, $(y^n)^{\star_{7.45}} = (y^n)$, $(z^r)^{\star_{7.45}} = (z^r)$, $(x^m + ay^n)^{\star_{7.45}} = (x^m, y^n)^{\star_{7.45}} = (x, y)$, $(x^m + az^r)^{\star_{7.45}} = (x^m, z^r)^{\star_{7.45}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.45}} = (y^n, z^r)^{\star_{7.45}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.45}} = R_3$ $\forall$ other I

555. $\star_{7.46}$:$(0)^{\star_{7.46}} = (0)$, $(x^m)^{\star_{7.46}} = (x^m)$, $(y^n)^{\star_{7.46}} = (y)$, $(z^r)^{\star_{7.46}} = (z^r)$, $(x^m + ay^n)^{\star_{7.46}} = (x^m, y^n)^{\star_{7.46}} = (x, y)$, $(x^m + az^r)^{\star_{7.46}} = (x^m, z^r)^{\star_{7.46}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.46}} = (y^n, z^r)^{\star_{7.46}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.46}} = R_3$ $\forall$

other I

556. $\star_{7.47}$:$(0)^{\star_{7.47}} = (0)$, $(x^m)^{\star_{7.47}} = (x^m)$, $(y^n)^{\star_{7.47}} = (y^n)$, $(z^r)^{\star_{7.47}} = (z^r)$, $(x^m + ay^n)^{\star_{7.47}} = (x^m, y^n)^{\star_{7.47}} = (x, y)$, $(x^m + az^r)^{\star_{7.47}} = (x^m, z^r)^{\star_{7.47}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.47}} = (y^n, z^r)^{\star_{7.47}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.47}} = R_3$ $\forall$ other I

557. $\star_{7.48}$:$(0)^{\star_{7.48}} = (0)$, $(x^m)^{\star_{7.48}} = (x^m)$, $(y^n)^{\star_{7.48}} = (y^n)$, $(z^r)^{\star_{7.48}} = (z^r)$, $(x^m + ay^n)^{\star_{7.48}} = (x^m, y^n)^{\star_{7.48}} = (x, y)$, $(x^m + az^r)^{\star_{7.48}} = (x^m, z^r)^{\star_{7.48}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.48}} = (y^n, z^r)^{\star_{7.48}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.48}} = R_3$ $\forall$ other I

558. $\star_{7.49}$:$(0)^{\star_{7.49}} = (0)$, $(x^m)^{\star_{7.49}} = (x^m)$, $(y^n)^{\star_{7.49}} = (y^n)$, $(z^r)^{\star_{7.49}} = (z^r)$, $(x^m + ay^n)^{\star_{7.49}} = (x^m, y^n)^{\star_{7.49}} = (x, y)$, $(x^m + az^r)^{\star_{7.49}} = (x^m, z^r)^{\star_{7.49}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.49}} = (y^n, z^r)^{\star_{7.49}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.49}} = R_3$ $\forall$ other I

559. $\star_{7.50}$:$(0)^{\star_{7.50}} = (0)$, $(x^m)^{\star_{7.50}} = (x^m)$, $(y^n)^{\star_{7.50}} = (y^n)$, $(z^r)^{\star_{7.50}} = (z^r)$, $(x^m + ay^n)^{\star_{7.50}} = (x^m, y^n)^{\star_{7.50}} = (x, y)$, $(x^m + az^r)^{\star_{7.50}} = (x^m, z^r)^{\star_{7.50}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.50}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.50}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.50}} = R_3$ $\forall$ other I

560. $\star_{7.51}$ through $\star_{7.57}$ see proof

527. $\star_{7.58}$:$(0)^{\star_{7.58}} = (0)$, $(x^m)^{\star_{7.58}} = (x)$, $(y^n)^{\star_{7.58}} = (y^n)$, $(z^r)^{\star_{7.58}} = (z)$, $(x^m + ay^n)^{\star_{7.58}} = (x^m, y^n)^{\star_{7.58}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.58}} = (x^m, z^r)^{\star_{7.58}} = (x, z)$, $(y^n + az^r)^{\star_{7.58}} = (y^n, z^r)^{\star_{7.58}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.58}} = R_3$ $\forall$ other I

528. $\star_{7.59}$:$(0)^{\star_{7.59}} = (0)$, $(x^m)^{\star_{7.59}} = (x)$, $(y^n)^{\star_{7.59}} = (y^n)$, $(z^r)^{\star_{7.59}} = (z^r)$, $(x^m + ay^n)^{\star_{7.59}} = (x^m, y^n)^{\star_{7.59}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.59}} = (x^m, z^r)^{\star_{7.59}} = (x, z)$, $(y^n + az^r)^{\star_{7.59}} = (y^n, z^r)^{\star_{7.59}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.59}} = R_3$ $\forall$ other I

529. $\star_{7.60}$:$(0)^{\star_{7.60}} = (0)$, $(x^m)^{\star_{7.60}} = (x^m)$, $(y^n)^{\star_{7.60}} = (y^n)$, $(z^r)^{\star_{7.60}} = (z)$, $(x^m + ay^n)^{\star_{7.60}} = (x^m, y^n)^{\star_{7.60}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.60}} = (x^m, z^r)^{\star_{7.60}} = (x, z)$,

$(y^n + az^r)^{\star_{7.60}} = (y^n, z^r)^{\star_{7.60}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.60}} = R_3$ $\forall$ other I

530. $\star_{7.61}$:$(0)^{\star_{7.61}} = (0)$, $(x^m)^{\star_{7.61}} = (x^m)$, $(y^n)^{\star_{7.61}} = (y^n)$, $(z^r)^{\star_{7.61}} = (z^r)$, $(x^m + ay^n)^{\star_{7.61}} = (x^m, y^n)^{\star_{7.61}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.61}} = (x^m, z^r)^{\star_{7.61}} = (x, z)$, $(y^n + az^r)^{\star_{7.61}} = (y^n, z^r)^{\star_{7.61}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.61}} = R_3$ $\forall$ other I
531. $\star_{7.62}$:$(0)^{\star_{7.62}} = (0)$, $(x^m)^{\star_{7.62}} = (x)$, $(y^n)^{\star_{7.62}} = (y^n)$, $(z^r)^{\star_{7.62}} = (z^r)$, $(x^m + ay^n)^{\star_{7.62}} = (x^m, y^n)^{\star_{7.62}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.62}} = (x^m, z^r)^{\star_{7.62}} = (x, z)$, $(y^n + az^r)^{\star_{7.62}} = (y^n, z^r)^{\star_{7.62}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.62}} = R_3$ $\forall$ other I
532. $\star_{7.63}$:$(0)^{\star_{7.63}} = (0)$, $(x^m)^{\star_{7.63}} = (x^m)$, $(y^n)^{\star_{7.63}} = (y^n)$, $(z^r)^{\star_{7.63}} = (z^r)$, $(x^m + ay^n)^{\star_{7.63}} = (x^m, y^n)^{\star_{7.63}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.63}} = (x^m, z^r)^{\star_{7.63}} = (x, z)$, $(y^n + az^r)^{\star_{7.63}} = (y^n, z^r)^{\star_{7.63}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.63}} = R_3$ $\forall$ other I
533. $\star_{7.64}$:$(0)^{\star_{7.64}} = (0)$, $(x^m)^{\star_{7.64}} = (x)$, $(y^n)^{\star_{7.64}} = (y^n)$, $(z^r)^{\star_{7.64}} = (z)$, $(x^m + ay^n)^{\star_{7.64}} = (x^m, y^n)^{\star_{7.64}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.64}} = (x^m, z^r)^{\star_{7.64}} = (x, z)$, $(y^n + az^r)^{\star_{7.64}} = (y^n, z^r)^{\star_{7.64}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.64}} = R_3$ $\forall$ other I
534. $\star_{7.65}$:$(0)^{\star_{7.65}} = (0)$, $(x^m)^{\star_{7.65}} = (x)$, $(y^n)^{\star_{7.65}} = (y^n)$, $(z^r)^{\star_{7.65}} = (z^r)$, $(x^m + ay^n)^{\star_{7.65}} = (x^m, y^n)^{\star_{7.65}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.65}} = (x^m, z^r)^{\star_{7.65}} = (x, z)$, $(y^n + az^r)^{\star_{7.65}} = (y^n, z^r)^{\star_{7.65}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.65}} = R_3$ $\forall$ other I
535. $\star_{7.66}$:$(0)^{\star_{7.66}} = (0)$, $(x^m)^{\star_{7.66}} = (x^m)$, $(y^n)^{\star_{7.66}} = (y^n)$, $(z^r)^{\star_{7.66}} = (z)$, $(x^m + ay^n)^{\star_{7.66}} = (x^m, y^n)^{\star_{7.66}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.66}} = (x^m, z^r)^{\star_{7.66}} = (x, z)$, $(y^n + az^r)^{\star_{7.66}} = (y^n, z^r)^{\star_{7.66}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.66}} = R_3$ $\forall$ other I
536. $\star_{7.67}$:$(0)^{\star_{7.67}} = (0)$, $(x^m)^{\star_{7.67}} = (x^m)$, $(y^n)^{\star_{7.67}} = (y^n)$, $(z^r)^{\star_{7.67}} = (z^r)$, $(x^m + ay^n)^{\star_{7.67}} = (x^m, y^n)^{\star_{7.67}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.67}} = (x^m, z^r)^{\star_{7.67}} = (x, z)$,

$(y^n + az^r)^{\star 7.67} = (y^n, z^r)^{\star 7.67} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.67} = R_3$ $\forall$ other I

537. $\star_{7.68}$:$(0)^{\star 7.68} = (0)$, $(x^m)^{\star 7.68} = (x)$, $(y^n)^{\star 7.68} = (y^n)$, $(z^r)^{\star 7.68} = (z^r)$, $(x^m + ay^n)^{\star 7.68} = (x^m, y^n)^{\star 7.68} = (x, y^n)$, $(x^m + az^r)^{\star 7.68} = (x^m, z^r)^{\star 7.68} = (x, z)$, $(y^n + az^r)^{\star 7.68} = (y^n, z^r)^{\star 7.68} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.68} = R_3$ $\forall$ other I

538. $\star_{7.69}$:$(0)^{\star 7.69} = (0)$, $(x^m)^{\star 7.69} = (x^m)$, $(y^n)^{\star 7.69} = (y^n)$, $(z^r)^{\star 7.69} = (z^r)$, $(x^m + ay^n)^{\star 7.69} = (x^m, y^n)^{\star 7.69} = (x, y^n)$, $(x^m + az^r)^{\star 7.69} = (x^m, z^r)^{\star 7.69} = (x, z)$, $(y^n + az^r)^{\star 7.69} = (y^n, z^r)^{\star 7.69} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.69} = R_3$ $\forall$ other I

539. $\star_{7.70}$:$(0)^{\star 7.70} = (0)$, $(x^m)^{\star 7.70} = (x)$, $(y^n)^{\star 7.70} = (y^n)$, $(z^r)^{\star 7.70} = (z^r)$, $(x^m + ay^n)^{\star 7.70} = (x^m, y^n)^{\star 7.70} = (x, y^n)$, $(x^m + az^r)^{\star 7.70} = (x^m, z^r)^{\star 7.70} = (x, z)$, $(y^n + az^r)^{\star 7.70} = (y^n + az^r)$, $(y^n, z^r)^{\star 7.70} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.70} = R_3$ $\forall$ other I

540. $\star_{7.71}$:$(0)^{\star 7.71} = (0)$, $(x^m)^{\star 7.71} = (x^m)$, $(y^n)^{\star 7.71} = (y^n)$, $(z^r)^{\star 7.71} = (z^r)$, $(x^m + ay^n)^{\star 7.71} = (x^m, y^n)^{\star 7.71} = (x, y^n)$, $(x^m + az^r)^{\star 7.71} = (x^m, z^r)^{\star 7.71} = (x, z)$, $(y^n + az^r)^{\star 7.71} = (y^n + az^r)$, $(y^n, z^r)^{\star 7.71} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.71} = R_3$ $\forall$ other I

541. $\star_{7.72}$:$(0)^{\star 7.72} = (0)$, $(x^m)^{\star 7.72} = (x)$, $(y^n)^{\star 7.72} = (y^n)$, $(z^r)^{\star 7.72} = (z^r)$, $(x^m + ay^n)^{\star 7.72} = (x^m, y^n)^{\star 7.72} = (x, y^n)$, $(x^m + az^r)^{\star 7.72} = (x^m, z^r)^{\star 7.72} = (x, z^r)$, $(y^n + az^r)^{\star 7.72} = (y^n, z^r)^{\star 7.72} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.72} = R_3$ $\forall$ other I

542. $\star_{7.73}$:$(0)^{\star 7.73} = (0)$, $(x^m)^{\star 7.73} = (x^m)$, $(y^n)^{\star 7.73} = (y^n)$, $(z^r)^{\star 7.73} = (z^r)$, $(x^m + ay^n)^{\star 7.73} = (x^m, y^n)^{\star 7.73} = (x, y^n)$, $(x^m + az^r)^{\star 7.73} = (x^m, z^r)^{\star 7.73} = (x, z^r)$, $(y^n + az^r)^{\star 7.73} = (y^n, z^r)^{\star 7.73} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.73} = R_3$ $\forall$ other I

543. $\star_{7.74}$:$(0)^{\star 7.74} = (0)$, $(x^m)^{\star 7.74} = (x)$, $(y^n)^{\star 7.74} = (y^n)$, $(z^r)^{\star 7.74} = (z^r)$, $(x^m + ay^n)^{\star 7.74} = (x^m, y^n)^{\star 7.74} = (x, y^n)$, $(x^m + az^r)^{\star 7.74} = (x^m, z^r)^{\star 7.74} = (x, z^r)$,

$(y^n + az^r)^{\star 7.74} = (y^n, z^r)^{\star 7.74} = (y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.74} = R_3\ \forall$ other I

544. $\star_{7.75}$:$(0)^{\star 7.75} = (0)$, $(x^m)^{\star 7.75} = (x^m)$, $(y^n)^{\star 7.75} = (y^n)$, $(z^r)^{\star 7.75} = (z^r)$, $(x^m + ay^n)^{\star 7.75} = (x^m, y^n)^{\star 7.75} = (x, y^n)$, $(x^m + az^r)^{\star 7.75} = (x^m, z^r)^{\star 7.75} = (x, z^r)$, $(y^n + az^r)^{\star 7.75} = (y^n, z^r)^{\star 7.75} = (y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.75} = R_3\ \forall$ other I

545. $\star_{7.76}$:$(0)^{\star 7.76} = (0)$, $(x^m)^{\star 7.76} = (x)$, $(y^n)^{\star 7.76} = (y^n)$, $(z^r)^{\star 7.76} = (z^r)$, $(x^m + ay^n)^{\star 7.76} = (x^m, y^n)^{\star 7.76} = (x, y^n)$, $(x^m + az^r)^{\star 7.76} = (x^m, z^r)^{\star 7.76} = (x, z^r)$, $(y^n + az^r)^{\star 7.76} = (y^n, z^r)^{\star 7.76} = (y^n, z)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.76} = R_3\ \forall$ other I

546. $\star_{7.77}$:$(0)^{\star 7.77} = (0)$, $(x^m)^{\star 7.77} = (x^m)$, $(y^n)^{\star 7.77} = (y^n)$, $(z^r)^{\star 7.77} = (z^r)$, $(x^m + ay^n)^{\star 7.77} = (x^m, y^n)^{\star 7.77} = (x, y^n)$, $(x^m + az^r)^{\star 7.77} = (x^m, z^r)^{\star 7.77} = (x, z^r)$, $(y^n + az^r)^{\star 7.77} = (y^n, z^r)^{\star 7.77} = (y^n, z)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.77} = R_3\ \forall$ other I

547. $\star_{7.78}$:$(0)^{\star 7.78} = (0)$, $(x^m)^{\star 7.78} = (x)$, $(y^n)^{\star 7.78} = (y^n)$, $(z^r)^{\star 7.78} = (z^r)$, $(x^m + ay^n)^{\star 7.78} = (x^m, y^n)^{\star 7.78} = (x, y^n)$, $(x^m + az^r)^{\star 7.78} = (x^m, z^r)^{\star 7.78} = (x, z^r)$, $(y^n + az^r)^{\star 7.78} = (y^n, z^r)^{\star 7.78} = (y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.78} = R_3\ \forall$ other I

548. $\star_{7.79}$:$(0)^{\star 7.79} = (0)$, $(x^m)^{\star 7.79} = (x^m)$, $(y^n)^{\star 7.79} = (y^n)$, $(z^r)^{\star 7.79} = (z^r)$, $(x^m + ay^n)^{\star 7.79} = (x^m, y^n)^{\star 7.79} = (x, y^n)$, $(x^m + az^r)^{\star 7.79} = (x^m, z^r)^{\star 7.79} = (x, z^r)$, $(y^n + az^r)^{\star 7.79} = (y^n, z^r)^{\star 7.79} = (y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.79} = R_3\ \forall$ other I

549. $\star_{7.80}$:$(0)^{\star 7.80} = (0)$, $(x^m)^{\star 7.80} = (x)$, $(y^n)^{\star 7.80} = (y^n)$, $(z^r)^{\star 7.80} = (z^r)$, $(x^m + ay^n)^{\star 7.80} = (x^m, y^n)^{\star 7.80} = (x, y^n)$, $(x^m + az^r)^{\star 7.80} = (x^m, z^r)^{\star 7.80} = (x, z^r)$, $(y^n + az^r)^{\star 7.80} = (y^n + az^r)$, $(y^n, z^r)^{\star 7.80} = (y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star 7.80} = R_3\ \forall$ other I

550. $\star_{7.81}$:$(0)^{\star 7.81} = (0)$, $(x^m)^{\star 7.81} = (x^m)$, $(y^n)^{\star 7.81} = (y^n)$, $(z^r)^{\star 7.81} = (z^r)$, $(x^m + ay^n)^{\star 7.81} = (x^m, y^n)^{\star 7.81} = (x, y^n)$, $(x^m + az^r)^{\star 7.81} = (x^m, z^r)^{\star 7.81} = (x, z^r)$,

$(y^n + az^r)^{\star_{7.81}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.81}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.81}} = R_3$ $\forall$ other I

551. $\star_{7.82}$:$(0)^{\star_{7.82}} = (0)$, $(x^m)^{\star_{7.82}} = (x^m)$, $(y^n)^{\star_{7.82}} = (y^n)$, $(z^r)^{\star_{7.82}} = (z)$, $(x^m + ay^n)^{\star_{7.82}} = (x^m, y^n)^{\star_{7.82}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.82}} = (x^m, z^r)^{\star_{7.82}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.82}} = (y^n, z^r)^{\star_{7.82}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.82}} = R_3$ $\forall$ other I

552. $\star_{7.83}$:$(0)^{\star_{7.83}} = (0)$, $(x^m)^{\star_{7.83}} = (x^m)$, $(y^n)^{\star_{7.83}} = (y^n)$, $(z^r)^{\star_{7.83}} = (z^r)$, $(x^m + ay^n)^{\star_{7.83}} = (x^m, y^n)^{\star_{7.83}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.83}} = (x^m, z^r)^{\star_{7.83}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.83}} = (y^n, z^r)^{\star_{7.83}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.83}} = R_3$ $\forall$ other I

553. $\star_{7.84}$:$(0)^{\star_{7.84}} = (0)$, $(x^m)^{\star_{7.84}} = (x^m)$, $(y^n)^{\star_{7.84}} = (y^n)$, $(z^r)^{\star_{7.84}} = (z^r)$, $(x^m + ay^n)^{\star_{7.84}} = (x^m, y^n)^{\star_{7.84}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.84}} = (x^m, z^r)^{\star_{7.84}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.84}} = (y^n, z^r)^{\star_{7.84}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.84}} = R_3$ $\forall$ other I

554. $\star_{7.85}$:$(0)^{\star_{7.85}} = (0)$, $(x^m)^{\star_{7.85}} = (x^m)$, $(y^n)^{\star_{7.85}} = (y^n)$, $(z^r)^{\star_{7.85}} = (z)$, $(x^m + ay^n)^{\star_{7.85}} = (x^m, y^n)^{\star_{7.85}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.85}} = (x^m, z^r)^{\star_{7.85}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.85}} = (y^n, z^r)^{\star_{7.85}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.85}} = R_3$ $\forall$ other I

555. $\star_{7.86}$:$(0)^{\star_{7.86}} = (0)$, $(x^m)^{\star_{7.86}} = (x^m)$, $(y^n)^{\star_{7.86}} = (y^n)$, $(z^r)^{\star_{7.86}} = (z^r)$, $(x^m + ay^n)^{\star_{7.86}} = (x^m, y^n)^{\star_{7.86}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.86}} = (x^m, z^r)^{\star_{7.86}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.86}} = (y^n, z^r)^{\star_{7.86}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.86}} = R_3$ $\forall$ other I

556. $\star_{7.87}$:$(0)^{\star_{7.87}} = (0)$, $(x^m)^{\star_{7.87}} = (x^m)$, $(y^n)^{\star_{7.87}} = (y^n)$, $(z^r)^{\star_{7.87}} = (z^r)$, $(x^m + ay^n)^{\star_{7.87}} = (x^m, y^n)^{\star_{7.87}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.87}} = (x^m, z^r)^{\star_{7.87}} = (x^m, z)$, $(y^n + az^r)^{\star_{7.87}} = (y^n, z^r)^{\star_{7.87}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.87}} = R_3$ $\forall$ other I

557. $\star_{7.88}$:$(0)^{\star_{7.88}} = (0)$, $(x^m)^{\star_{7.88}} = (x^m)$, $(y^n)^{\star_{7.88}} = (y^n)$, $(z^r)^{\star_{7.88}} = (z^r)$, $(x^m + ay^n)^{\star_{7.88}} = (x^m, y^n)^{\star_{7.88}} = (x, y^n)$, $(x^m + az^r)^{\star_{7.88}} = (x^m, z^r)^{\star_{7.88}} = (x^m, z)$,

$(y^n + az^r)^{\star_{7.88}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.88}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.88}} = R_3$ $\forall$ other I

558. $\star_{7.89}$:$(0)^{\star_{7.89}} = (0)$, $(x^m)^{\star_{7.89}} = (x^m)$, $(y^n)^{\star_{7.89}} = (y^n)$, $(z^r)^{\star_{7.89}} = (z^r)$, $(x^m+ay^n)^{\star_{7.89}} = (x^m, y^n)^{\star_{7.89}} = (x, y^n)$, $(x^m+az^r)^{\star_{7.89}} = (x^m, z^r)^{\star_{7.89}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.89}} = (y^n, z^r)^{\star_{7.89}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.89}} = R_3$ $\forall$ other I
559. $\star_{7.90}$:$(0)^{\star_{7.90}} = (0)$, $(x^m)^{\star_{7.90}} = (x^m)$, $(y^n)^{\star_{7.90}} = (y^n)$, $(z^r)^{\star_{7.90}} = (z^r)$, $(x^m+ay^n)^{\star_{7.90}} = (x^m, y^n)^{\star_{7.90}} = (x, y^n)$, $(x^m+az^r)^{\star_{7.90}} = (x^m, z^r)^{\star_{7.90}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.90}} = (y^n, z^r)^{\star_{7.90}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.90}} = R_3$ $\forall$ other I
560. $\star_{7.91}$:$(0)^{\star_{7.91}} = (0)$, $(x^m)^{\star_{7.91}} = (x^m)$, $(y^n)^{\star_{7.91}} = (y^n)$, $(z^r)^{\star_{7.91}} = (z^r)$, $(x^m+ay^n)^{\star_{7.91}} = (x^m, y^n)^{\star_{7.91}} = (x, y^n)$, $(x^m+az^r)^{\star_{7.91}} = (x^m, z^r)^{\star_{7.91}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.91}} = (y^n, z^r)^{\star_{7.91}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.91}} = R_3$ $\forall$ other I
561. $\star_{7.92}$:$(0)^{\star_{7.92}} = (0)$, $(x^m)^{\star_{7.92}} = (x^m)$, $(y^n)^{\star_{7.92}} = (y^n)$, $(z^r)^{\star_{7.92}} = (z^r)$, $(x^m+ay^n)^{\star_{7.92}} = (x^m, y^n)^{\star_{7.92}} = (x, y^n)$, $(x^m+az^r)^{\star_{7.92}} = (x^m, z^r)^{\star_{7.92}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.92}} = (y^n, z^r)^{\star_{7.92}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.92}} = R_3$ $\forall$ other I
562. $\star_{7.93}$:$(0)^{\star_{7.93}} = (0)$, $(x^m)^{\star_{7.93}} = (x^m)$, $(y^n)^{\star_{7.93}} = (y^n)$, $(z^r)^{\star_{7.93}} = (z^r)$, $(x^m+ay^n)^{\star_{7.93}} = (x^m, y^n)^{\star_{7.93}} = (x, y^n)$, $(x^m+az^r)^{\star_{7.93}} = (x^m, z^r)^{\star_{7.93}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{7.93}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.93}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.93}} = R_3$ $\forall$ other I
563. $\star_{7.94}$ through $\star_{7.98}$ see proof
608. $\star_{7.99}$ through $\star_{7.139}$ see proof
649. $\star_{7.140}$:$(0)^{\star_{7.140}} = (0)$, $(x^m)^{\star_{7.140}} = (x^m)$, $(y^n)^{\star_{7.140}} = (y^n)$, $(z^r)^{\star_{7.140}} = (z)$, $(x^m + ay^n)^{\star_{7.140}} = (x^m, y^n)^{\star_{7.140}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.140}} = (x^m, z^r)^{\star_{7.140}} = (x, z)$, $(y^n + az^r)^{\star_{7.140}} = (y^n, z^r)^{\star_{7.140}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.140}} = R_3$ $\forall$ other I

650. $\star_{7.141}$:$(0)^{\star_{7.141}} = (0)$, $(x^m)^{\star_{7.141}} = (x^m)$, $(y^n)^{\star_{7.141}} = (y^n)$, $(z^r)^{\star_{7.141}} = (z^r)$, $(x^m + ay^n)^{\star_{7.141}} = (x^m, y^n)^{\star_{7.141}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.141}} = (x^m, z^r)^{\star_{7.141}} = (x, z)$, $(y^n + az^r)^{\star_{7.141}} = (y^n, z^r)^{\star_{7.141}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.141}} = R_3$ $\forall$ other I
651. $\star_{7.142}$:$(0)^{\star_{7.142}} = (0)$, $(x^m)^{\star_{7.142}} = (x^m)$, $(y^n)^{\star_{7.142}} = (y^n)$, $(z^r)^{\star_{7.142}} = (z^r)$, $(x^m + ay^n)^{\star_{7.142}} = (x^m, y^n)^{\star_{7.142}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.142}} = (x^m, z^r)^{\star_{7.142}} = (x, z)$, $(y^n + az^r)^{\star_{7.142}} = (y^n, z^r)^{\star_{7.142}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.142}} = R_3$ $\forall$ other I
652. $\star_{7.143}$:$(0)^{\star_{7.143}} = (0)$, $(x^m)^{\star_{7.143}} = (x^m)$, $(y^n)^{\star_{7.143}} = (y^n)$, $(z^r)^{\star_{7.143}} = (z)$, $(x^m + ay^n)^{\star_{7.143}} = (x^m, y^n)^{\star_{7.143}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.143}} = (x^m, z^r)^{\star_{7.143}} = (x, z)$, $(y^n + az^r)^{\star_{7.143}} = (y^n, z^r)^{\star_{7.143}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.143}} = R_3$ $\forall$ other I
653. $\star_{7.144}$:$(0)^{\star_{7.144}} = (0)$, $(x^m)^{\star_{7.144}} = (x^m)$, $(y^n)^{\star_{7.144}} = (y^n)$, $(z^r)^{\star_{7.144}} = (z^r)$, $(x^m + ay^n)^{\star_{7.144}} = (x^m, y^n)^{\star_{7.144}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.144}} = (x^m, z^r)^{\star_{7.144}} = (x, z)$, $(y^n + az^r)^{\star_{7.144}} = (y^n, z^r)^{\star_{7.144}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.144}} = R_3$ $\forall$ other I
654. $\star_{7.145}$:$(0)^{\star_{7.145}} = (0)$, $(x^m)^{\star_{7.145}} = (x^m)$, $(y^n)^{\star_{7.145}} = (y^n)$, $(z^r)^{\star_{7.145}} = (z^r)$, $(x^m + ay^n)^{\star_{7.145}} = (x^m, y^n)^{\star_{7.145}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.145}} = (x^m, z^r)^{\star_{7.145}} = (x, z)$, $(y^n + az^r)^{\star_{7.145}} = (y^n, z^r)^{\star_{7.145}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.145}} = R_3$ $\forall$ other I
655. $\star_{7.146}$:$(0)^{\star_{7.146}} = (0)$, $(x^m)^{\star_{7.146}} = (x^m)$, $(y^n)^{\star_{7.146}} = (y^n)$, $(z^r)^{\star_{7.146}} = (z^r)$, $(x^m + ay^n)^{\star_{7.146}} = (x^m, y^n)^{\star_{7.146}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.146}} = (x^m, z^r)^{\star_{7.146}} = (x, z)$, $(y^n + az^r)^{\star_{7.146}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.146}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.146}} = R_3$ $\forall$ other I
656. $\star_{7.147}$:$(0)^{\star_{7.147}} = (0)$, $(x^m)^{\star_{7.147}} = (x^m)$, $(y^n)^{\star_{7.147}} = (y^n)$, $(z^r)^{\star_{7.147}} = (z^r)$, $(x^m + ay^n)^{\star_{7.147}} = (x^m, y^n)^{\star_{7.147}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.147}} = (x^m, z^r)^{\star_{7.147}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.147}} = (y^n, z^r)^{\star_{7.147}} = (y, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.147}} = R_3$ $\forall$ other I

657. $\star_{7.148}$:$(0)^{\star_{7.148}} = (0)$, $(x^m)^{\star_{7.148}} = (x^m)$, $(y^n)^{\star_{7.148}} = (y^n)$, $(z^r)^{\star_{7.148}} = (z^r)$, $(x^m + ay^n)^{\star_{7.148}} = (x^m, y^n)^{\star_{7.148}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.148}} = (x^m, z^r)^{\star_{7.148}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.148}} = (y^n, z^r)^{\star_{7.148}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.148}} = R_3$ $\forall$ other I
658. $\star_{7.149}$:$(0)^{\star_{7.149}} = (0)$, $(x^m)^{\star_{7.149}} = (x^m)$, $(y^n)^{\star_{7.149}} = (y^n)$, $(z^r)^{\star_{7.149}} = (z^r)$, $(x^m + ay^n)^{\star_{7.149}} = (x^m, y^n)^{\star_{7.149}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.149}} = (x^m, z^r)^{\star_{7.149}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.149}} = (y^n, z^r)^{\star_{7.149}} = (y^n, z)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.149}} = R_3$ $\forall$ other I
659. $\star_{7.150}$:$(0)^{\star_{7.150}} = (0)$, $(x^m)^{\star_{7.150}} = (x^m)$, $(y^n)^{\star_{7.150}} = (y^n)$, $(z^r)^{\star_{7.150}} = (z^r)$, $(x^m + ay^n)^{\star_{7.150}} = (x^m, y^n)^{\star_{7.150}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.150}} = (x^m, z^r)^{\star_{7.150}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.150}} = (y^n, z^r)^{\star_{7.150}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.150}} = R_3$ $\forall$ other I
660. $\star_{7.151}$:$(0)^{\star_{7.151}} = (0)$, $(x^m)^{\star_{7.151}} = (x^m)$, $(y^n)^{\star_{7.151}} = (y^n)$, $(z^r)^{\star_{7.151}} = (z^r)$, $(x^m + ay^n)^{\star_{7.151}} = (x^m, y^n)^{\star_{7.151}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{7.151}} = (x^m, z^r)^{\star_{7.151}} = (x, z^r)$, $(y^n + az^r)^{\star_{7.151}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{7.151}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a \in k^\times$, $I^{\star_{7.151}} = R_3$ $\forall$ other I
661. $\star_{7.152}$ through $\star_{7.163}$ see proof
673. $\star_8$:$(0)^{\star_8} = (x^m)^{\star_8} = (y^n)^{\star_8} = (x^m + ay^n)^{\star_8} = (x^m, y^n)^{\star_8} = (x, y)$, $(z^r)^{\star_8} = (x^m + az^r)^{\star_8} = (y^n + az^r)^{\star_8} = (x^m + ay^n + bz^r)^{\star_8} = (x^m, z^r)^{\star_8} = (y^n, z^r)^{\star_8} = (x^m + az^r, y^n + bz^r)^{\star_8} = (x^m, y^n + az^r)^{\star_8} = (x^m + ay^n, z^r)^{\star_8} = (x^m + az^r, y^n)^{\star_8} = (x^m, y^n, z^r)^{\star_8} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
674. $\star_{8.1}$:$(0)^{\star_{8.1}} = (0)$, $(x^m)^{\star_{8.1}} = (y^n)^{\star_{8.1}} = (x^m + ay^n)^{\star_{8.1}} = (x^m, y^n)^{\star_{8.1}} = (x, y)$, $(z^r)^{\star_{8.1}} = (x^m + az^r)^{\star_{8.1}} = (y^n + az^r)^{\star_{8.1}} = (x^m + ay^n + bz^r)^{\star_{8.1}} = (x^m, z^r)^{\star_{8.1}} = (y^n, z^r)^{\star_{8.1}} = (x^m + az^r, y^n + bz^r)^{\star_{8.1}} = (x^m, y^n + az^r)^{\star_{8.1}} = (x^m + ay^n, z^r)^{\star_{8.1}} = (x^m + az^r, y^n)^{\star_{8.1}} = (x^m, y^n, z^r)^{\star_{8.1}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
675. $\star_{8.2}$:$(0)^{\star_{8.2}} = (0)$, $(x^m)^{\star_{8.2}} = (y^n)^{\star_{8.2}} = (x^m + ay^n)^{\star_{8.2}} = (x^m, y^n)^{\star_{8.2}} = (x, y)$, $(z^r)^{\star_{8.2}} = (z^r)$, $(x^m + az^r)^{\star_{8.2}} = (y^n + az^r)^{\star_{8.2}} = (x^m + ay^n + bz^r)^{\star_{8.2}} = (x^m, z^r)^{\star_{8.2}} = (y^n, z^r)^{\star_{8.2}} = (x^m + az^r, y^n + bz^r)^{\star_{8.2}} = (x^m, y^n + az^r)^{\star_{8.2}} =$

$(x^m+ay^n,z^r)^{\star_{8.2}} = (x^m+az^r,y^n)^{\star_{8.2}} = (x^m,y^n,z^r)^{\star_{8.2}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

676. $\star_{8.3}$:$(0)^{\star_{8.3}} = (y^n)^{\star_{8.3}} = (y)$, $(x^m)^{\star_{8.3}} = (x^m+ay^n)^{\star_{8.3}} = (x^m,y^n)^{\star_{8.3}} = (x,y)$, $(z^r)^{\star_{8.3}} = (x^m+az^r)^{\star_{8.3}} = (y^n+az^r)^{\star_{8.3}} = (x^m+ay^n+bz^r)^{\star_{8.3}} = (x^m,z^r)^{\star_{8.3}} = (y^n,z^r)^{\star_{8.3}} = (x^m+az^r,y^n+bz^r)^{\star_{8.3}} = (x^m,y^n+az^r)^{\star_{8.3}} = (x^m+ay^n,z^r)^{\star_{8.3}} = (x^m+az^r,y^n)^{\star_{8.3}} = (x^m,y^n,z^r)^{\star_{8.3}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

677. $\star_{8.4}$:$(0)^{\star_{8.4}} = (0)$, $(x^m)^{\star_{8.4}} = (x^m+ay^n)^{\star_{8.4}} = (x^m,y^n)^{\star_{8.4}} = (x,y)$, $(y^n)^{\star_{8.4}} = (y)$, $(z^r)^{\star_{8.4}} = (x^m+az^r)^{\star_{8.4}} = (y^n+az^r)^{\star_{8.4}} = (x^m+ay^n+bz^r)^{\star_{8.4}} = (x^m,z^r)^{\star_{8.4}} = (y^n,z^r)^{\star_{8.4}} = (x^m+az^r,y^n+bz^r)^{\star_{8.4}} = (x^m,y^n+az^r)^{\star_{8.4}} = (x^m+ay^n,z^r)^{\star_{8.4}} = (x^m+az^r,y^n)^{\star_{8.4}} = (x^m,y^n,z^r)^{\star_{8.4}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

678. $\star_{8.5}$:$(0)^{\star_{8.5}} = (0)$, $(x^m)^{\star_{8.5}} = (x^m+ay^n)^{\star_{8.5}} = (x^m,y^n)^{\star_{8.5}} = (x,y)$, $(y^n)^{\star_{8.5}} = (y)$, $(z^r)^{\star_{8.5}} = (z^r)$, $(x^m+az^r)^{\star_{8.5}} = (y^n+az^r)^{\star_{8.5}} = (x^m+ay^n+bz^r)^{\star_{8.5}} = (x^m,z^r)^{\star_{8.5}} = (y^n,z^r)^{\star_{8.5}} = (x^m+az^r,y^n+bz^r)^{\star_{8.5}} = (x^m,y^n+az^r)^{\star_{8.5}} = (x^m+ay^n,z^r)^{\star_{8.5}} = (x^m+az^r,y^n)^{\star_{8.5}} = (x^m,y^n,z^r)^{\star_{8.5}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

679. $\star_{8.6}$:$(0)^{\star_{8.6}} = (y^n)^{\star_{8.6}} = (y)$, $(x^m)^{\star_{8.6}} = (x^m+ay^n)^{\star_{8.6}} = (x^m,y^n)^{\star_{8.6}} = (x,y)$, $(z^r)^{\star_{8.6}} = (y^n+az^r)^{\star_{8.6}} = (y^n,z^r)^{\star_{8.6}} = (y,z^r)$, $(x^m+az^r)^{\star_{8.6}}$ $= (x^m+ay^n+bz^r)^{\star_{8.6}} = (x^m,z^r)^{\star_{8.6}} = (x^m+az^r,y^n+bz^r)^{\star_{8.6}}$ $= (x^m,y^n+az^r)^{\star_{8.6}} = (x^m+ay^n,z^r)^{\star_{8.6}} = (x^m+az^r,y^n)^{\star_{8.6}} = (x^m,y^n,z^r)^{\star_{8.6}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

680. $\star_{8.7}$:$(0)^{\star_{8.7}} = (0)$, $(x^m)^{\star_{8.7}} = (x^m+ay^n)^{\star_{8.7}} = (x^m,y^n)^{\star_{8.7}} = (x,y)$, $(y^n)^{\star_{8.7}} = (y)$, $(z^r)^{\star_{8.7}} == (y^n+az^r)^{\star_{8.7}} = (y^n,z^r)^{\star_{8.7}} = (y,z^r)$, $(x^m+az^r)^{\star_{8.7}}$ $= (x^m+ay^n+bz^r)^{\star_{8.7}} = (x^m,z^r)^{\star_{8.7}} = (x^m+az^r,y^n+bz^r)^{\star_{8.7}}$ $= (x^m,y^n+az^r)^{\star_{8.7}} = (x^m+ay^n,z^r)^{\star_{8.7}} = (x^m+az^r,y^n)^{\star_{8.7}} = (x^m,y^n,z^r)^{\star_{8.7}} = (x,y,z^r)\ \forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

681. $\star_{8.8}$:$(0)^{\star_{8.8}} = (0)$, $(x^m)^{\star_{8.8}} = (x^m+ay^n)^{\star_{8.8}} = (x^m,y^n)^{\star_{8.8}} = (x,y)$, $(y^n)^{\star_{8.8}} = (y)$, $(z^r)^{\star_{8.8}} = (z^r)$, $(y^n+az^r)^{\star_{8.8}} = (y^n,z^r)^{\star_{8.8}} = (y,z^r)$, $(x^m+az^r)^{\star_{8.8}}$ $= (x^m+ay^n+bz^r)^{\star_{8.8}} = (x^m,z^r)^{\star_{8.8}} = (x^m+az^r,y^n+bz^r)^{\star_{8.8}}$

$= (x^m, y^n + az^r)^{\star_{8.8}} = (x^m + ay^n, z^r)^{\star_{8.8}} = (x^m + az^r, y^n)^{\star_{8.8}} = (x^m, y^n, z^r)^{\star_{8.8}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

682. $\star_{8.9}$:$(0)^{\star_{8.9}} = (0)$, $(x^m)^{\star_{8.9}} = (x^m + ay^n)^{\star_{8.9}} = (x^m, y^n)^{\star_{8.9}} = (x, y)$, $(y^n)^{\star_{8.9}} = (y^n)$, $(z^r)^{\star_{8.9}} = (x^m + az^r)^{\star_{8.9}} = (y^n + az^r)^{\star_{8.9}} = (x^m + ay^n + bz^r)^{\star_{8.9}} = (x^m, z^r)^{\star_{8.9}} = (y^n, z^r)^{\star_{8.9}} = (x^m + az^r, y^n + bz^r)^{\star_{8.9}} = (x^m, y^n + az^r)^{\star_{8.9}} = (x^m + ay^n, z^r)^{\star_{8.9}} = (x^m + az^r, y^n)^{\star_{8.9}} = (x^m, y^n, z^r)^{\star_{8.9}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

683. $\star_{8.10}$:$(0)^{\star_{8.10}} = (0)$, $(x^m)^{\star_{8.10}} = (x^m + ay^n)^{\star_{8.10}} = (x^m, y^n)^{\star_{8.10}} = (x, y)$, $(y^n)^{\star_{8.10}} = (y^n)$, $(z^r)^{\star_{8.10}} = (z^r)$, $(x^m + az^r)^{\star_{8.10}} = (y^n + az^r)^{\star_{8.10}} = (x^m + ay^n + bz^r)^{\star_{8.10}} = (x^m, z^r)^{\star_{8.10}} = (y^n, z^r)^{\star_{8.10}} = (x^m + az^r, y^n + bz^r)^{\star_{8.10}} = (x^m, y^n + az^r)^{\star_{8.10}} = (x^m + ay^n, z^r)^{\star_{8.10}} = (x^m + az^r, y^n)^{\star_{8.10}} = (x^m, y^n, z^r)^{\star_{8.10}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

684. $\star_{8.11}$:$(0)^{\star_{8.11}} = (0)$, $(x^m)^{\star_{8.11}} = (x^m + ay^n)^{\star_{8.11}} = (x^m, y^n)^{\star_{8.11}} = (x, y)$, $(y^n)^{\star_{8.11}} = (y^n)$, $(z^r)^{\star_{8.11}} = (y^n + az^r)^{\star_{8.11}} = (y^n, z^r)^{\star_{8.11}} = (y, z^r)$, $(x^m + az^r)^{\star_{8.11}} = (x^m + ay^n + bz^r)^{\star_{8.11}} = (x^m, z^r)^{\star_{8.11}} = (x^m + az^r, y^n + bz^r)^{\star_{8.11}} = (x^m, y^n + az^r)^{\star_{8.11}} = (x^m + ay^n, z^r)^{\star_{8.11}} = (x^m + az^r, y^n)^{\star_{8.11}} = (x^m, y^n, z^r)^{\star_{8.11}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

685. $\star_{8.12}$:$(0)^{\star_{8.12}} = (0)$, $(x^m)^{\star_{8.12}} = (x^m + ay^n)^{\star_{8.12}} = (x^m, y^n)^{\star_{8.12}} = (x, y)$, $(y^n)^{\star_{8.12}} = (y^n)$, $(z^r)^{\star_{8.12}} = (z^r)$, $(x^m + az^r)^{\star_{8.12}} = (x^m + ay^n + bz^r)^{\star_{8.12}} = (x^m, z^r)^{\star_{8.12}} = (x^m + az^r, y^n + bz^r)^{\star_{8.12}} = (x^m, y^n + az^r)^{\star_{8.12}} = (x^m + ay^n, z^r)^{\star_{8.12}} = (x^m + az^r, y^n)^{\star_{8.12}} = (x^m, y^n, z^r)^{\star_{8.12}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.12}} = (y^n, z^r)^{\star_{8.12}} = (y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

686. $\star_{8.13}$:$(0)^{\star_{8.13}} = (0)$, $(x^m)^{\star_{8.13}} = (x^m + ay^n)^{\star_{8.13}} = (x^m, y^n)^{\star_{8.13}} = (x, y)$, $(y^n)^{\star_{8.13}} = (y^n)$, $(z^r)^{\star_{8.13}} = (z^r)$, $(x^m + az^r)^{\star_{8.13}} = (x^m + ay^n + bz^r)^{\star_{8.13}} = (x^m, z^r)^{\star_{8.13}} = (x^m + az^r, y^n + bz^r)^{\star_{8.13}} = (x^m, y^n + az^r)^{\star_{8.13}} = (x^m + ay^n, z^r)^{\star_{8.13}} = (x^m + az^r, y^n)^{\star_{8.13}} = (x^m, y^n, z^r)^{\star_{8.13}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.13}} = (y^n, z^r)^{\star_{8.13}} = (y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

687. $\star_{8.14}$:$(0)^{\star_{8.14}} = (0)$, $(x^m)^{\star_{8.14}} = (x^m + ay^n)^{\star_{8.14}} = (x^m, y^n)^{\star_{8.14}} = (x, y)$, $(y^n)^{\star_{8.14}} =$

(y^n), $(z^r)^{\star_{8.14}} = (z^r)$, $(x^m + az^r)^{\star_{8.14}} = (x^m + ay^n + bz^r)^{\star_{8.14}} = (x^m, z^r)^{\star_{8.14}} = (x^m + az^r, y^n + bz^r)^{\star_{8.14}} = (x^m, y^n + az^r)^{\star_{8.14}} = (x^m + ay^n, z^r)^{\star_{8.14}} = (x^m + az^r, y^n)^{\star_{8.14}} = (x^m, y^n, z^r)^{\star_{8.14}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.14}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.14}} = (y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

688. $\star_{8.15}$:$(0)^{\star_{8.15}} = (x^m)^{\star_{8.15}} = (x)$, $(y^n)^{\star_{8.15}} = (x^m+ay^n)^{\star_{8.15}} = (x^m, y^n)^{\star_{8.15}} = (x, y)$, $(z^r)^{\star_{8.15}} = (x^m+az^r)^{\star_{8.15}} = (y^n+az^r)^{\star_{8.15}} = (x^m+ay^n+bz^r)^{\star_{8.15}} = (x^m, z^r)^{\star_{8.15}} = (y^n, z^r)^{\star_{8.15}} = (x^m+az^r, y^n+bz^r)^{\star_{8.15}} = (x^m, y^n+az^r)^{\star_{8.15}} = (x^m+ay^n, z^r)^{\star_{8.15}} = (x^m + az^r, y^n)^{\star_{8.15}} = (x^m, y^n, z^r)^{\star_{8.15}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

689. $\star_{8.16}$:$(0)^{\star_{8.16}} = (0)$, $(x^m)^{\star_{8.16}} = (x)$, $(y^n)^{\star_{8.16}} = (x^m + ay^n)^{\star_{8.16}} = (x^m, y^n)^{\star_{8.16}} = (x, y)$, $(z^r)^{\star_{8.16}} = (x^m + az^r)^{\star_{8.16}} = (y^n + az^r)^{\star_{8.16}} = (x^m + ay^n + bz^r)^{\star_{8.16}} = (x^m, z^r)^{\star_{8.16}} = (y^n, z^r)^{\star_{8.16}} = (x^m + az^r, y^n + bz^r)^{\star_{8.16}} = (x^m, y^n + az^r)^{\star_{8.16}} = (x^m+ay^n, z^r)^{\star_{8.16}} = (x^m+az^r, y^n)^{\star_{8.16}} = (x^m, y^n, z^r)^{\star_{8.16}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

690. $\star_{8.17}$:$(0)^{\star_{8.17}} = (0)$, $(x^m)^{\star_{8.17}} = (x)$, $(y^n)^{\star_{8.17}} = (x^m + ay^n)^{\star_{8.17}} = (x^m, y^n)^{\star_{8.17}} = (x, y)$, $(z^r)^{\star_{8.17}} = (z^r)$, $(x^m+az^r)^{\star_{8.17}} = (y^n+az^r)^{\star_{8.17}} = (x^m+ay^n+bz^r)^{\star_{8.17}} = (x^m, z^r)^{\star_{8.17}} = (y^n, z^r)^{\star_{8.17}} = (x^m + az^r, y^n + bz^r)^{\star_{8.17}} = (x^m, y^n + az^r)^{\star_{8.17}} = (x^m+ay^n, z^r)^{\star_{8.17}} = (x^m+az^r, y^n)^{\star_{8.17}} = (x^m, y^n, z^r)^{\star_{8.17}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

691. $\star_{8.18}$:$(0)^{\star_{8.18}} = (0)$, $(x^m)^{\star_{8.18}} = (x)$, $(y^n)^{\star_{8.18}} = (y)$, $(z^r)^{\star_{8.18}} = (x^m + az^r)^{\star_{8.18}} = (y^n + az^r)^{\star_{8.18}} = (x^m + ay^n + bz^r)^{\star_{8.18}} = (x^m, z^r)^{\star_{8.18}} = (y^n, z^r)^{\star_{8.18}} = (x^m + az^r, y^n + bz^r)^{\star_{8.18}} = (x^m, y^n + az^r)^{\star_{8.18}} = (x^m + ay^n, z^r)^{\star_{8.18}} = (x^m+az^r, y^n)^{\star_{8.18}} = (x^m, y^n, z^r)^{\star_{8.18}} = (x, y, z^r)$, $(x^m+ay^n)^{\star_{8.18}} = (x^m, y^n)^{\star_{8.18}} = (x, y)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

692. $\star_{8.19}$:$(0)^{\star_{8.19}} = (0)$, $(x^m)^{\star_{8.19}} = (x)$, $(y^n)^{\star_{8.19}} = (y)$, $(z^r)^{\star_{8.19}} = (z^r)$, $(x^m + ay^n)^{\star_{8.19}} = (x^m, y^n)^{\star_{8.19}} = (x, y)$, $(x^m + az^r)^{\star_{8.19}} = (y^n + az^r)^{\star_{8.19}} = (x^m + ay^n + bz^r)^{\star_{8.19}} = (x^m, z^r)^{\star_{8.19}} = (y^n, z^r)^{\star_{8.19}} = (x^m + az^r, y^n + bz^r)^{\star_{8.19}} = (x^m, y^n+az^r)^{\star_{8.19}} = (x^m+ay^n, z^r)^{\star_{8.19}} = (x^m+az^r, y^n)^{\star_{8.19}} = (x^m, y^n, z^r)^{\star_{8.19}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

693. $\star_{8.20}$:$(0)^{\star_{8.20}} = (0)$, $(x^m)^{\star_{8.20}} = (x)$, $(y^n)^{\star_{8.20}} = (y)$, $(z^r)^{\star_{8.20}} = (y^n + az^r)^{\star_{8.20}} = (y^n, z^r)^{\star_{8.20}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.20}} = (x^m, y^n)^{\star_{8.20}} = (x, y)$, $(x^m + az^r)^{\star_{8.20}} = (x^m + ay^n + bz^r)^{\star_{8.20}} = (x^m, z^r)^{\star_{8.20}} = (x^m + az^r, y^n + bz^r)^{\star_{8.20}} = (x^m, y^n + az^r)^{\star_{8.20}} = (x^m + ay^n, z^r)^{\star_{8.20}} = (x^m + az^r, y^n)^{\star_{8.20}} = (x^m, y^n, z^r)^{\star_{8.20}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

694. $\star_{8.21}$:$(0)^{\star_{8.21}} = (0)$, $(x^m)^{\star_{8.21}} = (x)$, $(y^n)^{\star_{8.21}} = (y)$, $(z^r)^{\star_{8.21}} = (z^r)$, $(x^m + ay^n)^{\star_{8.21}} = (x^m, y^n)^{\star_{8.21}} = (x, y)$, $(x^m + az^r)^{\star_{8.21}} = (x^m + ay^n + bz^r)^{\star_{8.21}} = (x^m, z^r)^{\star_{8.21}} = (x^m+az^r, y^n+bz^r)^{\star_{8.21}} = (x^m, y^n+az^r)^{\star_{8.21}} = (x^m+ay^n, z^r)^{\star_{8.21}} = (x^m + az^r, y^n)^{\star_{8.21}} = (x^m, y^n, z^r)^{\star_{8.21}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.21}} = (y^n, z^r)^{\star_{8.21}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

695. $\star_{8.22}$:$(0)^{\star_{8.22}} = (0)$, $(x^m)^{\star_{8.22}} = (x)$, $(y^n)^{\star_{8.22}} = (y^n)$, $(z^r)^{\star_{8.22}} = (x^m + az^r)^{\star_{8.22}} = (y^n + az^r)^{\star_{8.22}} = (x^m + ay^n + bz^r)^{\star_{8.22}} = (x^m, z^r)^{\star_{8.22}} = (y^n, z^r)^{\star_{8.22}} = (x^m + az^r, y^n + bz^r)^{\star_{8.22}} = (x^m, y^n + az^r)^{\star_{8.22}} = (x^m + ay^n, z^r)^{\star_{8.22}} = (x^m + az^r, y^n)^{\star_{8.22}} = (x^m, y^n, z^r)^{\star_{8.22}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.22}} = (x^m, y^n)^{\star_{8.22}} = (x, y)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

696. $\star_{8.23}$:$(0)^{\star_{8.23}} = (0)$, $(x^m)^{\star_{8.23}} = (x)$, $(y^n)^{\star_{8.23}} = (y^n)$, $(z^r)^{\star_{8.23}} = (z^r)$, $(x^m + ay^n)^{\star_{8.23}} = (x^m, y^n)^{\star_{8.23}} = (x, y)$, $(x^m + az^r)^{\star_{8.23}} = (y^n + az^r)^{\star_{8.23}} = (x^m + ay^n + bz^r)^{\star_{8.23}} = (x^m, z^r)^{\star_{8.23}} = (y^n, z^r)^{\star_{8.23}} = (x^m + az^r, y^n + bz^r)^{\star_{8.23}} = (x^m, y^n + az^r)^{\star_{8.23}} = (x^m + ay^n, z^r)^{\star_{8.23}} = (x^m + az^r, y^n)^{\star_{8.23}} = (x^m, y^n, z^r)^{\star_{8.23}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

697. $\star_{8.24}$:$(0)^{\star_{8.24}} = (0)$, $(x^m)^{\star_{8.24}} = (x)$, $(y^n)^{\star_{8.24}} = (y^n)$, $(z^r)^{\star_{8.24}} = (y^n + az^r)^{\star_{8.24}}) = (y^n, z^r)^{\star_{8.24}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.24}} = (x^m, y^n)^{\star_{8.24}} = (x, y)$, $(x^m + az^r)^{\star_{8.24}} = (x^m + ay^n + bz^r)^{\star_{8.24}} = (x^m, z^r)^{\star_{8.24}} = (x^m + az^r, y^n + bz^r)^{\star_{8.24}} = (x^m, y^n + az^r)^{\star_{8.24}} = (x^m + ay^n, z^r)^{\star_{8.24}} = (x^m + az^r, y^n)^{\star_{8.24}} = (x^m, y^n, z^r)^{\star_{8.24}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

698. $\star_{8.25}$:$(0)^{\star_{8.25}} = (0)$, $(x^m)^{\star_{8.25}} = (x)$, $(y^n)^{\star_{8.25}} = (y^n)$, $(z^r)^{\star_{8.25}} = (z^r)$, $(x^m + ay^n)^{\star_{8.25}} = (x^m, y^n)^{\star_{8.25}} = (x, y)$, $(x^m + az^r)^{\star_{8.25}} = (x^m + ay^n + bz^r)^{\star_{8.25}} = (x^m, z^r)^{\star_{8.25}} = (x^m+az^r, y^n+bz^r)^{\star_{8.25}} = (x^m, y^n+az^r)^{\star_{8.25}} = (x^m+ay^n, z^r)^{\star_{8.25}} =$

$(x^m + az^r, y^n)^{\star_{8.25}} = (x^m, y^n, z^r)^{\star_{8.25}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.25}} = (y^n, z^r)^{\star_{8.25}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

699. $\star_{8.26}$:$(0)^{\star_{8.26}} = (0)$, $(x^m)^{\star_{8.26}} = (x)$, $(y^n)^{\star_{8.26}} = (y^n)$, $(z^r)^{\star_{8.26}} = (z^r)$, $(x^m + ay^n)^{\star_{8.26}} = (x^m, y^n)^{\star_{8.26}} = (x, y)$, $(x^m + az^r)^{\star_{8.26}} = (x^m + ay^n + bz^r)^{\star_{8.26}} = (x^m, z^r)^{\star_{8.26}} = (x^m + az^r, y^n + bz^r)^{\star_{8.26}} = (x^m, y^n + az^r)^{\star_{8.26}} = (x^m + ay^n, z^r)^{\star_{8.26}} = (x^m + az^r, y^n)^{\star_{8.26}} = (x^m, y^n, z^r)^{\star_{8.26}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.26}} = (y^n, z^r)^{\star_{8.26}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

700. $\star_{8.27}$:$(0)^{\star_{8.27}} = (0)$, $(x^m)^{\star_{8.27}} = (x)$, $(y^n)^{\star_{8.27}} = (y^n)$, $(z^r)^{\star_{8.27}} = (z^r)$, $(x^m + ay^n)^{\star_{8.27}} = (x^m, y^n)^{\star_{8.27}} = (x, y)$, $(x^m + az^r)^{\star_{8.27}} = (x^m + ay^n + bz^r)^{\star_{8.27}} = (x^m, z^r)^{\star_{8.27}} = (x^m + az^r, y^n + bz^r)^{\star_{8.27}} = (x^m, y^n + az^r)^{\star_{8.27}} = (x^m + ay^n, z^r)^{\star_{8.27}} = (x^m + az^r, y^n)^{\star_{8.27}} = (x^m, y^n, z^r)^{\star_{8.27}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.27}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.27}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

701. $\star_{8.28}$:$(0)^{\star_{8.28}} = (x^m)^{\star_{8.28}} = (x)$, $(y^n)^{\star_{8.28}} = (x^m + ay^n)^{\star_{8.28}} = (x^m, y^n)^{\star_{8.28}} = (x, y)$, $(z^r)^{\star_{8.28}} = (x^m + az^r)^{\star_{8.28}} = (x^m, z^r)^{\star_{8.28}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.28}} = (x^m + ay^n + bz^r)^{\star_{8.28}} = (y^n, z^r)^{\star_{8.28}} = (x^m + az^r, y^n + bz^r)^{\star_{8.28}} = (x^m, y^n + az^r)^{\star_{8.28}} = (x^m + ay^n, z^r)^{\star_{8.28}} = (x^m + az^r, y^n)^{\star_{8.28}} = (x^m, y^n, z^r)^{\star_{8.28}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

702. $\star_{8.29}$:$(0)^{\star_{8.29}} = (0)$, $(x^m)^{\star_{8.29}} = (x)$, $(y^n)^{\star_{8.29}} = (x^m + ay^n)^{\star_{8.29}} = (x^m, y^n)^{\star_{8.29}} = (x, y)$, $(z^r)^{\star_{8.29}} = (x^m + az^r)^{\star_{8.29}} = (x^m, z^r)^{\star_{8.29}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.29}} = (x^m + ay^n + bz^r)^{\star_{8.29}} = (y^n, z^r)^{\star_{8.29}} = (x^m + az^r, y^n + bz^r)^{\star_{8.29}} = (x^m, y^n + az^r)^{\star_{8.29}} = (x^m + ay^n, z^r)^{\star_{8.29}} = (x^m + az^r, y^n)^{\star_{8.29}} = (x^m, y^n, z^r)^{\star_{8.29}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

703. $\star_{8.30}$:$(0)^{\star_{8.30}} = (0)$, $(x^m)^{\star_{8.30}} = (x)$, $(y^n)^{\star_{8.30}} = (x^m + ay^n)^{\star_{8.30}} = (x^m, y^n)^{\star_{8.30}} = (x, y)$, $(z^r)^{\star_{8.30}} = (z^r)$, $(x^m + az^r)^{\star_{8.30}} = (x^m, z^r)^{\star_{8.30}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.30}} = (x^m + ay^n + bz^r)^{\star_{8.30}} = (y^n, z^r)^{\star_{8.30}} = (x^m + az^r, y^n + bz^r)^{\star_{8.30}} = (x^m, y^n + az^r)^{\star_{8.30}} = (x^m + ay^n, z^r)^{\star_{8.30}} = (x^m + az^r, y^n)^{\star_{8.30}} = (x^m, y^n, z^r)^{\star_{8.30}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

704. $\star_{8.31}$:$(0)^{\star_{8.31}} = (0)$, $(x^m)^{\star_{8.31}} = (x)$, $(y^n)^{\star_{8.31}} = (y)$, $(z^r)^{\star_{8.31}} = (x^m + az^r)^{\star_{8.31}} =$

$(x^m, z^r)^{\star_{8.31}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.31}} = (x^m, y^n)^{\star_{8.31}} = (x, y)$, $(y^n + az^r)^{\star_{8.31}} = (x^m + ay^n + bz^r)^{\star_{8.31}} = (y^n, z^r)^{\star_{8.31}} = (x^m + az^r, y^n + bz^r)^{\star_{8.31}} = (x^m, y^n + az^r)^{\star_{8.31}} = (x^m + ay^n, z^r)^{\star_{8.31}} = (x^m + az^r, y^n)^{\star_{8.31}} = (x^m, y^n, z^r)^{\star_{8.31}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

705. $\star_{8.32}$:$(0)^{\star_{8.32}} = (0)$, $(x^m)^{\star_{8.32}} = (x)$, $(y^n)^{\star_{8.32}} = (y)$, $(z^r)^{\star_{8.32}} = (z^r)$, $(x^m + ay^n)^{\star_{8.32}} = (x^m, y^n)^{\star_{8.32}} = (x, y)$, $(x^m + az^r)^{\star_{8.32}} = (x^m, z^r)^{\star_{8.32}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.32}} = (x^m + ay^n + bz^r)^{\star_{8.32}} = (y^n, z^r)^{\star_{8.32}} = (x^m + az^r, y^n + bz^r)^{\star_{8.32}} = (x^m, y^n + az^r)^{\star_{8.32}} = (x^m + ay^n, z^r)^{\star_{8.32}} = (x^m + az^r, y^n)^{\star_{8.32}} = (x^m, y^n, z^r)^{\star_{8.32}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

706. $\star_{8.33}$:$(0)^{\star_{8.33}} = (0)$, $(x^m)^{\star_{8.33}} = (x)$, $(y^n)^{\star_{8.33}} = (y)$, $(z^r)^{\star_{8.33}} = (z^r)$, $(x^m + ay^n)^{\star_{8.33}} = (x^m, y^n)^{\star_{8.33}} = (x, y)$, $(x^m + az^r)^{\star_{8.33}} = (x^m, z^r)^{\star_{8.33}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.33}} = (y^n, z^r)^{\star_{8.33}} = (y, z^r)$ $(x^m + ay^n + bz^r)^{\star_{8.33}} = (x^m + az^r, y^n + bz^r)^{\star_{8.33}} = (x^m, y^n + az^r)^{\star_{8.33}} = (x^m + ay^n, z^r)^{\star_{8.33}} = (x^m + az^r, y^n)^{\star_{8.33}} = (x^m, y^n, z^r)^{\star_{8.33}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

707. $\star_{8.34}$:$(0)^{\star_{8.34}} = (0)$, $(x^m)^{\star_{8.34}} = (x)$, $(y^n)^{\star_{8.34}} = (y^n)$, $(z^r)^{\star_{8.34}} = (x^m + az^r)^{\star_{8.34}} = (x^m, z^r)^{\star_{8.34}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.34}} = (x^m, y^n)^{\star_{8.34}} = (x, y)$, $(y^n + az^r)^{\star_{8.34}} = (x^m + ay^n + bz^r)^{\star_{8.34}} = (y^n, z^r)^{\star_{8.34}} = (x^m + az^r, y^n + bz^r)^{\star_{8.34}} = (x^m, y^n + az^r)^{\star_{8.34}} = (x^m + ay^n, z^r)^{\star_{8.34}} = (x^m + az^r, y^n)^{\star_{8.34}} = (x^m, y^n, z^r)^{\star_{8.34}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

708. $\star_{8.35}$:$(0)^{\star_{8.35}} = (0)$, $(x^m)^{\star_{8.35}} = (x)$, $(y^n)^{\star_{8.35}} = (y^n)$, $(z^r)^{\star_{8.35}} = (z^r)$, $(x^m + ay^n)^{\star_{8.35}} = (x^m, y^n)^{\star_{8.35}} = (x, y)$, $(x^m + az^r)^{\star_{8.35}} = (x^m, z^r)^{\star_{8.35}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.35}} = (x^m + ay^n + bz^r)^{\star_{8.35}} = (y^n, z^r)^{\star_{8.35}} = (x^m + az^r, y^n + bz^r)^{\star_{8.35}} = (x^m, y^n + az^r)^{\star_{8.35}} = (x^m + ay^n, z^r)^{\star_{8.35}} = (x^m + az^r, y^n)^{\star_{8.35}} = (x^m, y^n, z^r)^{\star_{8.35}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

709. $\star_{8.36}$:$(0)^{\star_{8.36}} = (0)$, $(x^m)^{\star_{8.36}} = (x)$, $(y^n)^{\star_{8.36}} = (y^n)$, $(z^r)^{\star_{8.36}} = (z^r)$, $(x^m + ay^n)^{\star_{8.36}} = (x^m, y^n)^{\star_{8.36}} = (x, y)$, $(x^m + az^r)^{\star_{8.36}} = (x^m, z^r)^{\star_{8.36}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.36}} = (y^n, z^r)^{\star_{8.36}} = (y, z^r)$ $(x^m + ay^n + bz^r)^{\star_{8.36}} = (x^m + az^r, y^n + bz^r)^{\star_{8.36}} = (x^m, y^n + az^r)^{\star_{8.36}} = (x^m + ay^n, z^r)^{\star_{8.36}}$

$= (x^m + az^r, y^n)^{\star_{8.36}} = (x^m, y^n, z^r)^{\star_{8.36}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$.

710. $\star_{8.37}$:$(0)^{\star_{8.37}} = (0)$, $(x^m)^{\star_{8.37}} = (x)$, $(y^n)^{\star_{8.37}} = (y^n)$, $(z^r)^{\star_{8.37}} = (z^r)$, $(x^m + ay^n)^{\star_{8.37}} = (x^m, y^n)^{\star_{8.37}} = (x, y)$, $(x^m + az^r)^{\star_{8.37}} = (x^m, z^r)^{\star_{8.37}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.37}}) = (y^n, z^r)^{\star_{8.37}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.37}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.37}} = (x^m, y^n + az^r)^{\star_{8.37}} = (x^m + ay^n, z^r)^{\star_{8.37}}$ $= (x^m + az^r, y^n)^{\star_{8.37}} = (x^m, y^n, z^r)^{\star_{8.37}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

711. $\star_{8.38}$:$(0)^{\star_{8.38}} = (0)$, $(x^m)^{\star_{8.38}} = (x)$, $(y^n)^{\star_{8.38}} = (y^n)$, $(z^r)^{\star_{8.38}} = (z^r)$, $(x^m + ay^n)^{\star_{8.38}} = (x^m, y^n)^{\star_{8.38}} = (x, y)$, $(x^m + az^r)^{\star_{8.38}} = (x^m, z^r)^{\star_{8.38}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.38}}) = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.38}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.38}} =$ $(x^m + az^r, y^n + bz^r)^{\star_{8.38}} = (x^m, y^n + az^r)^{\star_{8.38}} = (x^m + ay^n, z^r)^{\star_{8.38}}$ $= (x^m + az^r, y^n)^{\star_{8.38}} = (x^m, y^n, z^r)^{\star_{8.38}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

712. $\star_{8.39}$:$(0)^{\star_{8.39}} = (0)$, $(x^m)^{\star_{8.39}} = (x^m)$, $(y^n)^{\star_{8.39}} = (x^m + ay^n)^{\star_{8.39}} = (x^m, y^n)^{\star_{8.39}} =$ (x, y), $(z^r)^{\star_{8.39}} = (x^m + az^r)^{\star_{8.39}} = (y^n + az^r)^{\star_{8.39}}) = (x^m + ay^n + bz^r)^{\star_{8.39}} =$ $(x^m, z^r)^{\star_{8.39}} = (y^n, z^r)^{\star_{8.39}} = (x^m + az^r, y^n + bz^r)^{\star_{8.39}} = (x^m, y^n + az^r)^{\star_{8.39}} =$ $(x^m + ay^n, z^r)^{\star_{8.39}} = (x^m + az^r, y^n)^{\star_{8.39}} = (x^m, y^n, z^r)^{\star_{8.39}} = (x, y, z^r)\ \forall\ m, n, r \in$ $\mathbb{N}$ and $a, b \in k^\times$

713. $\star_{8.40}$:$(0)^{\star_{8.40}} = (0)$, $(x^m)^{\star_{8.40}} = (x^m)$, $(y^n)^{\star_{8.40}} = (x^m + ay^n)^{\star_{8.40}} = (x^m, y^n)^{\star_{8.40}} =$ (x, y), $(z^r)^{\star_{8.40}} = (z^r)$, $(x^m + az^r)^{\star_{8.40}} = (y^n + az^r)^{\star_{8.40}}) = (x^m + ay^n + bz^r)^{\star_{8.40}} =$ $(x^m, z^r)^{\star_{8.40}} = (y^n, z^r)^{\star_{8.40}} = (x^m + az^r, y^n + bz^r)^{\star_{8.40}} = (x^m, y^n + az^r)^{\star_{8.40}}$ $= (x^m + ay^n, z^r)^{\star_{8.40}} = (x^m + az^r, y^n)^{\star_{8.40}} = (x^m, y^n, z^r)^{\star_{8.40}} = (x, y, z^r)\ \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

714. $\star_{8.41}$:$(0)^{\star_{8.41}} = (0)$, $(x^m)^{\star_{8.41}} = (x^m)$, $(y^n)^{\star_{8.41}} = (y)$, $(z^r)^{\star_{8.41}} = (x^m + az^r)^{\star_{8.41}} =$ $(y^n + az^r)^{\star_{8.41}}) = (x^m + ay^n + bz^r)^{\star_{8.41}} = (x^m, z^r)^{\star_{8.41}} = (y^n, z^r)^{\star_{8.41}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.41}} = (x^m, y^n + az^r)^{\star_{8.41}} = (x^m + ay^n, z^r)^{\star_{8.41}}$ $= (x^m + az^r, y^n)^{\star_{8.41}} = (x^m, y^n, z^r)^{\star_{8.41}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.41}} = (x^m, y^n)^{\star_{8.41}}$ $= (x, y)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

715. $\star_{8.42}$:$(0)^{\star_{8.42}} = (0)$, $(x^m)^{\star_{8.42}} = (x^m)$, $(y^n)^{\star_{8.42}} = (y)$, $(z^r)^{\star_{8.42}} = (z^r)$, $(x^m + ay^n)^{\star_{8.42}} = (x^m, y^n)^{\star_{8.42}} = (x, y)$, $(x^m + az^r)^{\star_{8.42}} = (y^n + az^r)^{\star_{8.42}}) =$

$(x^m + ay^n + bz^r)^{\star_{8.42}} = (x^m, z^r)^{\star_{8.42}} = (y^n, z^r)^{\star_{8.42}} = (x^m + az^r, y^n + bz^r)^{\star_{8.42}} = (x^m, y^n + az^r)^{\star_{8.42}} = (x^m + ay^n, z^r)^{\star_{8.42}} = (x^m + az^r, y^n)^{\star_{8.42}} = (x^m, y^n, z^r)^{\star_{8.42}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

716. $\star_{8.43}$:$(0)^{\star_{8.43}} = (0)$, $(x^m)^{\star_{8.43}} = (x^m)$, $(y^n)^{\star_{8.43}} = (y)$, $(z^r)^{\star_{8.43}} = (y^n + az^r)^{\star_{8.43}}) = (y^n, z^r)^{\star_{8.43}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.43}} = (x^m, y^n)^{\star_{8.43}} = (x, y)$, $(x^m + az^r)^{\star_{8.43}} = (x^m + ay^n + bz^r)^{\star_{8.43}} = (x^m, z^r)^{\star_{8.43}} = (x^m + az^r, y^n + bz^r)^{\star_{8.43}} = (x^m, y^n + az^r)^{\star_{8.43}} = (x^m + ay^n, z^r)^{\star_{8.43}} = (x^m + az^r, y^n)^{\star_{8.43}} = (x^m, y^n, z^r)^{\star_{8.43}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

717. $\star_{8.44}$:$(0)^{\star_{8.44}} = (0)$, $(x^m)^{\star_{8.44}} = (x^m)$, $(y^n)^{\star_{8.44}} = (y)$, $(z^r)^{\star_{8.44}} = (z^r)$, $(x^m + ay^n)^{\star_{8.44}} = (x^m, y^n)^{\star_{8.44}} = (x, y)$, $(x^m + az^r)^{\star_{8.44}} = (x^m + ay^n + bz^r)^{\star_{8.44}} = (x^m, z^r)^{\star_{8.44}} = (x^m + az^r, y^n + bz^r)^{\star_{8.44}} = (x^m, y^n + az^r)^{\star_{8.44}} = (x^m + ay^n, z^r)^{\star_{8.44}} = (x^m + az^r, y^n)^{\star_{8.44}} = (x^m, y^n, z^r)^{\star_{8.44}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.44}}) = (y^n, z^r)^{\star_{8.44}} = (y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

718. $\star_{8.45}$:$(0)^{\star_{8.45}} = (0)$, $(x^m)^{\star_{8.45}} = (x^m)$, $(y^n)^{\star_{8.45}} = (y^n)$, $(z^r)^{\star_{8.45}} = (x^m + az^r)^{\star_{8.45}} = (y^n + az^r)^{\star_{8.45}}) = (x^m + ay^n + bz^r)^{\star_{8.45}} = (x^m, z^r)^{\star_{8.45}} = (y^n, z^r)^{\star_{8.45}} = (x^m + az^r, y^n + bz^r)^{\star_{8.45}} = (x^m, y^n + az^r)^{\star_{8.45}} = (x^m + ay^n, z^r)^{\star_{8.45}} = (x^m + az^r, y^n)^{\star_{8.45}} = (x^m, y^n, z^r)^{\star_{8.45}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.45}} = (x^m, y^n)^{\star_{8.45}} = (x, y)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

719. $\star_{8.46}$:$(0)^{\star_{8.46}} = (0)$, $(x^m)^{\star_{8.46}} = (x^m)$, $(y^n)^{\star_{8.46}} = (y^n)$, $(z^r)^{\star_{8.46}} = (z^r)$, $(x^m + ay^n)^{\star_{8.46}} = (x^m, y^n)^{\star_{8.46}} = (x, y)$, $(x^m + az^r)^{\star_{8.46}} = (y^n + az^r)^{\star_{8.46}}) = (x^m + ay^n + bz^r)^{\star_{8.46}} = (x^m, z^r)^{\star_{8.46}} = (y^n, z^r)^{\star_{8.46}} = (x^m + az^r, y^n + bz^r)^{\star_{8.46}} = (x^m, y^n + az^r)^{\star_{8.46}} = (x^m + ay^n, z^r)^{\star_{8.46}} = (x^m + az^r, y^n)^{\star_{8.46}} = (x^m, y^n, z^r)^{\star_{8.46}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

720. $\star_{8.47}$:$(0)^{\star_{8.47}} = (0)$, $(x^m)^{\star_{8.47}} = (x^m)$, $(y^n)^{\star_{8.47}} = (y^n)$, $(z^r)^{\star_{8.47}} = (y^n + az^r)^{\star_{8.47}} = (y^n, z^r)^{\star_{8.47}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.47}} = (x^m, y^n)^{\star_{8.47}} = (x, y)$, $(x^m + az^r)^{\star_{8.47}} = (x^m + ay^n + bz^r)^{\star_{8.47}} = (x^m, z^r)^{\star_{8.47}} = (x^m + az^r, y^n + bz^r)^{\star_{8.47}} = (x^m, y^n + az^r)^{\star_{8.47}} = (x^m + ay^n, z^r)^{\star_{8.47}} = (x^m + az^r, y^n)^{\star_{8.47}} = (x^m, y^n, z^r)^{\star_{8.47}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

721. $\star_{8.48}$:$(0)^{\star_{8.48}} = (0)$, $(x^m)^{\star_{8.48}} = (x^m)$, $(y^n)^{\star_{8.48}} = (y^n)$, $(z^r)^{\star_{8.48}} = (z^r)$, $(x^m + ay^n)^{\star_{8.48}} = (x^m, y^n)^{\star_{8.48}} = (x, y)$,$(y^n + az^r)^{\star_{8.48}} = (y^n, z^r)^{\star_{8.48}} = (y, z^r)$, $(x^m+az^r)^{\star_{8.48}} = (x^m+ay^n+bz^r)^{\star_{8.48}} = (x^m, z^r)^{\star_{8.48}} = (x^m+az^r, y^n+bz^r)^{\star_{8.48}} = (x^m, y^n+az^r)^{\star_{8.48}} = (x^m+ay^n, z^r)^{\star_{8.48}} = (x^m+az^r, y^n)^{\star_{8.48}} = (x^m, y^n, z^r)^{\star_{8.48}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
722. $\star_{8.49}$:$(0)^{\star_{8.49}} = (0)$, $(x^m)^{\star_{8.49}} = (x^m)$, $(y^n)^{\star_{8.49}} = (y^n)$, $(z^r)^{\star_{8.49}} = (z^r)$, $(x^m + ay^n)^{\star_{8.49}} = (x^m, y^n)^{\star_{8.49}} = (x, y)$,$(y^n + az^r)^{\star_{8.49}} = (y^n, z^r)^{\star_{8.49}} = (y^n, z^r)$, $(x^m+az^r)^{\star_{8.49}} = (x^m+ay^n+bz^r)^{\star_{8.49}} = (x^m, z^r)^{\star_{8.49}} = (x^m+az^r, y^n+bz^r)^{\star_{8.49}} = (x^m, y^n+az^r)^{\star_{8.49}} = (x^m+ay^n, z^r)^{\star_{8.49}} = (x^m+az^r, y^n)^{\star_{8.49}} = (x^m, y^n, z^r)^{\star_{8.49}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
723. $\star_{8.50}$:$(0)^{\star_{8.50}} = (0)$, $(x^m)^{\star_{8.50}} = (x^m)$, $(y^n)^{\star_{8.50}} = (y^n)$, $(z^r)^{\star_{8.50}} = (z^r)$, $(x^m+ay^n)^{\star_{8.50}} = (x^m, y^n)^{\star_{8.50}} = (x, y)$,$(y^n+az^r)^{\star_{8.50}} = (y^n+az^r)^{\star_{8.50}}$, $(y^n, z^r)^{\star_{8.50}} = (y^n, z^r)$, $(x^m + az^r)^{\star_{8.50}} = (x^m + ay^n + bz^r)^{\star_{8.50}} = (x^m, z^r)^{\star_{8.50}} = (x^m + az^r, y^n + bz^r)^{\star_{8.50}} = (x^m, y^n + az^r)^{\star_{8.50}} = (x^m + ay^n, z^r)^{\star_{8.50}} = (x^m + az^r, y^n)^{\star_{8.50}} = (x^m, y^n, z^r)^{\star_{8.50}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
724. $\star_{8.51}$:$(0)^{\star_{8.51}} = (0)$, $(x^m)^{\star_{8.51}} = (x^m)$, $(y^n)^{\star_{8.51}} = (x^m+ay^n)^{\star_{8.51}} = (x^m, y^n)^{\star_{8.51}} = (x, y)$, $(z^r)^{\star_{8.51}} = (x^m + az^r)^{\star_{8.51}} = (x^m, z^r)^{\star_{8.51}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.51}} = (x^m + ay^n + bz^r)^{\star_{8.51}} = (y^n, z^r)^{\star_{8.51}} = (x^m + az^r, y^n + bz^r)^{\star_{8.51}} = (x^m, y^n+az^r)^{\star_{8.51}} = (x^m+ay^n, z^r)^{\star_{8.51}} = (x^m+az^r, y^n)^{\star_{8.51}} = (x^m, y^n, z^r)^{\star_{8.51}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
725. $\star_{8.52}$:$(0)^{\star_{8.52}} = (0)$, $(x^m)^{\star_{8.52}} = (x^m)$, $(y^n)^{\star_{8.52}} = (x^m+ay^n)^{\star_{8.52}} = (x^m, y^n)^{\star_{8.52}} = (x, y)$, $(z^r)^{\star_{8.52}} = (z^r)$, $(x^m + az^r)^{\star_{8.52}} = (x^m, z^r)^{\star_{8.52}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.52}} = (x^m + ay^n + bz^r)^{\star_{8.52}} = (y^n, z^r)^{\star_{8.52}} = (x^m + az^r, y^n + bz^r)^{\star_{8.52}} = (x^m, y^n+az^r)^{\star_{8.52}} = (x^m+ay^n, z^r)^{\star_{8.52}} = (x^m+az^r, y^n)^{\star_{8.52}} = (x^m, y^n, z^r)^{\star_{8.52}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
726. $\star_{8.53}$:$(0)^{\star_{8.53}} = (0)$, $(x^m)^{\star_{8.53}} = (x^m)$, $(y^n)^{\star_{8.53}} = (y)$, $(z^r)^{\star_{8.53}} = (x^m+az^r)^{\star_{8.53}} = (x^m, z^r)^{\star_{8.53}} = (x, z^r)$,$(x^m + ay^n)^{\star_{8.53}} = (x^m, y^n)^{\star_{8.53}} = (x, y)$, $(y^n + az^r)^{\star_{8.53}} = (x^m + ay^n + bz^r)^{\star_{8.53}} = (y^n, z^r)^{\star_{8.53}} = (x^m + az^r, y^n + bz^r)^{\star_{8.53}}$

$= (x^m, y^n + az^r)^{\star_{8.53}} = (x^m + ay^n, z^r)^{\star_{8.53}} = (x^m + az^r, y^n)^{\star_{8.53}} = (x^m, y^n, z^r)^{\star_{8.53}}$
$= (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

727. $\star_{8.54}$:$(0)^{\star_{8.54}} = (0)$, $(x^m)^{\star_{8.54}} = (x^m)$, $(y^n)^{\star_{8.54}} = (y)$, $(z^r)^{\star_{8.54}} = (z^r)$, $(x^m + ay^n)^{\star_{8.54}} = (x^m, y^n)^{\star_{8.54}} = (x, y)$, $(x^m + az^r)^{\star_{8.54}} = (x^m, z^r)^{\star_{8.54}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.54}} = (x^m + ay^n + bz^r)^{\star_{8.54}} = (y^n, z^r)^{\star_{8.54}} = (x^m + az^r, y^n + bz^r)^{\star_{8.54}} = (x^m, y^n + az^r)^{\star_{8.54}} = (x^m + ay^n, z^r)^{\star_{8.54}} = (x^m + az^r, y^n)^{\star_{8.54}} = (x^m, y^n, z^r)^{\star_{8.54}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

728. $\star_{8.55}$:$(0)^{\star_{8.55}} = (0)$, $(x^m)^{\star_{8.55}} = (x^m)$, $(y^n)^{\star_{8.55}} = (y)$, $(z^r)^{\star_{8.55}} = (z^r)$, $(x^m + ay^n)^{\star_{8.55}} = (x^m, y^n)^{\star_{8.55}} = (x, y)$, $(x^m + az^r)^{\star_{8.55}} = (x^m, z^r)^{\star_{8.55}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.55}} = (y^n, z^r)^{\star_{8.55}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.55}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.55}} = (x^m, y^n + az^r)^{\star_{8.55}} = (x^m + ay^n, z^r)^{\star_{8.55}}$
$= (x^m + az^r, y^n)^{\star_{8.55}} = (x^m, y^n, z^r)^{\star_{8.55}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

729. $\star_{8.56}$:$(0)^{\star_{8.56}} = (0)$, $(x^m)^{\star_{8.56}} = (x^m)$, $(y^n)^{\star_{8.56}} = (y^n)$, $(z^r)^{\star_{8.56}} = (x^m + az^r)^{\star_{8.56}} = (x^m, z^r)^{\star_{8.56}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.56}} = (x^m, y^n)^{\star_{8.56}} = (x, y)$, $(y^n + az^r)^{\star_{8.56}} = (x^m + ay^n + bz^r)^{\star_{8.56}} = (y^n, z^r)^{\star_{8.56}}(x^m + az^r, y^n + bz^r)^{\star_{8.56}} = (x^m, y^n + az^r)^{\star_{8.56}} = (x^m + ay^n, z^r)^{\star_{8.56}} = (x^m + az^r, y^n)^{\star_{8.56}} = (x^m, y^n, z^r)^{\star_{8.56}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

730. $\star_{8.57}$:$(0)^{\star_{8.57}} = (0)$, $(x^m)^{\star_{8.57}} = (x^m)$, $(y^n)^{\star_{8.57}} = (y^n)$, $(z^r)^{\star_{8.57}} = (z^r)$, $(x^m + ay^n)^{\star_{8.57}} = (x^m, y^n)^{\star_{8.57}} = (x, y)$, $(x^m + az^r)^{\star_{8.57}} = (x^m, z^r)^{\star_{8.57}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.57}} = (x^m + ay^n + bz^r)^{\star_{8.57}} = (y^n, z^r)^{\star_{8.57}}(x^m + az^r, y^n + bz^r)^{\star_{8.57}} = (x^m, y^n + az^r)^{\star_{8.57}} = (x^m + ay^n, z^r)^{\star_{8.57}} = (x^m + az^r, y^n)^{\star_{8.57}} = (x^m, y^n, z^r)^{\star_{8.57}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

731. $\star_{8.58}$:$(0)^{\star_{8.58}} = (0)$, $(x^m)^{\star_{8.58}} = (x^m)$, $(y^n)^{\star_{8.58}} = (y^n)$, $(z^r)^{\star_{8.58}} = (z^r)$, $(x^m + ay^n)^{\star_{8.58}} = (x^m, y^n)^{\star_{8.58}} = (x, y)$, $(x^m + az^r)^{\star_{8.58}} = (x^m, z^r)^{\star_{8.58}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.58}} = (y^n, z^r)^{\star_{8.58}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.58}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.58}} = (x^m, y^n + az^r)^{\star_{8.58}} = (x^m + ay^n, z^r)^{\star_{8.58}}$
$= (x^m + az^r, y^n)^{\star_{8.58}} = (x^m, y^n, z^r)^{\star_{8.58}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

732. $\star_{8.59}$:$(0)^{\star_{8.59}} = (0)$, $(x^m)^{\star_{8.59}} = (x^m)$, $(y^n)^{\star_{8.59}} = (y^n)$, $(z^r)^{\star_{8.59}} = (z^r)$,

$(x^m + ay^n)^{\star_{8.59}} = (x^m, y^n)^{\star_{8.59}} = (x, y)$, $(x^m + az^r)^{\star_{8.59}} = (x^m, z^r)^{\star_{8.59}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.59}} = (y^n, z^r)^{\star_{8.59}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.59}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.59}} = (x^m, y^n + az^r)^{\star_{8.59}} = (x^m + ay^n, z^r)^{\star_{8.59}}$ $= (x^m + az^r, y^n)^{\star_{8.59}} = (x^m, y^n, z^r)^{\star_{8.59}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

733. $\star_{8.60}$:$(0)^{\star_{8.60}} = (0)$, $(x^m)^{\star_{8.60}} = (x^m)$, $(y^n)^{\star_{8.60}} = (y^n)$, $(z^r)^{\star_{8.60}} = (z^r)$, $(x^m + ay^n)^{\star_{8.60}} = (x^m, y^n)^{\star_{8.60}} = (x, y)$, $(x^m + az^r)^{\star_{8.60}} = (x^m, z^r)^{\star_{8.60}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.60}} = (y^n + az^r)^{\star_{8.60}}$, $(y^n, z^r)^{\star_{8.60}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.60}} =$ $(x^m + az^r, y^n + bz^r)^{\star_{8.60}} = (x^m, y^n + az^r)^{\star_{8.60}} = (x^m + ay^n, z^r)^{\star_{8.60}}$ $= (x^m + az^r, y^n)^{\star_{8.60}} = (x^m, y^n, z^r)^{\star_{8.60}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

734. $\star_{8.61}$:$(0)^{\star_{8.61}} = (0)$, $(x^m)^{\star_{8.61}} = (x^m)$, $(y^n)^{\star_{8.61}} = (x^m + ay^n)^{\star_{8.61}} = (x^m, y^n)^{\star_{8.61}} =$ (x, y), $(z^r)^{\star_{8.61}} = (z^r)$, $(x^m + az^r)^{\star_{8.61}} = (x^m, z^r)^{\star_{8.61}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.61}} =$ $(x^m + ay^n + bz^r)^{\star_{8.61}} = (y^n, z^r)^{\star_{8.61}} = (x^m + az^r, y^n + bz^r)^{\star_{8.61}}$ $= (x^m, y^n + az^r)^{\star_{8.61}} = (x^m + ay^n, z^r)^{\star_{8.61}} = (x^m + az^r, y^n)^{\star_{8.61}} = (x^m, y^n, z^r)^{\star_{8.61}}$ $= (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

735. $\star_{8.62}$:$(0)^{\star_{8.62}} = (0)$, $(x^m)^{\star_{8.62}} = (x^m)$, $(y^n)^{\star_{8.62}} = (y)$, $(z^r)^{\star_{8.62}} = (z^r)$, $(x^m + ay^n)^{\star_{8.62}} = (x^m, y^n)^{\star_{8.62}} = (x, y)$, $(x^m + az^r)^{\star_{8.62}} = (x^m, z^r)^{\star_{8.62}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.62}} = (x^m + ay^n + bz^r)^{\star_{8.62}} = (y^n, z^r)^{\star_{8.62}} = (x^m + az^r, y^n + bz^r)^{\star_{8.62}} =$ $(x^m, y^n + az^r)^{\star_{8.62}} = (x^m + ay^n, z^r)^{\star_{8.62}} = (x^m + az^r, y^n)^{\star_{8.62}} = (x^m, y^n, z^r)^{\star_{8.62}} =$ (x, y, z^r) $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

736. $\star_{8.63}$:$(0)^{\star_{8.63}} = (0)$, $(x^m)^{\star_{8.63}} = (x^m)$, $(y^n)^{\star_{8.63}} = (y)$, $(z^r)^{\star_{8.63}} = (z^r)$, $(x^m + ay^n)^{\star_{8.63}} = (x^m, y^n)^{\star_{8.63}} = (x, y)$, $(x^m + az^r)^{\star_{8.63}} = (x^m, z^r)^{\star_{8.63}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.63}} = (y^n, z^r)^{\star_{8.63}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.63}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.63}} = (x^m, y^n + az^r)^{\star_{8.63}} = (x^m + ay^n, z^r)^{\star_{8.63}}$ $= (x^m + az^r, y^n)^{\star_{8.63}} = (x^m, y^n, z^r)^{\star_{8.63}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

737. $\star_{8.64}$:$(0)^{\star_{8.64}} = (0)$, $(x^m)^{\star_{8.64}} = (x^m)$, $(y^n)^{\star_{8.64}} = (y^n)$, $(z^r)^{\star_{8.64}} = (z^r)$, $(x^m + ay^n)^{\star_{8.64}} = (x^m, y^n)^{\star_{8.64}} = (x, y)$, $(x^m + az^r)^{\star_{8.64}} = (x^m, z^r)^{\star_{8.64}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.64}} = (x^m + ay^n + bz^r)^{\star_{8.64}} = (y^n, z^r)^{\star_{8.64}} = (x^m + az^r, y^n + bz^r)^{\star_{8.64}} =$ $(x^m, y^n + az^r)^{\star_{8.64}} = (x^m + ay^n, z^r)^{\star_{8.64}} = (x^m + az^r, y^n)^{\star_{8.64}} = (x^m, y^n, z^r)^{\star_{8.64}} =$

(x, y, z^r) $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

738. $\star_{8.65}$:$(0)^{\star_{8.65}} = (0)$, $(x^m)^{\star_{8.65}} = (x^m)$, $(y^n)^{\star_{8.65}} = (y^n)$, $(z^r)^{\star_{8.65}} = (z^r)$, $(x^m + ay^n)^{\star_{8.65}} = (x^m, y^n)^{\star_{8.65}} = (x, y)$, $(x^m + az^r)^{\star_{8.65}} = (x^m, z^r)^{\star_{8.65}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.65}} = (y^n, z^r)^{\star_{8.65}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.65}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.65}} = (x^m, y^n + az^r)^{\star_{8.65}} = (x^m + ay^n, z^r)^{\star_{8.65}}$ $= (x^m + az^r, y^n)^{\star_{8.65}} = (x^m, y^n, z^r)^{\star_{8.65}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

739. $\star_{8.66}$:$(0)^{\star_{8.66}} = (0)$, $(x^m)^{\star_{8.66}} = (x^m)$, $(y^n)^{\star_{8.66}} = (y^n)$, $(z^r)^{\star_{8.66}} = (z^r)$, $(x^m + ay^n)^{\star_{8.66}} = (x^m, y^n)^{\star_{8.66}} = (x, y)$, $(x^m + az^r)^{\star_{8.66}} = (x^m, z^r)^{\star_{8.66}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.66}} = (y^n, z^r)^{\star_{8.66}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.66}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.66}} = (x^m, y^n + az^r)^{\star_{8.66}} = (x^m + ay^n, z^r)^{\star_{8.66}}$ $= (x^m + az^r, y^n)^{\star_{8.66}} = (x^m, y^n, z^r)^{\star_{8.66}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

740. $\star_{8.67}$:$(0)^{\star_{8.67}} = (0)$, $(x^m)^{\star_{8.67}} = (x^m)$, $(y^n)^{\star_{8.67}} = (y^n)$, $(z^r)^{\star_{8.67}} = (z^r)$, $(x^m + ay^n)^{\star_{8.67}} = (x^m, y^n)^{\star_{8.67}} = (x, y)$, $(x^m + az^r)^{\star_{8.67}} = (x^m, z^r)^{\star_{8.67}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.67}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.67}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.67}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.67}} = (x^m, y^n + az^r)^{\star_{8.67}} = (x^m + ay^n, z^r)^{\star_{8.67}}$ $= (x^m + az^r, y^n)^{\star_{8.67}} = (x^m, y^n, z^r)^{\star_{8.67}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

741. $\star_{8.68}$:$(0)^{\star_{8.68}} = (0)$, $(x^m)^{\star_{8.68}} = (x^m)$, $(y^n)^{\star_{8.68}} = (x^m + ay^n)^{\star_{8.68}} = (x^m, y^n)^{\star_{8.68}} =$ (x, y), $(z^r)^{\star_{8.68}} = (z^r)$, $(x^m + az^r)^{\star_{8.68}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.68}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.68}} = (x^m + ay^n + bz^r)^{\star_{8.68}} = (y^n, z^r)^{\star_{8.68}} = (x^m + az^r, y^n + bz^r)^{\star_{8.68}} =$ $(x^m, y^n + az^r)^{\star_{8.68}} = (x^m + ay^n, z^r)^{\star_{8.68}} = (x^m + az^r, y^n)^{\star_{8.68}} = (x^m, y^n, z^r)^{\star_{8.68}} =$ (x, y, z^r) $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

742. $\star_{8.69}$:$(0)^{\star_{8.69}} = (0)$, $(x^m)^{\star_{8.69}} = (x^m)$, $(y^n)^{\star_{8.69}} = (y)$, $(z^r)^{\star_{8.69}} = (z^r)$, $(x^m + ay^n)^{\star_{8.69}} = (x^m, y^n)^{\star_{8.69}} = (x, y)$, $(x^m + az^r)^{\star_{8.69}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.69}}$ $= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.69}} = (x^m + ay^n + bz^r)^{\star_{8.69}} = (y^n, z^r)^{\star_{8.69}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.69}} = (x^m, y^n + az^r)^{\star_{8.69}} = (x^m + ay^n, z^r)^{\star_{8.69}}$ $= (x^m + az^r, y^n)^{\star_{8.69}} = (x^m, y^n, z^r)^{\star_{8.69}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

743. $\star_{8.70}$:$(0)^{\star_{8.70}} = (0)$, $(x^m)^{\star_{8.70}} = (x^m)$, $(y^n)^{\star_{8.70}} = (y)$, $(z^r)^{\star_{8.70}} = (z^r)$, $(x^m + ay^n)^{\star_{8.70}} = (x^m, y^n)^{\star_{8.70}} = (x, y)$, $(x^m + az^r)^{\star_{8.70}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.70}}$

$= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.70}} = (y^n, z^r)^{\star_{8.70}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.70}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.70}} = (x^m, y^n + az^r)^{\star_{8.70}} = (x^m + ay^n, z^r)^{\star_{8.70}}$
$= (x^m + az^r, y^n)^{\star_{8.70}} = (x^m, y^n, z^r)^{\star_{8.70}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

744. $\star_{8.71}$:$(0)^{\star_{8.71}} = (0)$, $(x^m)^{\star_{8.71}} = (x^m)$, $(y^n)^{\star_{8.71}} = (y^n)$, $(z^r)^{\star_{8.71}} = (z^r)$,
$(x^m + ay^n)^{\star_{8.71}} = (x^m, y^n)^{\star_{8.71}} = (x, y)$, $(x^m + az^r)^{\star_{8.71}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.71}}$
$= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.71}} = (x^m + ay^n + bz^r)^{\star_{8.71}} = (y^n, z^r)^{\star_{8.71}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.71}} = (x^m, y^n + az^r)^{\star_{8.71}} = (x^m + ay^n, z^r)^{\star_{8.71}}$
$= (x^m + az^r, y^n)^{\star_{8.71}} = (x^m, y^n, z^r)^{\star_{8.71}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

745. $\star_{8.72}$:$(0)^{\star_{8.72}} = (0)$, $(x^m)^{\star_{8.72}} = (x^m)$, $(y^n)^{\star_{8.72}} = (y^n)$, $(z^r)^{\star_{8.72}} = (z^r)$,
$(x^m + ay^n)^{\star_{8.72}} = (x^m, y^n)^{\star_{8.72}} = (x, y)$, $(x^m + az^r)^{\star_{8.72}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.72}}$
$= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.72}} = (y^n, z^r)^{\star_{8.72}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.72}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.72}} = (x^m, y^n + az^r)^{\star_{8.72}} = (x^m + ay^n, z^r)^{\star_{8.72}}$
$= (x^m + az^r, y^n)^{\star_{8.72}} = (x^m, y^n, z^r)^{\star_{8.72}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

746. $\star_{8.73}$:$(0)^{\star_{8.73}} = (0)$, $(x^m)^{\star_{8.73}} = (x^m)$, $(y^n)^{\star_{8.73}} = (y^n)$, $(z^r)^{\star_{8.73}} = (z^r)$,
$(x^m + ay^n)^{\star_{8.73}} = (x^m, y^n)^{\star_{8.73}} = (x, y)$, $(x^m + az^r)^{\star_{8.73}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.73}}$
$= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.73}} = (y^n, z^r)^{\star_{8.73}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.73}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.73}} = (x^m, y^n + az^r)^{\star_{8.73}} = (x^m + ay^n, z^r)^{\star_{8.73}}$
$= (x^m + az^r, y^n)^{\star_{8.73}} = (x^m, y^n, z^r)^{\star_{8.73}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

747. $\star_{8.74}$:$(0)^{\star_{8.74}} = (0)$, $(x^m)^{\star_{8.74}} = (x^m)$, $(y^n)^{\star_{8.74}} = (y^n)$, $(z^r)^{\star_{8.74}} = (z^r)$,
$(x^m + ay^n)^{\star_{8.74}} = (x^m, y^n)^{\star_{8.74}} = (x, y)$, $(x^m + az^r)^{\star_{8.74}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.74}}$
$= (x^m, z^r)$, $(y^n + az^r)^{\star_{8.74}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.74}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.74}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.74}} = (x^m, y^n + az^r)^{\star_{8.74}} = (x^m + ay^n, z^r)^{\star_{8.74}}$
$= (x^m + az^r, y^n)^{\star_{8.74}} = (x^m, y^n, z^r)^{\star_{8.74}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

748. $\star_{8.75}$:$(0)^{\star_{8.75}} = (x^m)^{\star_{8.75}} = (x)$, $(y^n)^{\star_{8.75}} = (x^m + ay^n)^{\star_{8.75}} = (x^m, y^n)^{\star_{8.75}} = (x, y^n)$, $(z^r)^{\star_{8.75}} = (x^m + az^r)^{\star_{8.75}} = (y^n + az^r)^{\star_{8.75}} = (x^m + ay^n + bz^r)^{\star_{8.75}} = (x^m, z^r)^{\star_{8.75}} = (y^n, z^r)^{\star_{8.75}} = (x^m + az^r, y^n + bz^r)^{\star_{8.75}} = (x^m, y^n + az^r)^{\star_{8.75}} = (x^m + ay^n, z^r)^{\star_{8.75}} = (x^m + az^r, y^n)^{\star_{8.75}} = (x^m, y^n, z^r)^{\star_{8.75}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

749. $\star_{8.76}$:$(0)^{\star_{8.76}} = (0)$, $(x^m)^{\star_{8.76}} = (x)$, $(y^n)^{\star_{8.76}} = (x^m + ay^n)^{\star_{8.76}} = (x^m, y^n)^{\star_{8.76}} = (x, y^n)$, $(z^r)^{\star_{8.76}} = (x^m + az^r)^{\star_{8.76}} = (y^n + az^r)^{\star_{8.76}} = (x^m + ay^n + bz^r)^{\star_{8.76}} = (x^m, z^r)^{\star_{8.76}} = (y^n, z^r)^{\star_{8.76}} = (x^m + az^r, y^n + bz^r)^{\star_{8.76}} = (x^m, y^n + az^r)^{\star_{8.76}} = (x^m + ay^n, z^r)^{\star_{8.76}} = (x^m + az^r, y^n)^{\star_{8.76}} = (x^m, y^n, z^r)^{\star_{8.76}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

750. $\star_{8.77}$:$(0)^{\star_{8.77}} = (0)$, $(x^m)^{\star_{8.77}} = (x)$, $(y^n)^{\star_{8.77}} = (x^m + ay^n)^{\star_{8.77}} = (x^m, y^n)^{\star_{8.77}} = (x, y^n)$, $(z^r)^{\star_{8.77}} = (z^r)$, $(x^m + az^r)^{\star_{8.77}} = (y^n + az^r)^{\star_{8.77}} = (x^m + ay^n + bz^r)^{\star_{8.77}} = (x^m, z^r)^{\star_{8.77}} = (y^n, z^r)^{\star_{8.77}} = (x^m + az^r, y^n + bz^r)^{\star_{8.77}} = (x^m, y^n + az^r)^{\star_{8.77}} = (x^m + ay^n, z^r)^{\star_{8.77}} = (x^m + az^r, y^n)^{\star_{8.77}} = (x^m, y^n, z^r)^{\star_{8.77}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

751. $\star_{8.78}$:$(0)^{\star_{8.78}} = (0)$, $(x^m)^{\star_{8.78}} = (x)$, $(y^n)^{\star_{8.78}} = (y^n)$, $(z^r)^{\star_{8.78}} = (x^m + az^r)^{\star_{8.78}} = (y^n + az^r)^{\star_{8.78}} = (x^m + ay^n + bz^r)^{\star_{8.78}} = (x^m, z^r)^{\star_{8.78}} = (y^n, z^r)^{\star_{8.78}} = (x^m + az^r, y^n + bz^r)^{\star_{8.78}} = (x^m, y^n + az^r)^{\star_{8.78}} = (x^m + ay^n, z^r)^{\star_{8.78}} = (x^m + az^r, y^n)^{\star_{8.78}} = (x^m, y^n, z^r)^{\star_{8.78}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.78}} = (x^m, y^n)^{\star_{8.78}} = (x, y^n)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

752. $\star_{8.79}$:$(0)^{\star_{8.79}} = (0)$, $(x^m)^{\star_{8.79}} = (x)$, $(y^n)^{\star_{8.79}} = (y^n)$, $(z^r)^{\star_{8.79}} = (z^r)$, $(x^m + ay^n)^{\star_{8.79}} = (x^m, y^n)^{\star_{8.79}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.79}} = (y^n + az^r)^{\star_{8.79}} = (x^m + ay^n + bz^r)^{\star_{8.79}} = (x^m, z^r)^{\star_{8.79}} = (y^n, z^r)^{\star_{8.79}} = (x^m + az^r, y^n + bz^r)^{\star_{8.79}} = (x^m, y^n + az^r)^{\star_{8.79}} = (x^m + ay^n, z^r)^{\star_{8.79}} = (x^m + az^r, y^n)^{\star_{8.79}} = (x^m, y^n, z^r)^{\star_{8.79}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

753. $\star_{8.80}$:$(0)^{\star_{8.80}} = (0)$, $(x^m)^{\star_{8.80}} = (x)$, $(y^n)^{\star_{8.80}} = (y^n)$, $(z^r)^{\star_{8.80}} = (y^n + az^r)^{\star_{8.80}} = (y^n, z^r)^{\star_{8.80}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.80}} = (x^m, y^n)^{\star_{8.80}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.80}} = (x^m + ay^n + bz^r)^{\star_{8.80}} = (x^m, z^r)^{\star_{8.80}} = (x^m + az^r, y^n + bz^r)^{\star_{8.80}} = (x^m, y^n + az^r)^{\star_{8.80}} = (x^m + ay^n, z^r)^{\star_{8.80}} = (x^m + az^r, y^n)^{\star_{8.80}} = (x^m, y^n, z^r)^{\star_{8.80}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

754. $\star_{8.81}$:$(0)^{\star_{8.81}} = (0)$, $(x^m)^{\star_{8.81}} = (x)$, $(y^n)^{\star_{8.81}} = (y^n)$, $(z^r)^{\star_{8.81}} = (z^r)$, $(x^m + ay^n)^{\star_{8.81}} = (x^m, y^n)^{\star_{8.81}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.81}} = (x^m + ay^n + bz^r)^{\star_{8.81}} = (x^m, z^r)^{\star_{8.81}} = (x^m + az^r, y^n + bz^r)^{\star_{8.81}} = (x^m, y^n + az^r)^{\star_{8.81}} = (x^m + ay^n, z^r)^{\star_{8.81}} =$

$(x^m+az^r,y^n)^{\star_{8.81}} = (x^m,y^n,z^r)^{\star_{8.81}} = (x,y,z^r)$, $(y^n+az^r)^{\star_{8.81}} = (y^n,z^r)^{\star_{8.81}} = (y,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

755. $\star_{8.82}$:$(0)^{\star_{8.82}} = (0)$, $(x^m)^{\star_{8.82}} = (x)$, $(y^n)^{\star_{8.82}} = (y^n)$, $(z^r)^{\star_{8.82}} = (z^r)$, $(x^m+ay^n)^{\star_{8.82}} = (x^m,y^n)^{\star_{8.82}} = (x,y^n)$, $(x^m+az^r)^{\star_{8.82}} = (x^m+ay^n+bz^r)^{\star_{8.82}} = (x^m,z^r)^{\star_{8.82}} = (x^m+az^r,y^n+bz^r)^{\star_{8.82}} = (x^m,y^n+az^r)^{\star_{8.82}} = (x^m+ay^n,z^r)^{\star_{8.82}} = (x^m+az^r,y^n)^{\star_{8.82}} = (x^m,y^n,z^r)^{\star_{8.82}} = (x,y,z^r)$, $(y^n+az^r)^{\star_{8.82}} = (y^n,z^r)^{\star_{8.82}} = (y^n,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

756. $\star_{8.83}$:$(0)^{\star_{8.83}} = (0)$, $(x^m)^{\star_{8.83}} = (x)$, $(y^n)^{\star_{8.83}} = (y^n)$, $(z^r)^{\star_{8.83}} = (z^r)$, $(x^m+ay^n)^{\star_{8.83}} = (x^m,y^n)^{\star_{8.83}} = (x,y^n)$, $(x^m+az^r)^{\star_{8.83}} = (x^m+ay^n+bz^r)^{\star_{8.83}} = (x^m,z^r)^{\star_{8.83}} = (x^m+az^r,y^n+bz^r)^{\star_{8.83}} = (x^m,y^n+az^r)^{\star_{8.83}} = (x^m+ay^n,z^r)^{\star_{8.83}} = (x^m+az^r,y^n)^{\star_{8.83}} = (x^m,y^n,z^r)^{\star_{8.83}} = (x,y,z^r)$, $(y^n+az^r)^{\star_{8.83}} = (y^n+az^r)$, $(y^n,z^r)^{\star_{8.83}} = (y^n,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

757. $\star_{8.84}$:$(0)^{\star_{8.84}} = (x^m)^{\star_{8.84}} = (x)$, $(y^n)^{\star_{8.84}} = (x^m+ay^n)^{\star_{8.84}} = (x^m,y^n)^{\star_{8.84}} = (x,y^n)$, $(z^r)^{\star_{8.84}} = (x^m+az^r)^{\star_{8.84}} = (x^m,z^r)^{\star_{8.84}} = (x,z^r)$, $(y^n+az^r)^{\star_{8.84}} = (x^m+ay^n+bz^r)^{\star_{8.84}} = (y^n,z^r)^{\star_{8.84}} = (x^m+az^r,y^n+bz^r)^{\star_{8.84}} = (x^m,y^n+az^r)^{\star_{8.84}} = (x^m+ay^n,z^r)^{\star_{8.84}} = (x^m+az^r,y^n)^{\star_{8.84}} = (x^m,y^n,z^r)^{\star_{8.84}} = (x,y,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

758. $\star_{8.85}$:$(0)^{\star_{8.85}} = (0)$, $(x^m)^{\star_{8.85}} = (x)$, $(y^n)^{\star_{8.85}} = (x^m+ay^n)^{\star_{8.85}} = (x^m,y^n)^{\star_{8.85}} = (x,y^n)$, $(z^r)^{\star_{8.85}} = (x^m+az^r)^{\star_{8.85}} = (x^m,z^r)^{\star_{8.85}} = (x,z^r)$, $(y^n+az^r)^{\star_{8.85}} = (x^m+ay^n+bz^r)^{\star_{8.85}} = (y^n,z^r)^{\star_{8.85}} = (x^m+az^r,y^n+bz^r)^{\star_{8.85}} = (x^m,y^n+az^r)^{\star_{8.85}} = (x^m+ay^n,z^r)^{\star_{8.85}} = (x^m+az^r,y^n)^{\star_{8.85}} = (x^m,y^n,z^r)^{\star_{8.85}} = (x,y,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

759. $\star_{8.86}$:$(0)^{\star_{8.86}} = (0)$, $(x^m)^{\star_{8.86}} = (x)$, $(y^n)^{\star_{8.86}} = (x^m+ay^n)^{\star_{8.86}} = (x^m,y^n)^{\star_{8.86}} = (x,y^n)$, $(z^r)^{\star_{8.86}} = (z^r)$, $(x^m+az^r)^{\star_{8.86}} = (x^m,z^r)^{\star_{8.86}} = (x,z^r)$, $(y^n+az^r)^{\star_{8.86}} = (x^m+ay^n+bz^r)^{\star_{8.86}} = (y^n,z^r)^{\star_{8.86}} = (x^m+az^r,y^n+bz^r)^{\star_{8.86}} = (x^m,y^n+az^r)^{\star_{8.86}} = (x^m+ay^n,z^r)^{\star_{8.86}} = (x^m+az^r,y^n)^{\star_{8.86}} = (x^m,y^n,z^r)^{\star_{8.86}} = (x,y,z^r)$ $\forall$ $m,n,r\in\mathbb{N}$ and $a,b\in k^\times$

760. $\star_{8.87}$:$(0)^{\star_{8.87}} = (0)$, $(x^m)^{\star_{8.87}} = (x)$, $(y^n)^{\star_{8.87}} = (y^n)$, $(z^r)^{\star_{8.87}} = (x^m+az^r)^{\star_{8.87}} =$

$(x^m, z^r)^{\star_{8.87}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.87}} = (x^m, y^n)^{\star_{8.87}} = (x, y^n)$, $(y^n + az^r)^{\star_{8.87}} = (x^m + ay^n + bz^r)^{\star_{8.87}} = (y^n, z^r)^{\star_{8.87}} = (x^m + az^r, y^n + bz^r)^{\star_{8.87}}$
$= (x^m, y^n + az^r)^{\star_{8.87}} = (x^m + ay^n, z^r)^{\star_{8.87}} = (x^m + az^r, y^n)^{\star_{8.87}} = (x^m, y^n, z^r)^{\star_{8.87}}$
$= (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

761. $\star_{8.88}$:$(0)^{\star_{8.88}} = (0)$, $(x^m)^{\star_{8.88}} = (x)$, $(y^n)^{\star_{8.88}} = (y^n)$, $(z^r)^{\star_{8.88}} = (z^r)$, $(x^m + ay^n)^{\star_{8.88}} = (x^m, y^n)^{\star_{8.88}} = (x, y^n)$,$(x^m + az^r)^{\star_{8.88}} = (x^m, z^r)^{\star_{8.88}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.88}} = (x^m + ay^n + bz^r)^{\star_{8.88}} = (y^n, z^r)^{\star_{8.88}} = (x^m + az^r, y^n + bz^r)^{\star_{8.88}} = (x^m, y^n + az^r)^{\star_{8.88}} = (x^m + ay^n, z^r)^{\star_{8.88}} = (x^m + az^r, y^n)^{\star_{8.88}} = (x^m, y^n, z^r)^{\star_{8.88}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

762. $\star_{8.89}$:$(0)^{\star_{8.89}} = (0)$, $(x^m)^{\star_{8.89}} = (x)$, $(y^n)^{\star_{8.89}} = (y^n)$, $(z^r)^{\star_{8.89}} = (z^r)$, $(x^m + ay^n)^{\star_{8.89}} = (x^m, y^n)^{\star_{8.89}} = (x, y^n)$,$(x^m + az^r)^{\star_{8.89}} = (x^m, z^r)^{\star_{8.89}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.89}} = (y^n, z^r)^{\star_{8.89}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.89}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.89}} = (x^m, y^n + az^r)^{\star_{8.89}} = (x^m + ay^n, z^r)^{\star_{8.89}}$
$= (x^m + az^r, y^n)^{\star_{8.89}} = (x^m, y^n, z^r)^{\star_{8.89}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

763. $\star_{8.90}$:$(0)^{\star_{8.90}} = (0)$, $(x^m)^{\star_{8.90}} = (x)$, $(y^n)^{\star_{8.90}} = (y^n)$, $(z^r)^{\star_{8.90}} = (z^r)$, $(x^m + ay^n)^{\star_{8.90}} = (x^m, y^n)^{\star_{8.90}} = (x, y^n)$,$(x^m + az^r)^{\star_{8.90}} = (x^m, z^r)^{\star_{8.90}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.90}} = (y^n, z^r)^{\star_{8.90}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.90}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.90}} = (x^m, y^n + az^r)^{\star_{8.90}} = (x^m + ay^n, z^r)^{\star_{8.90}}$
$= (x^m + az^r, y^n)^{\star_{8.90}} = (x^m, y^n, z^r)^{\star_{8.90}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

764. $\star_{8.91}$:$(0)^{\star_{8.91}} = (0)$, $(x^m)^{\star_{8.91}} = (x)$, $(y^n)^{\star_{8.91}} = (y^n)$, $(z^r)^{\star_{8.91}} = (z^r)$, $(x^m + ay^n)^{\star_{8.91}} = (x^m, y^n)^{\star_{8.91}} = (x, y^n)$,$(x^m + az^r)^{\star_{8.91}} = (x^m, z^r)^{\star_{8.91}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.91}} = (y^n + az^r)^{\star_{8.91}}$, $(y^n, z^r)^{\star_{8.91}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.91}}$
$= (x^m + az^r, y^n + bz^r)^{\star_{8.91}} = (x^m, y^n + az^r)^{\star_{8.91}} = (x^m + ay^n, z^r)^{\star_{8.91}}$
$= (x^m + az^r, y^n)^{\star_{8.91}} = (x^m, y^n, z^r)^{\star_{8.91}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

765. $\star_{8.92}$:$(0)^{\star_{8.92}} = (0)$, $(x^m)^{\star_{8.92}} = (x^m)$, $(y^n)^{\star_{8.92}} = (x^m + ay^n)^{\star_{8.92}} = (x^m, y^n)^{\star_{8.92}} = (x, y^n)$, $(z^r)^{\star_{8.92}} = (x^m + az^r)^{\star_{8.92}} = (y^n + az^r)^{\star_{8.92}} = (x^m + ay^n + bz^r)^{\star_{8.92}} = (x^m, z^r)^{\star_{8.92}} = (y^n, z^r)^{\star_{8.92}} = (x^m + az^r, y^n + bz^r)^{\star_{8.92}} = (x^m, y^n + az^r)^{\star_{8.92}} = (x^m + ay^n, z^r)^{\star_{8.92}} = (x^m + az^r, y^n)^{\star_{8.92}} = (x^m, y^n, z^r)^{\star_{8.92}} = (x, y, z^r)$ $\forall\, m, n, r \in$

$\mathbb{N}$ and $a, b \in k^\times$

766. $\star_{8.93}$:$(0)^{\star 8.93} = (0)$, $(x^m)^{\star 8.93} = (x^m)$, $(y^n)^{\star 8.93} = (y^n)$, $(x^m + ay^n)^{\star 8.93} = (x^m, y^n)^{\star 8.93} = (x, y^n)$, $(z^r)^{\star 8.93} = (x^m + az^r)^{\star 8.93} = (y^n + az^r)^{\star 8.93} = (x^m + ay^n + bz^r)^{\star 8.93} = (x^m, z^r)^{\star 8.93} = (y^n, z^r)^{\star 8.93} = (x^m + az^r, y^n + bz^r)^{\star 8.93} = (x^m, y^n + az^r)^{\star 8.93} = (x^m + ay^n, z^r)^{\star 8.93} = (x^m + az^r, y^n)^{\star 8.93} = (x^m, y^n, z^r)^{\star 8.93} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

767. $\star_{8.94}$:$(0)^{\star 8.94} = (0)$, $(x^m)^{\star 8.94} = (x^m)$, $(y^n)^{\star 8.94} = (y^n)$, $(z^r)^{\star 8.94} = (z^r)$, $(x^m + ay^n)^{\star 8.94} = (x^m, y^n)^{\star 8.94} = (x, y^n)$, $(x^m + az^r)^{\star 8.94} = (y^n + az^r)^{\star 8.94} = (x^m + ay^n + bz^r)^{\star 8.94} = (x^m, z^r)^{\star 8.94} = (y^n, z^r)^{\star 8.94} = (x^m + az^r, y^n + bz^r)^{\star 8.94} = (x^m, y^n + az^r)^{\star 8.94} = (x^m + ay^n, z^r)^{\star 8.94} = (x^m + az^r, y^n)^{\star 8.94} = (x^m, y^n, z^r)^{\star 8.94} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

768. $\star_{8.95}$:$(0)^{\star 8.95} = (0)$, $(x^m)^{\star 8.95} = (x^m)$, $(y^n)^{\star 8.95} = (y^n)$, $(z^r)^{\star 8.95} = (y^n + az^r)^{\star 8.95} = (y^n, z^r)^{\star 8.95} = (y, z^r)$, $(x^m + ay^n)^{\star 8.95} = (x^m, y^n)^{\star 8.95} = (x, y^n)$, $(x^m + az^r)^{\star 8.95} = (x^m + ay^n + bz^r)^{\star 8.95} = (x^m, z^r)^{\star 8.95} = (x^m + az^r, y^n + bz^r)^{\star 8.95} = (x^m, y^n + az^r)^{\star 8.95} = (x^m + ay^n, z^r)^{\star 8.95} = (x^m + az^r, y^n)^{\star 8.95} = (x^m, y^n, z^r)^{\star 8.95} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

769. $\star_{8.96}$:$(0)^{\star 8.96} = (0)$, $(x^m)^{\star 8.96} = (x^m)$, $(y^n)^{\star 8.96} = (y^n)$, $(z^r)^{\star 8.96} = (z^r)$, $(x^m + ay^n)^{\star 8.96} = (x^m, y^n)^{\star 8.96} = (x, y^n)$, $(x^m + az^r)^{\star 8.96} = (x^m + ay^n + bz^r)^{\star 8.96} = (x^m, z^r)^{\star 8.96} = (x^m + az^r, y^n + bz^r)^{\star 8.96} = (x^m, y^n + az^r)^{\star 8.96} = (x^m + ay^n, z^r)^{\star 8.96} = (x^m + az^r, y^n)^{\star 8.96} = (x^m, y^n, z^r)^{\star 8.96} = (x, y, z^r)$, $(y^n + az^r)^{\star 8.96} = (y^n, z^r)^{\star 8.96} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

770. $\star_{8.97}$:$(0)^{\star 8.97} = (0)$, $(x^m)^{\star 8.97} = (x^m)$, $(y^n)^{\star 8.97} = (y^n)$, $(z^r)^{\star 8.97} = (z^r)$, $(x^m + ay^n)^{\star 8.97} = (x^m, y^n)^{\star 8.97} = (x, y^n)$, $(x^m + az^r)^{\star 8.97} = (x^m + ay^n + bz^r)^{\star 8.97} = (x^m, z^r)^{\star 8.97} = (x^m + az^r, y^n + bz^r)^{\star 8.97} = (x^m, y^n + az^r)^{\star 8.97} = (x^m + ay^n, z^r)^{\star 8.97} = (x^m + az^r, y^n)^{\star 8.97} = (x^m, y^n, z^r)^{\star 8.97} = (x, y, z^r)$, $(y^n + az^r)^{\star 8.97} = (y^n, z^r)^{\star 8.97} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

771. $\star_{8.98}$:$(0)^{\star 8.98} = (0)$, $(x^m)^{\star 8.98} = (x^m)$, $(y^n)^{\star 8.98} = (y^n)$, $(z^r)^{\star 8.98} = (z^r)$, $(x^m + ay^n)^{\star 8.98} = (x^m, y^n)^{\star 8.98} = (x, y^n)$, $(x^m + az^r)^{\star 8.98} = (x^m + ay^n + bz^r)^{\star 8.98} =$

$(x^m, z^r)^{\star_{8.98}} = (x^m + az^r, y^n + bz^r)^{\star_{8.98}} = (x^m, y^n + az^r)^{\star_{8.98}} = (x^m + ay^n, z^r)^{\star_{8.98}} = (x^m + az^r, y^n)^{\star_{8.98}} = (x^m, y^n, z^r)^{\star_{8.98}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.98}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.98}} = (y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

772. $\star_{8.99}$:$(0)^{\star_{8.99}} = (0)$, $(x^m)^{\star_{8.99}} = (x^m)$, $(y^n)^{\star_{8.99}} = (x^m + ay^n)^{\star_{8.99}} = (x^m, y^n)^{\star_{8.99}} = (x, y^n)$, $(z^r)^{\star_{8.99}} = (x^m + az^r)^{\star_{8.99}} = (x^m, z^r)^{\star_{8.99}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.99}} = (x^m + ay^n + bz^r)^{\star_{8.99}} = (y^n, z^r)^{\star_{8.99}} = (x^m + az^r, y^n + bz^r)^{\star_{8.99}} = (x^m, y^n + az^r)^{\star_{8.99}} = (x^m + ay^n, z^r)^{\star_{8.99}} = (x^m + az^r, y^n)^{\star_{8.99}} = (x^m, y^n, z^r)^{\star_{8.99}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

773. $\star_{8.100}$:$(0)^{\star_{8.100}} = (0)$, $(x^m)^{\star_{8.100}} = (x^m)$, $(y^n)^{\star_{8.100}} = (x^m + ay^n)^{\star_{8.100}} = (x^m, y^n)^{\star_{8.100}} = (x, y^n)$, $(z^r)^{\star_{8.100}} = (z^r)$, $(x^m + az^r)^{\star_{8.100}} = (x^m, z^r)^{\star_{8.100}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.100}} = (x^m + ay^n + bz^r)^{\star_{8.100}} = (y^n, z^r)^{\star_{8.100}} = (x^m + az^r, y^n + bz^r)^{\star_{8.100}} = (x^m, y^n + az^r)^{\star_{8.100}} = (x^m + ay^n, z^r)^{\star_{8.100}} = (x^m + az^r, y^n)^{\star_{8.100}} = (x^m, y^n, z^r)^{\star_{8.100}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

774. $\star_{8.101}$:$(0)^{\star_{8.101}} = (0)$, $(x^m)^{\star_{8.101}} = (x^m)$, $(y^n)^{\star_{8.101}} = (y^n)$, $(z^r)^{\star_{8.101}} = (x^m + az^r)^{\star_{8.101}} = (x^m, z^r)^{\star_{8.101}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.101}} = (x^m, y^n)^{\star_{8.101}} = (x, y^n)$, $(y^n + az^r)^{\star_{8.101}} = (x^m + ay^n + bz^r)^{\star_{8.101}} = (y^n, z^r)^{\star_{8.101}} = (x^m + az^r, y^n + bz^r)^{\star_{8.101}} = (x^m, y^n + az^r)^{\star_{8.101}} = (x^m + ay^n, z^r)^{\star_{8.101}} = (x^m + az^r, y^n)^{\star_{8.101}} = (x^m, y^n, z^r)^{\star_{8.101}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

775. $\star_{8.102}$:$(0)^{\star_{8.102}} = (0)$, $(x^m)^{\star_{8.102}} = (x^m)$, $(y^n)^{\star_{8.102}} = (y^n)$, $(z^r)^{\star_{8.102}} = (z^r)$, $(x^m + ay^n)^{\star_{8.102}} = (x^m, y^n)^{\star_{8.102}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.102}} = (x^m, z^r)^{\star_{8.102}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.102}} = (x^m + ay^n + bz^r)^{\star_{8.102}} = (y^n, z^r)^{\star_{8.102}} = (x^m + az^r, y^n + bz^r)^{\star_{8.102}} = (x^m, y^n + az^r)^{\star_{8.102}} = (x^m + ay^n, z^r)^{\star_{8.102}} = (x^m + az^r, y^n)^{\star_{8.102}} = (x^m, y^n, z^r)^{\star_{8.102}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

776. $\star_{8.103}$:$(0)^{\star_{8.103}} = (0)$, $(x^m)^{\star_{8.103}} = (x^m)$, $(y^n)^{\star_{8.103}} = (y^n)$, $(z^r)^{\star_{8.103}} = (z^r)$, $(x^m + ay^n)^{\star_{8.103}} = (x^m, y^n)^{\star_{8.103}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.103}} = (x^m, z^r)^{\star_{8.103}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.103}} = (y^n, z^r)^{\star_{8.103}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.103}} = (x^m + az^r, y^n + bz^r)^{\star_{8.103}} = (x^m, y^n + az^r)^{\star_{8.103}} = (x^m + ay^n, z^r)^{\star_{8.103}} = (x^m + az^r, y^n)^{\star_{8.103}}$

$= (x^m, y^n, z^r)^{\star_{8.103}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

777. $\star_{8.104}$:$(0)^{\star_{8.104}} = (0)$, $(x^m)^{\star_{8.104}} = (x^m)$, $(y^n)^{\star_{8.104}} = (y^n)$, $(z^r)^{\star_{8.104}} = (z^r)$, $(x^m + ay^n)^{\star_{8.104}} = (x^m, y^n)^{\star_{8.104}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.104}} = (x^m, z^r)^{\star_{8.104}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.104}} = (y^n, z^r)^{\star_{8.104}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.104}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.104}} = (x^m, y^n + az^r)^{\star_{8.104}} = (x^m + ay^n, z^r)^{\star_{8.104}}$ $= (x^m + az^r, y^n)^{\star_{8.104}} = (x^m, y^n, z^r)^{\star_{8.104}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

778. $\star_{8.105}$:$(0)^{\star_{8.105}} = (0)$, $(x^m)^{\star_{8.105}} = (x^m)$, $(y^n)^{\star_{8.105}} = (y^n)$, $(z^r)^{\star_{8.105}} = (z^r)$, $(x^m + ay^n)^{\star_{8.105}} = (x^m, y^n)^{\star_{8.105}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.105}} = (x^m, z^r)^{\star_{8.105}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.105}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.105}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.105}} = (x^m + az^r, y^n + bz^r)^{\star_{8.105}} = (x^m, y^n + az^r)^{\star_{8.105}}$ $= (x^m + ay^n, z^r)^{\star_{8.105}} = (x^m + az^r, y^n)^{\star_{8.105}} = (x^m, y^n, z^r)^{\star_{8.105}} = (x, y, z^r)\ \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

779. $\star_{8.106}$:$(0)^{\star_{8.106}} = (0)$, $(x^m)^{\star_{8.106}} = (x^m)$, $(y^n)^{\star_{8.106}} = (x^m + ay^n)^{\star_{8.106}}$ $= (x^m, y^n)^{\star_{8.106}} = (x, y^n)$, $(z^r)^{\star_{8.106}} = (z^r)$, $(x^m + az^r)^{\star_{8.106}} = (x^m, z^r)^{\star_{8.106}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.106}} = (x^m + ay^n + bz^r)^{\star_{8.106}} = (y^n, z^r)^{\star_{8.106}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.106}} = (x^m, y^n + az^r)^{\star_{8.106}} = (x^m + ay^n, z^r)^{\star_{8.106}}$ $= (x^m + az^r, y^n)^{\star_{8.106}} = (x^m, y^n, z^r)^{\star_{8.106}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

780. $\star_{8.107}$:$(0)^{\star_{8.107}} = (0)$, $(x^m)^{\star_{8.107}} = (x^m)$, $(y^n)^{\star_{8.107}} = (y^n)$, $(z^r)^{\star_{8.107}} = (z^r)$, $(x^m + ay^n)^{\star_{8.107}} = (x^m, y^n)^{\star_{8.107}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.107}} = (x^m, z^r)^{\star_{8.107}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.107}} = (x^m + ay^n + bz^r)^{\star_{8.107}} = (y^n, z^r)^{\star_{8.107}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.107}} = (x^m, y^n + az^r)^{\star_{8.107}} = (x^m + ay^n, z^r)^{\star_{8.107}}$ $= (x^m + az^r, y^n)^{\star_{8.107}} = (x^m, y^n, z^r)^{\star_{8.107}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

781. $\star_{8.108}$:$(0)^{\star_{8.108}} = (0)$, $(x^m)^{\star_{8.108}} = (x^m)$, $(y^n)^{\star_{8.108}} = (y^n)$, $(z^r)^{\star_{8.108}} = (z^r)$, $(x^m + ay^n)^{\star_{8.108}} = (x^m, y^n)^{\star_{8.108}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.108}} = (x^m, z^r)^{\star_{8.108}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.108}} = (y^n, z^r)^{\star_{8.108}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.108}} = (x^m + az^r, y^n + bz^r)^{\star_{8.108}} = (x^m, y^n + az^r)^{\star_{8.108}} = (x^m + ay^n, z^r)^{\star_{8.108}}$ $= (x^m + az^r, y^n)^{\star_{8.108}} = (x^m, y^n, z^r)^{\star_{8.108}} = (x, y, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

782. $\star_{8.109}$:$(0)^{\star_{8.109}} = (0)$, $(x^m)^{\star_{8.109}} = (x^m)$, $(y^n)^{\star_{8.109}} = (y^n)$, $(z^r)^{\star_{8.109}} = (z^r)$,

$(x^m + ay^n)^{\star_{8.109}} = (x^m, y^n)^{\star_{8.109}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.109}} = (x^m, z^r)^{\star_{8.109}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.109}} = (y^n, z^r)^{\star_{8.109}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.109}} = (x^m + az^r, y^n + bz^r)^{\star_{8.109}} = (x^m, y^n + az^r)^{\star_{8.109}} = (x^m + ay^n, z^r)^{\star_{8.109}} = (x^m + az^r, y^n)^{\star_{8.109}} = (x^m, y^n, z^r)^{\star_{8.109}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

783. $\star_{8.110}$:$(0)^{\star_{8.110}} = (0)$, $(x^m)^{\star_{8.110}} = (x^m)$, $(y^n)^{\star_{8.110}} = (y^n)$, $(z^r)^{\star_{8.110}} = (z^r)$, $(x^m + ay^n)^{\star_{8.110}} = (x^m, y^n)^{\star_{8.110}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.110}} = (x^m, z^r)^{\star_{8.110}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.110}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.110}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.110}} = (x^m + az^r, y^n + bz^r)^{\star_{8.110}} = (x^m, y^n + az^r)^{\star_{8.110}} = (x^m + ay^n, z^r)^{\star_{8.110}} = (x^m + az^r, y^n)^{\star_{8.110}} = (x^m, y^n, z^r)^{\star_{8.109}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

784. $\star_{8.111}$:$(0)^{\star_{8.111}} = (0)$, $(x^m)^{\star_{8.111}} = (x^m)$, $(y^n)^{\star_{8.111}} = (x^m + ay^n)^{\star_{8.111}} = (x^m, y^n)^{\star_{8.111}} = (x, y^n)$, $(z^r)^{\star_{8.111}} = (z^r)$, $(x^m + az^r)^{\star_{8.111}} = (x^m, z^r)^{\star_{8.111}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.111}} = (x^m + ay^n + bz^r)^{\star_{8.111}} = (y^n, z^r)^{\star_{8.111}} = (x^m + az^r, y^n + bz^r)^{\star_{8.111}} = (x^m, y^n + az^r)^{\star_{8.111}} = (x^m + ay^n, z^r)^{\star_{8.111}} = (x^m + az^r, y^n)^{\star_{8.111}} = (x^m, y^n, z^r)^{\star_{8.111}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

785. $\star_{8.112}$:$(0)^{\star_{8.112}} = (0)$, $(x^m)^{\star_{8.112}} = (x^m)$, $(y^n)^{\star_{8.112}} = (y^n)$, $(z^r)^{\star_{8.112}} = (z^r)$, $(x^m + ay^n)^{\star_{8.112}} = (x^m, y^n)^{\star_{8.112}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.112}} = (x^m, z^r)^{\star_{8.112}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.112}} = (x^m + ay^n + bz^r)^{\star_{8.112}} = (y^n, z^r)^{\star_{8.112}} = (x^m + az^r, y^n + bz^r)^{\star_{8.112}} = (x^m, y^n + az^r)^{\star_{8.112}} = (x^m + ay^n, z^r)^{\star_{8.112}} = (x^m + az^r, y^n)^{\star_{8.112}} = (x^m, y^n, z^r)^{\star_{8.112}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

786. $\star_{8.113}$:$(0)^{\star_{8.113}} = (0)$, $(x^m)^{\star_{8.113}} = (x^m)$, $(y^n)^{\star_{8.113}} = (y^n)$, $(z^r)^{\star_{8.113}} = (z^r)$, $(x^m + ay^n)^{\star_{8.113}} = (x^m, y^n)^{\star_{8.113}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.113}} = (x^m, z^r)^{\star_{8.113}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.113}} = (y^n, z^r)^{\star_{8.113}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.113}} = (x^m + az^r, y^n + bz^r)^{\star_{8.113}} = (x^m, y^n + az^r)^{\star_{8.113}} = (x^m + ay^n, z^r)^{\star_{8.113}} = (x^m + az^r, y^n)^{\star_{8.113}} = (x^m, y^n, z^r)^{\star_{8.113}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

787. $\star_{8.114}$:$(0)^{\star_{8.114}} = (0)$, $(x^m)^{\star_{8.114}} = (x^m)$, $(y^n)^{\star_{8.114}} = (y^n)$, $(z^r)^{\star_{8.114}} = (z^r)$, $(x^m + ay^n)^{\star_{8.114}} = (x^m, y^n)^{\star_{8.114}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.114}} = (x^m, z^r)^{\star_{8.114}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.114}} = (y^n, z^r)^{\star_{8.114}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.114}} =$

$(x^m + az^r, y^n + bz^r)^{\star_{8.114}} = (x^m, y^n + az^r)^{\star_{8.114}} = (x^m + ay^n, z^r)^{\star_{8.114}}$
$= (x^m + az^r, y^n)^{\star_{8.114}} = (x^m, y^n, z^r)^{\star_{8.114}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

788. $\star_{8.115}$:$(0)^{\star_{8.115}} = (0)$, $(x^m)^{\star_{8.115}} = (x^m)$, $(y^n)^{\star_{8.115}} = (y^n)$, $(z^r)^{\star_{8.115}} = (z^r)$, $(x^m + ay^n)^{\star_{8.115}} = (x^m, y^n)^{\star_{8.115}} = (x, y^n)$, $(x^m + az^r)^{\star_{8.115}} = (x^m, z^r)^{\star_{8.115}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.115}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.115}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.115}} = (x^m + az^r, y^n + bz^r)^{\star_{8.115}} = (x^m, y^n + az^r)^{\star_{8.115}}$ $= (x^m + ay^n, z^r)^{\star_{8.115}} = (x^m + az^r, y^n)^{\star_{8.115}} = (x^m, y^n, z^r)^{\star_{8.115}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

789. $\star_{8.116}$ through $\star_{8.156}$ see proof

830. $\star_{8.157}$:$(0)^{\star_{8.157}} = (0)$, $(x^m)^{\star_{8.157}} = (x^m)$, $(y^n)^{\star_{8.157}} = (y^n)$, $(z^r)^{\star_{8.157}}$ $= (x^m + az^r)^{\star_{8.157}} = (y^n + az^r)^{\star_{8.157}} = (x^m + ay^n + bz^r)^{\star_{8.157}} = (x^m, z^r)^{\star_{8.157}} = (y^n, z^r)^{\star_{8.157}} = (x^m + az^r, y^n + bz^r)^{\star_{8.157}} = (x^m, y^n + az^r)^{\star_{8.157}} = (x^m + ay^n, z^r)^{\star_{8.157}}$ $= (x^m + az^r, y^n)^{\star_{8.157}} = (x^m, y^n, z^r)^{\star_{8.157}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.157}}$ $= (x^m, y^n)^{\star_{8.157}} = (x^m, y^n)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

831. $\star_{8.158}$:$(0)^{\star_{8.158}} = (0)$, $(x^m)^{\star_{8.158}} = (x^m)$, $(y^n)^{\star_{8.158}} = (y^n)$, $(z^r)^{\star_{8.158}} = (z^r)$, $(x^m + az^r)^{\star_{8.158}} = (y^n + az^r)^{\star_{8.158}} = (x^m + ay^n + bz^r)^{\star_{8.158}} = (x^m, z^r)^{\star_{8.158}} = (y^n, z^r)^{\star_{8.158}} = (x^m + az^r, y^n + bz^r)^{\star_{8.158}} = (x^m, y^n + az^r)^{\star_{8.158}} = (x^m + ay^n, z^r)^{\star_{8.158}}$ $= (x^m + az^r, y^n)^{\star_{8.158}} = (x^m, y^n, z^r)^{\star_{8.158}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.158}}$ $= (x^m, y^n)^{\star_{8.158}} = (x^m, y^n)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

832. $\star_{8.159}$:$(0)^{\star_{8.159}} = (0)$, $(x^m)^{\star_{8.159}} = (x^m)$, $(y^n)^{\star_{8.159}} = (y^n)$, $(z^r)^{\star_{8.159}}$ $= (y^n + az^r)^{\star_{8.159}} = (y^n, z^r)^{\star_{8.159}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.159}} = (x^m, y^n)^{\star_{8.159}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.159}} = (x^m + ay^n + bz^r)^{\star_{8.159}} = (x^m, z^r)^{\star_{8.159}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.159}} = (x^m, y^n + az^r)^{\star_{8.159}} = (x^m + ay^n, z^r)^{\star_{8.159}}$ $= (x^m + az^r, y^n)^{\star_{8.159}} = (x^m, y^n, z^r)^{\star_{8.159}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

833. $\star_{8.160}$:$(0)^{\star_{8.160}} = (0)$, $(x^m)^{\star_{8.160}} = (x^m)$, $(y^n)^{\star_{8.160}} = (y^n)$, $(z^r)^{\star_{8.160}} = (z^r)$, $(x^m + ay^n)^{\star_{8.160}} = (x^m, y^n)^{\star_{8.160}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.160}}$ $= (x^m + ay^n + bz^r)^{\star_{8.160}} = (x^m, z^r)^{\star_{8.160}} = (x^m + az^r, y^n + bz^r)^{\star_{8.160}}$ $= (x^m, y^n + az^r)^{\star_{8.160}} = (x^m + ay^n, z^r)^{\star_{8.160}} = (x^m + az^r, y^n)^{\star_{8.160}}$

$= (x^m, y^n, z^r)^{\star_{8.160}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.160}} = (y^n, z^r)^{\star_{8.160}} = (y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

834. $\star_{8.161}$:$(0)^{\star_{8.161}} = (0)$, $(x^m)^{\star_{8.161}} = (x^m)$, $(y^n)^{\star_{8.161}} = (y^n)$, $(z^r)^{\star_{8.161}} = (z^r)$, $(x^m + ay^n)^{\star_{8.161}} = (x^m, y^n)^{\star_{8.161}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.161}}$ $= (x^m + ay^n + bz^r)^{\star_{8.161}} = (x^m, z^r)^{\star_{8.161}} = (x^m + az^r, y^n + bz^r)^{\star_{8.161}}$ $= (x^m, y^n + az^r)^{\star_{8.161}} = (x^m + ay^n, z^r)^{\star_{8.161}} = (x^m + az^r, y^n)^{\star_{8.161}}$ $= (x^m, y^n, z^r)^{\star_{8.161}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.161}} = (y^n, z^r)^{\star_{8.161}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

835. $\star_{8.162}$:$(0)^{\star_{8.162}} = (0)$, $(x^m)^{\star_{8.162}} = (x^m)$, $(y^n)^{\star_{8.162}} = (y^n)$, $(z^r)^{\star_{8.162}} = (z^r)$, $(x^m + ay^n)^{\star_{8.162}} = (x^m, y^n)^{\star_{8.162}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.162}}$ $= (x^m + ay^n + bz^r)^{\star_{8.162}} = (x^m, z^r)^{\star_{8.162}} = (x^m + az^r, y^n + bz^r)^{\star_{8.162}}$ $= (x^m, y^n + az^r)^{\star_{8.162}} = (x^m + ay^n, z^r)^{\star_{8.162}} = (x^m + az^r, y^n)^{\star_{8.162}}$ $= (x^m, y^n, z^r)^{\star_{8.162}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.162}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.162}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

836. $\star_{8.163}$:$(0)^{\star_{8.163}} = (0)$, $(x^m)^{\star_{8.163}} = (x^m)$, $(y^n)^{\star_{8.163}} = (y^n)$, $(z^r)^{\star_{8.163}}$ $= (x^m + az^r)^{\star_{8.163}} = (x^m, z^r)^{\star_{8.163}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.163}} = (x^m, y^n)^{\star_{8.163}} = (x^m, y^n)$, $(y^n + az^r)^{\star_{8.163}} = (x^m + ay^n + bz^r)^{\star_{8.163}} = (y^n, z^r)^{\star_{8.163}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.163}} = (x^m, y^n + az^r)^{\star_{8.163}} = (x^m + ay^n, z^r)^{\star_{8.163}}$ $= (x^m + az^r, y^n)^{\star_{8.163}} = (x^m, y^n, z^r)^{\star_{8.163}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

837. $\star_{8.164}$:$(0)^{\star_{8.164}} = (0)$, $(x^m)^{\star_{8.164}} = (x^m)$, $(y^n)^{\star_{8.164}} = (y^n)$, $(z^r)^{\star_{8.164}} = (z^r)$, $(x^m + ay^n)^{\star_{8.164}} = (x^m, y^n)^{\star_{8.164}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.164}} = (x^m, z^r)^{\star_{8.164}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.164}} = (x^m + ay^n + bz^r)^{\star_{8.164}} = (y^n, z^r)^{\star_{8.164}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.164}} = (x^m, y^n + az^r)^{\star_{8.164}} = (x^m + ay^n, z^r)^{\star_{8.164}}$ $= (x^m + az^r, y^n)^{\star_{8.164}} = (x^m, y^n, z^r)^{\star_{8.164}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

838. $\star_{8.165}$:$(0)^{\star_{8.165}} = (0)$, $(x^m)^{\star_{8.165}} = (x^m)$, $(y^n)^{\star_{8.165}} = (y^n)$, $(z^r)^{\star_{8.165}} = (z^r)$, $(x^m + ay^n)^{\star_{8.165}} = (x^m, y^n)^{\star_{8.165}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.165}} = (x^m, z^r)^{\star_{8.165}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.165}} = (y^n, z^r)^{\star_{8.165}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.165}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.165}} = (x^m, y^n + az^r)^{\star_{8.165}} = (x^m + ay^n, z^r)^{\star_{8.165}}$

$= (x^m + az^r, y^n)^{\star_{8.165}} = (x^m, y^n, z^r)^{\star_{8.165}} = (x, y, z^r) \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

839. $\star_{8.166}$:$(0)^{\star_{8.166}} = (0)$, $(x^m)^{\star_{8.166}} = (x^m)$, $(y^n)^{\star_{8.166}} = (y^n)$, $(z^r)^{\star_{8.166}} = (z^r)$, $(x^m + ay^n)^{\star_{8.166}} = (x^m, y^n)^{\star_{8.166}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.166}} = (x^m, z^r)^{\star_{8.166}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.166}} = (y^n, z^r)^{\star_{8.166}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.166}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.166}} = (x^m, y^n + az^r)^{\star_{8.166}} = (x^m + ay^n, z^r)^{\star_{8.166}}$ $= (x^m + az^r, y^n)^{\star_{8.166}} = (x^m, y^n, z^r)^{\star_{8.166}} = (x, y, z^r) \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

840. $\star_{8.167}$:$(0)^{\star_{8.167}} = (0)$, $(x^m)^{\star_{8.167}} = (x^m)$, $(y^n)^{\star_{8.167}} = (y^n)$, $(z^r)^{\star_{8.167}} = (z^r)$, $(x^m + ay^n)^{\star_{8.167}} = (x^m, y^n)^{\star_{8.167}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.167}} = (x^m, z^r)^{\star_{8.167}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.167}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.167}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.167}} = (x^m + az^r, y^n + bz^r)^{\star_{8.167}} = (x^m, y^n + az^r)^{\star_{8.167}}$ $= (x^m + ay^n, z^r)^{\star_{8.167}} = (x^m + az^r, y^n)^{\star_{8.167}} = (x^m, y^n, z^r)^{\star_{8.167}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

841. $\star_{8.168}$:$(0)^{\star_{8.168}} = (0)$, $(x^m)^{\star_{8.168}} = (x^m)$, $(y^n)^{\star_{8.168}} = (y^n)$, $(z^r)^{\star_{8.168}} = (z^r)$, $(x^m + ay^n)^{\star_{8.168}} = (x^m, y^n)^{\star_{8.168}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.168}} = (x^m, z^r)^{\star_{8.168}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.168}} = (x^m + ay^n + bz^r)^{\star_{8.168}} = (y^n, z^r)^{\star_{8.168}}$ $= (x^m + az^r, y^n + bz^r)^{\star_{8.168}} = (x^m, y^n + az^r)^{\star_{8.168}} = (x^m + ay^n, z^r)^{\star_{8.168}}$ $= (x^m + az^r, y^n)^{\star_{8.168}} = (x^m, y^n, z^r)^{\star_{8.168}} = (x, y, z^r) \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

842. $\star_{8.169}$:$(0)^{\star_{8.169}} = (0)$, $(x^m)^{\star_{8.169}} = (x^m)$, $(y^n)^{\star_{8.169}} = (y^n)$, $(z^r)^{\star_{8.169}} = (z^r)$, $(x^m + ay^n)^{\star_{8.169}} = (x^m, y^n)^{\star_{8.169}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.169}} = (x^m, z^r)^{\star_{8.169}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.169}} = (y^n, z^r)^{\star_{8.169}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.169}} = (x^m + az^r, y^n + bz^r)^{\star_{8.169}} = (x^m, y^n + az^r)^{\star_{8.169}} = (x^m + ay^n, z^r)^{\star_{8.169}}$ $= (x^m + az^r, y^n)^{\star_{8.169}} = (x^m, y^n, z^r)^{\star_{8.169}} = (x, y, z^r) \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

843. $\star_{8.170}$:$(0)^{\star_{8.170}} = (0)$, $(x^m)^{\star_{8.170}} = (x^m)$, $(y^n)^{\star_{8.170}} = (y^n)$, $(z^r)^{\star_{8.170}} = (z^r)$, $(x^m + ay^n)^{\star_{8.170}} = (x^m, y^n)^{\star_{8.170}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.170}} = (x^m, z^r)^{\star_{8.170}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.170}} = (y^n, z^r)^{\star_{8.170}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.170}} = (x^m + az^r, y^n + bz^r)^{\star_{8.170}} = (x^m, y^n + az^r)^{\star_{8.170}} = (x^m + ay^n, z^r)^{\star_{8.170}}$ $= (x^m + az^r, y^n)^{\star_{8.170}} = (x^m, y^n, z^r)^{\star_{8.170}} = (x, y, z^r) \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

844. $\star_{8.171}$:$(0)^{\star_{8.171}} = (0)$, $(x^m)^{\star_{8.171}} = (x^m)$, $(y^n)^{\star_{8.171}} = (y^n)$, $(z^r)^{\star_{8.171}} = (z^r)$,

$(x^m + ay^n)^{\star_{8.171}} = (x^m, y^n)^{\star_{8.171}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.171}} = (x^m, z^r)^{\star_{8.171}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.171}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.171}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.171}} = (x^m + az^r, y^n + bz^r)^{\star_{8.171}} = (x^m, y^n + az^r)^{\star_{8.171}} = (x^m + ay^n, z^r)^{\star_{8.171}} = (x^m + az^r, y^n)^{\star_{8.171}} = (x^m, y^n, z^r)^{\star_{8.171}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

845. $\star_{8.172}$:$(0)^{\star_{8.172}} = (0)$, $(x^m)^{\star_{8.172}} = (x^m)$, $(y^n)^{\star_{8.172}} = (y^n)$, $(z^r)^{\star_{8.172}} = (z^r)$, $(x^m + ay^n)^{\star_{8.172}} = (x^m, y^n)^{\star_{8.172}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.172}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.172}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.172}} = (x^m + ay^n + bz^r)^{\star_{8.172}} = (y^n, z^r)^{\star_{8.172}} = (x^m + az^r, y^n + bz^r)^{\star_{8.172}} = (x^m, y^n + az^r)^{\star_{8.172}} = (x^m + ay^n, z^r)^{\star_{8.172}} = (x^m + az^r, y^n)^{\star_{8.172}} = (x^m, y^n, z^r)^{\star_{8.172}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

846. $\star_{8.173}$:$(0)^{\star_{8.173}} = (0)$, $(x^m)^{\star_{8.173}} = (x^m)$, $(y^n)^{\star_{8.173}} = (y^n)$, $(z^r)^{\star_{8.173}} = (z^r)$, $(x^m + ay^n)^{\star_{8.173}} = (x^m, y^n)^{\star_{8.173}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.173}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.173}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.173}} = (y^n, z^r)^{\star_{8.173}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.173}} = (x^m + az^r, y^n + bz^r)^{\star_{8.173}} = (x^m, y^n + az^r)^{\star_{8.173}} = (x^m + ay^n, z^r)^{\star_{8.173}} = (x^m + az^r, y^n)^{\star_{8.173}} = (x^m, y^n, z^r)^{\star_{8.173}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

847. $\star_{8.174}$:$(0)^{\star_{8.174}} = (0)$, $(x^m)^{\star_{8.174}} = (x^m)$, $(y^n)^{\star_{8.174}} = (y^n)$, $(z^r)^{\star_{8.174}} = (z^r)$, $(x^m + ay^n)^{\star_{8.174}} = (x^m, y^n)^{\star_{8.174}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.174}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.174}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.174}} = (y^n, z^r)^{\star_{8.174}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.174}} = (x^m + az^r, y^n + bz^r)^{\star_{8.174}} = (x^m, y^n + az^r)^{\star_{8.174}} = (x^m + ay^n, z^r)^{\star_{8.174}} = (x^m + az^r, y^n)^{\star_{8.174}} = (x^m, y^n, z^r)^{\star_{8.174}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

848. $\star_{8.175}$:$(0)^{\star_{8.175}} = (0)$, $(x^m)^{\star_{8.175}} = (x^m)$, $(y^n)^{\star_{8.175}} = (y^n)$, $(z^r)^{\star_{8.175}} = (z^r)$, $(x^m + ay^n)^{\star_{8.175}} = (x^m, y^n)^{\star_{8.175}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.175}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.175}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.175}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.175}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.175}} = (x^m + az^r, y^n + bz^r)^{\star_{8.175}} = (x^m, y^n + az^r)^{\star_{8.175}} = (x^m + ay^n, z^r)^{\star_{8.175}} = (x^m + az^r, y^n)^{\star_{8.175}} = (x^m, y^n, z^r)^{\star_{8.175}} = (x, y, z^r) \forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

849. $\star_{8.176}$:$(0)^{\star_{8.176}} = (0)$, $(x^m)^{\star_{8.176}} = (x^m)$, $(y^n)^{\star_{8.176}} = (y^n)$, $(z^r)^{\star_{8.176}} = (x^m + az^r)^{\star_{8.176}} = (y^n + az^r)^{\star_{8.176}} = (x^m + ay^n + bz^r)^{\star_{8.176}} = (x^m, z^r)^{\star_{8.176}} = (y^n, z^r)^{\star_{8.176}} = (x^m+az^r, y^n+bz^r)^{\star_{8.176}} = (x^m, y^n+az^r)^{\star_{8.176}} = (x^m+ay^n, z^r)^{\star_{8.176}} = (x^m + az^r, y^n)^{\star_{8.176}} = (x^m, y^n, z^r)^{\star_{8.176}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.176}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.176}} = (x^m, y^n)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

850. $\star_{8.177}$:$(0)^{\star_{8.177}} = (0)$, $(x^m)^{\star_{8.177}} = (x^m)$, $(y^n)^{\star_{8.177}} = (y^n)$, $(z^r)^{\star_{8.177}} = (z^r)$, $(x^m + az^r)^{\star_{8.177}} = (y^n + az^r)^{\star_{8.177}} = (x^m + ay^n + bz^r)^{\star_{8.177}} = (x^m, z^r)^{\star_{8.177}} = (y^n, z^r)^{\star_{8.177}} = (x^m+az^r, y^n+bz^r)^{\star_{8.177}} = (x^m, y^n+az^r)^{\star_{8.177}} = (x^m+ay^n, z^r)^{\star_{8.177}} = (x^m + az^r, y^n)^{\star_{8.177}} = (x^m, y^n, z^r)^{\star_{8.177}} = (x, y, z^r)$, $(x^m + ay^n)^{\star_{8.177}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.177}} = (x^m, y^n)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

851. $\star_{8.178}$:$(0)^{\star_{8.178}} = (0)$, $(x^m)^{\star_{8.178}} = (x^m)$, $(y^n)^{\star_{8.178}} = (y^n)$, $(z^r)^{\star_{8.178}} = (y^n + az^r)^{\star_{8.178}} = (y^n, z^r)^{\star_{8.178}} = (y, z^r)$, $(x^m + ay^n)^{\star_{8.178}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.178}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.178}} = (x^m + ay^n + bz^r)^{\star_{8.178}} = (x^m, z^r)^{\star_{8.178}} = (x^m + az^r, y^n + bz^r)^{\star_{8.178}} = (x^m, y^n + az^r)^{\star_{8.178}} = (x^m + ay^n, z^r)^{\star_{8.178}} = (x^m + az^r, y^n)^{\star_{8.178}} = (x^m, y^n, z^r)^{\star_{8.178}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

852. $\star_{8.179}$:$(0)^{\star_{8.179}} = (0)$, $(x^m)^{\star_{8.179}} = (x^m)$, $(y^n)^{\star_{8.179}} = (y^n)$, $(z^r)^{\star_{8.179}} = (z^r)$, $(x^m + ay^n)^{\star_{8.179}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.179}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.179}} = (x^m + ay^n + bz^r)^{\star_{8.179}} = (x^m, z^r)^{\star_{8.179}} = (x^m + az^r, y^n + bz^r)^{\star_{8.179}} = (x^m, y^n+az^r)^{\star_{8.179}} = (x^m+ay^n, z^r)^{\star_{8.179}} = (x^m+az^r, y^n)^{\star_{8.179}} = (x^m, y^n, z^r)^{\star_{8.179}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.179}} = (y^n, z^r)^{\star_{8.179}} = (y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

853. $\star_{8.180}$:$(0)^{\star_{8.180}} = (0)$, $(x^m)^{\star_{8.180}} = (x^m)$, $(y^n)^{\star_{8.180}} = (y^n)$, $(z^r)^{\star_{8.180}} = (z^r)$, $(x^m + ay^n)^{\star_{8.180}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.180}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.180}} = (x^m + ay^n + bz^r)^{\star_{8.180}} = (x^m, z^r)^{\star_{8.180}} = (x^m + az^r, y^n + bz^r)^{\star_{8.180}} = (x^m, y^n+az^r)^{\star_{8.180}} = (x^m+ay^n, z^r)^{\star_{8.180}} = (x^m+az^r, y^n)^{\star_{8.180}} = (x^m, y^n, z^r)^{\star_{8.180}} = (x, y, z^r)$, $(y^n+az^r)^{\star_{8.180}} = (y^n, z^r)^{\star_{8.180}} = (y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

854. $\star_{8.181}$:$(0)^{\star_{8.181}} = (0)$, $(x^m)^{\star_{8.181}} = (x^m)$, $(y^n)^{\star_{8.181}} = (y^n)$, $(z^r)^{\star_{8.181}} = (z^r)$, $(x^m + ay^n)^{\star_{8.181}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.181}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.181}} = (x^m + ay^n + bz^r)^{\star_{8.181}} = (x^m, z^r)^{\star_{8.181}} = (x^m + az^r, y^n + bz^r)^{\star_{8.181}} =$

$(x^m, y^n + az^r)^{\star_{8.181}} = (x^m + ay^n, z^r)^{\star_{8.181}} = (x^m + az^r, y^n)^{\star_{8.181}} = (x^m, y^n, z^r)^{\star_{8.181}} = (x, y, z^r)$, $(y^n + az^r)^{\star_{8.181}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.181}} = (y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

855. $\star_{8.182}$:$(0)^{\star_{8.182}} = (0)$, $(x^m)^{\star_{8.182}} = (x^m)$, $(y^n)^{\star_{8.182}} = (y^n)$, $(z^r)^{\star_{8.182}} = (x^m + az^r)^{\star_{8.182}} = (x^m, z^r)^{\star_{8.182}} = (x, z^r)$, $(x^m + ay^n)^{\star_{8.182}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.182}} = (x^m, y^n)$, $(y^n + az^r)^{\star_{8.182}} = (x^m + ay^n + bz^r)^{\star_{8.182}} = (y^n, z^r)^{\star_{8.182}} = (x^m + az^r, y^n + bz^r)^{\star_{8.182}} = (x^m, y^n + az^r)^{\star_{8.182}} = (x^m + ay^n, z^r)^{\star_{8.182}} = (x^m + az^r, y^n)^{\star_{8.182}} = (x^m, y^n, z^r)^{\star_{8.182}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
856. $\star_{8.183}$:$(0)^{\star_{8.183}} = (0)$, $(x^m)^{\star_{8.183}} = (x^m)$, $(y^n)^{\star_{8.183}} = (y^n)$, $(z^r)^{\star_{8.183}} = (z^r)$, $(x^m + ay^n)^{\star_{8.183}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.183}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.183}} = (x^m, z^r)^{\star_{8.183}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.183}} = (x^m + ay^n + bz^r)^{\star_{8.183}} = (y^n, z^r)^{\star_{8.183}} = (x^m + az^r, y^n + bz^r)^{\star_{8.183}} = (x^m, y^n + az^r)^{\star_{8.183}} = (x^m + ay^n, z^r)^{\star_{8.183}} = (x^m + az^r, y^n)^{\star_{8.183}} = (x^m, y^n, z^r)^{\star_{8.183}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
857. $\star_{8.184}$:$(0)^{\star_{8.184}} = (0)$, $(x^m)^{\star_{8.184}} = (x^m)$, $(y^n)^{\star_{8.184}} = (y^n)$, $(z^r)^{\star_{8.184}} = (z^r)$, $(x^m + ay^n)^{\star_{8.184}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.184}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.184}} = (x^m, z^r)^{\star_{8.184}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.184}} = (y^n, z^r)^{\star_{8.184}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.184}} = (x^m + az^r, y^n + bz^r)^{\star_{8.184}} = (x^m, y^n + az^r)^{\star_{8.184}} = (x^m + ay^n, z^r)^{\star_{8.184}} = (x^m + az^r, y^n)^{\star_{8.184}} = (x^m, y^n, z^r)^{\star_{8.184}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
858. $\star_{8.185}$:$(0)^{\star_{8.185}} = (0)$, $(x^m)^{\star_{8.185}} = (x^m)$, $(y^n)^{\star_{8.185}} = (y^n)$, $(z^r)^{\star_{8.185}} = (z^r)$, $(x^m + ay^n)^{\star_{8.185}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.185}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.185}} = (x^m, z^r)^{\star_{8.185}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.185}} = (y^n, z^r)^{\star_{8.185}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.185}} = (x^m + az^r, y^n + bz^r)^{\star_{8.185}} = (x^m, y^n + az^r)^{\star_{8.185}} = (x^m + ay^n, z^r)^{\star_{8.185}} = (x^m + az^r, y^n)^{\star_{8.185}} = (x^m, y^n, z^r)^{\star_{8.185}} = (x, y, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
859. $\star_{8.186}$:$(0)^{\star_{8.186}} = (0)$, $(x^m)^{\star_{8.186}} = (x^m)$, $(y^n)^{\star_{8.186}} = (y^n)$, $(z^r)^{\star_{8.186}} = (z^r)$, $(x^m + ay^n)^{\star_{8.186}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.186}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.186}} = (x^m, z^r)^{\star_{8.186}} = (x, z^r)$, $(y^n + az^r)^{\star_{8.186}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.186}} = (y^n, z^r)$,

$(x^m + ay^n + bz^r)^{\star_{8.186}} = (x^m + az^r, y^n + bz^r)^{\star_{8.186}} = (x^m, y^n + az^r)^{\star_{8.186}} = (x^m + ay^n, z^r)^{\star_{8.186}} = (x^m + az^r, y^n)^{\star_{8.186}} = (x^m, y^n, z^r)^{\star_{8.186}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

860. $\star_{8.187}$:$(0)^{\star_{8.187}} = (0)$, $(x^m)^{\star_{8.187}} = (x^m)$, $(y^n)^{\star_{8.187}} = (y^n)$, $(z^r)^{\star_{8.187}} = (z^r)$, $(x^m + ay^n)^{\star_{8.187}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.187}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.187}} = (x^m, z^r)^{\star_{8.187}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.187}} = (x^m + ay^n + bz^r)^{\star_{8.187}} = (y^n, z^r)^{\star_{8.187}} = (x^m + az^r, y^n + bz^r)^{\star_{8.187}} = (x^m, y^n + az^r)^{\star_{8.187}} = (x^m + ay^n, z^r)^{\star_{8.187}} = (x^m + az^r, y^n)^{\star_{8.187}} = (x^m, y^n, z^r)^{\star_{8.187}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
861. $\star_{8.188}$:$(0)^{\star_{8.188}} = (0)$, $(x^m)^{\star_{8.188}} = (x^m)$, $(y^n)^{\star_{8.188}} = (y^n)$, $(z^r)^{\star_{8.188}} = (z^r)$, $(x^m + ay^n)^{\star_{8.188}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.188}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.188}} = (x^m, z^r)^{\star_{8.188}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.188}} = (y^n, z^r)^{\star_{8.188}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.188}} = (x^m + az^r, y^n + bz^r)^{\star_{8.188}} = (x^m, y^n + az^r)^{\star_{8.188}} = (x^m + ay^n, z^r)^{\star_{8.188}} = (x^m + az^r, y^n)^{\star_{8.188}} = (x^m, y^n, z^r)^{\star_{8.188}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
862. $\star_{8.189}$:$(0)^{\star_{8.189}} = (0)$, $(x^m)^{\star_{8.189}} = (x^m)$, $(y^n)^{\star_{8.189}} = (y^n)$, $(z^r)^{\star_{8.189}} = (z^r)$, $(x^m + ay^n)^{\star_{8.189}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.189}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.189}} = (x^m, z^r)^{\star_{8.189}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.189}} = (y^n, z^r)^{\star_{8.189}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.189}} = (x^m + az^r, y^n + bz^r)^{\star_{8.189}} = (x^m, y^n + az^r)^{\star_{8.189}} = (x^m + ay^n, z^r)^{\star_{8.189}} = (x^m + az^r, y^n)^{\star_{8.189}} = (x^m, y^n, z^r)^{\star_{8.189}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
863. $\star_{8.190}$:$(0)^{\star_{8.190}} = (0)$, $(x^m)^{\star_{8.190}} = (x^m)$, $(y^n)^{\star_{8.190}} = (y^n)$, $(z^r)^{\star_{8.190}} = (z^r)$, $(x^m + ay^n)^{\star_{8.190}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.190}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.190}} = (x^m, z^r)^{\star_{8.190}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.190}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.190}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.190}} = (x^m + az^r, y^n + bz^r)^{\star_{8.190}} = (x^m, y^n + az^r)^{\star_{8.190}} = (x^m + ay^n, z^r)^{\star_{8.190}} = (x^m + az^r, y^n)^{\star_{8.190}} = (x^m, y^n, z^r)^{\star_{8.190}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
864. $\star_{8.191}$:$(0)^{\star_{8.191}} = (0)$, $(x^m)^{\star_{8.191}} = (x^m)$, $(y^n)^{\star_{8.191}} = (y^n)$, $(z^r)^{\star_{8.191}} = (z^r)$, $(x^m + ay^n)^{\star_{8.191}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.191}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.191}} =$

$(x^m + az^r)$, $(x^m, z^r)^{\star_{8.191}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.191}} = (x^m + ay^n + bz^r)^{\star_{8.191}} = (y^n, z^r)^{\star_{8.191}} = (x^m+az^r, y^n+bz^r)^{\star_{8.191}} = (x^m, y^n+az^r)^{\star_{8.191}} = (x^m+ay^n, z^r)^{\star_{8.191}} = (x^m + az^r, y^n)^{\star_{8.191}} = (x^m, y^n, z^r)^{\star_{8.191}} = (x, y, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

865. $\star_{8.192}$:$(0)^{\star_{8.192}} = (0)$, $(x^m)^{\star_{8.192}} = (x^m)$, $(y^n)^{\star_{8.192}} = (y^n)$, $(z^r)^{\star_{8.192}} = (z^r)$, $(x^m + ay^n)^{\star_{8.192}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.192}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.192}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.192}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.192}} = (y^n, z^r)^{\star_{8.192}} = (y, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.192}} = (x^m + az^r, y^n + bz^r)^{\star_{8.192}} = (x^m, y^n + az^r)^{\star_{8.192}} = (x^m + ay^n, z^r)^{\star_{8.192}} = (x^m + az^r, y^n)^{\star_{8.192}} = (x^m, y^n, z^r)^{\star_{8.192}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
866. $\star_{8.193}$:$(0)^{\star_{8.193}} = (0)$, $(x^m)^{\star_{8.193}} = (x^m)$, $(y^n)^{\star_{8.193}} = (y^n)$, $(z^r)^{\star_{8.193}} = (z^r)$, $(x^m + ay^n)^{\star_{8.193}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.193}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.193}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.193}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.193}} = (y^n, z^r)^{\star_{8.193}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.193}} = (x^m + az^r, y^n + bz^r)^{\star_{8.193}} = (x^m, y^n + az^r)^{\star_{8.193}} = (x^m + ay^n, z^r)^{\star_{8.193}} = (x^m + az^r, y^n)^{\star_{8.193}} = (x^m, y^n, z^r)^{\star_{8.193}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
867. $\star_{8.194}$:$(0)^{\star_{8.194}} = (0)$, $(x^m)^{\star_{8.194}} = (x^m)$, $(y^n)^{\star_{8.194}} = (y^n)$, $(z^r)^{\star_{8.194}} = (z^r)$, $(x^m + ay^n)^{\star_{8.194}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{8.194}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{8.194}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{8.194}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{8.194}} = (y^n + az^r)$, $(y^n, z^r)^{\star_{8.194}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{8.194}} = (x^m + az^r, y^n + bz^r)^{\star_{8.194}} = (x^m, y^n + az^r)^{\star_{8.194}} = (x^m + ay^n, z^r)^{\star_{8.194}} = (x^m + az^r, y^n)^{\star_{8.194}} = (x^m, y^n, z^r)^{\star_{8.194}} = (x, y, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
868. $\star_{8.195}$ through $\star_{8.389}$ see proof
1063. $\star_{8.390}$ through $\star_{8.584}$ see proof
1258. $\star_9$:$(0)^{\star_9} = (x^m)^{\star_9} = (x)$, $(y^n)^{\star_9} = (x^m + ay^n)^{\star_9} = (x^m, y^n)^{\star_9} = (x, y^n)$, $(z^r)^{\star_9} = (x^m + az^r)^{\star_9} = (x^m, z^r)^{\star_9} = (x, z^r)$, $(y^n + az^r)^{\star_9} = (x^m + ay^n + bz^r)^{\star_9} = (y^n, z^r)^{\star_9} = (x^m + az^r, y^n + bz^r)^{\star_9} = (x^m, y^n + az^r)^{\star_9} = (x^m + ay^n, z^r)^{\star_9} = (x^m + az^r, y^n)^{\star_9} = (x^m, y^n, z^r)^{\star_9} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
1259. $\star_{9.1}$:$(0)^{\star_{9.1}} = (0)$, $(x^m)^{\star_{9.1}} = (x)$, $(y^n)^{\star_{9.1}} = (x^m + ay^n)^{\star_{9.1}} = (x^m, y^n)^{\star_{9.1}} =$

(x, y^n), $(z^r)^{\star 9.1} = (x^m + az^r)^{\star 9.1} = (x^m, z^r)^{\star 9.1} = (x, z^r)$, $(y^n + az^r)^{\star 9.1} = (x^m + ay^n + bz^r)^{\star 9.1} = (y^n, z^r)^{\star 9.1} = (x^m + az^r, y^n + bz^r)^{\star 9.1} = (x^m, y^n + az^r)^{\star 9.1} = (x^m + ay^n, z^r)^{\star 9.1} = (x^m + az^r, y^n)^{\star 9.1} = (x^m, y^n, z^r)^{\star 9.1} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1260. $\star_{9.2}$:$(0)^{\star 9.2} = (0)$, $(x^m)^{\star 9.2} = (x)$, $(y^n)^{\star 9.2} = (x^m + ay^n)^{\star 9.2} = (x^m, y^n)^{\star 9.2} = (x, y^n)$, $(z^r)^{\star 9.2} = (z^r)$, $(x^m + az^r)^{\star 9.2} = (x^m, z^r)^{\star 9.2} = (x, z^r)$, $(y^n + az^r)^{\star 9.2} = (x^m + ay^n + bz^r)^{\star 9.2} = (y^n, z^r)^{\star 9.2} = (x^m + az^r, y^n + bz^r)^{\star 9.2} = (x^m, y^n + az^r)^{\star 9.2} = (x^m + ay^n, z^r)^{\star 9.2} = (x^m + az^r, y^n)^{\star 9.2} = (x^m, y^n, z^r)^{\star 9.2} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1261. $\star_{9.3}$:$(0)^{\star 9.3} = (x^m)^{\star 9.3} = (x)$, $(y^n)^{\star 9.3} = (y^n)$, $(z^r)^{\star 9.3} = (x^m + az^r)^{\star 9.3} = (x^m, z^r)^{\star 9.3} = (x, z^r)$, $(x^m + ay^n)^{\star 9.3} = (x^m, y^n)^{\star 9.3} = (x, y^n)$, $(y^n + az^r)^{\star 9.3} = (x^m + ay^n + bz^r)^{\star 9.3} = (y^n, z^r)^{\star 9.3} = (x^m + az^r, y^n + bz^r)^{\star 9.3} = (x^m, y^n + az^r)^{\star 9.3} = (x^m + ay^n, z^r)^{\star 9.3} = (x^m + az^r, y^n)^{\star 9.3} = (x^m, y^n, z^r)^{\star 9.3} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1262. $\star_{9.4}$:$(0)^{\star 9.4} = (0)$, $(x^m)^{\star 9.4} = (x)$, $(y^n)^{\star 9.4} = (y^n)$, $(z^r)^{\star 9.4} = (x^m + az^r)^{\star 9.4} = (x^m, z^r)^{\star 9.4} = (x, z^r)$, $(x^m + ay^n)^{\star 9.4} = (x^m, y^n)^{\star 9.4} = (x, y^n)$, $(y^n + az^r)^{\star 9.4} = (x^m + ay^n + bz^r)^{\star 9.4} = (y^n, z^r)^{\star 9.4} = (x^m + az^r, y^n + bz^r)^{\star 9.4} = (x^m, y^n + az^r)^{\star 9.4} = (x^m + ay^n, z^r)^{\star 9.4} = (x^m + az^r, y^n)^{\star 9.4} = (x^m, y^n, z^r)^{\star 9.4} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1263. $\star_{9.5}$:$(0)^{\star 9.5} = (0)$, $(x^m)^{\star 9.5} = (x)$, $(y^n)^{\star 9.5} = (y^n)$, $(z^r)^{\star 9.5} = (z^r)$, $(x^m + ay^n)^{\star 9.5} = (x^m, y^n)^{\star 9.5} = (x, y^n)$, $(x^m + az^r)^{\star 9.5} = (x^m, z^r)^{\star 9.5} = (x, z^r)$, $(y^n + az^r)^{\star 9.5} = (x^m + ay^n + bz^r)^{\star 9.5} = (y^n, z^r)^{\star 9.5} = (x^m + az^r, y^n + bz^r)^{\star 9.5} = (x^m, y^n + az^r)^{\star 9.5} = (x^m + ay^n, z^r)^{\star 9.5} = (x^m + az^r, y^n)^{\star 9.5} = (x^m, y^n, z^r)^{\star 9.5} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1264. $\star_{9.6}$:$(0)^{\star 9.6} = (0)$, $(x^m)^{\star 9.6} = (x)$, $(y^n)^{\star 9.6} = (y^n)$, $(z^r)^{\star 9.6} = (z^r)$, $(x^m + ay^n)^{\star 9.6} = (x^m, y^n)^{\star 9.6} = (x, y^n)$, $(x^m + az^r)^{\star 9.6} = (x^m, z^r)^{\star 9.6} = (x, z^r)$, $(y^n + az^r)^{\star 9.6} = (y^n, z^r)^{\star 9.6} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star 9.6} = (x^m + az^r, y^n + bz^r)^{\star 9.6} = (x^m, y^n + az^r)^{\star 9.6} = (x^m + ay^n, z^r)^{\star 9.6} = (x^m + az^r, y^n)^{\star 9.6} = (x^m, y^n, z^r)^{\star 9.6} =$

(x, y^n, z^r) $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1265. $\star_{9.7}$:$(0)^{\star_{9.7}} = (0)$, $(x^m)^{\star_{9.7}} = (x)$, $(y^n)^{\star_{9.7}} = (y^n)$, $(z^r)^{\star_{9.7}} = (z^r)$, $(x^m+ay^n)^{\star_{9.7}} = (x^m, y^n)^{\star_{9.7}} = (x, y^n)$, $(x^m + az^r)^{\star_{9.7}} = (x^m, z^r)^{\star_{9.7}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.7}} = (y^n+az^r)$, $(y^n, z^r)^{\star_{9.7}} = (y^n, z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.7}} = (x^m+az^r, y^n+bz^r)^{\star_{9.7}} = (x^m, y^n + az^r)^{\star_{9.7}} = (x^m + ay^n, z^r)^{\star_{9.7}} = (x^m + az^r, y^n)^{\star_{9.7}} = (x^m, y^n, z^r)^{\star_{9.7}} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1266. $\star_{9.8}$:$(0)^{\star_{9.8}} = (0)$, $(x^m)^{\star_{9.8}} = (x^m)$, $(y^n)^{\star_{9.8}} = (x^m + ay^n)^{\star_{9.8}} = (x^m, y^n)^{\star_{9.8}} = (x, y^n)$, $(z^r)^{\star_{9.8}} = (x^m + az^r)^{\star_{9.8}} = (x^m, z^r)^{\star_{9.8}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.8}} = (x^m+ay^n+bz^r)^{\star_{9.8}} = (y^n, z^r)^{\star_{9.8}} = (x^m+az^r, y^n+bz^r)^{\star_{9.8}} = (x^m, y^n+az^r)^{\star_{9.8}} = (x^m+ay^n, z^r)^{\star_{9.8}} = (x^m+az^r, y^n)^{\star_{9.8}} = (x^m, y^n, z^r)^{\star_{9.8}} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1267. $\star_{9.9}$:$(0)^{\star_{9.9}} = (0)$, $(x^m)^{\star_{9.9}} = (x^m)$, $(y^n)^{\star_{9.9}} = (x^m + ay^n)^{\star_{9.9}} = (x^m, y^n)^{\star_{9.9}} = (x, y^n)$, $(z^r)^{\star_{9.9}} = (z^r)$, $(x^m + az^r)^{\star_{9.9}} = (x^m, z^r)^{\star_{9.9}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.9}} = (x^m+ay^n+bz^r)^{\star_{9.9}} = (y^n, z^r)^{\star_{9.9}} = (x^m+az^r, y^n+bz^r)^{\star_{9.9}} = (x^m, y^n+az^r)^{\star_{9.9}} = (x^m+ay^n, z^r)^{\star_{9.9}} = (x^m+az^r, y^n)^{\star_{9.9}} = (x^m, y^n, z^r)^{\star_{9.9}} = (x, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1268. $\star_{9.10}$:$(0)^{\star_{9.10}} = (0)$, $(x^m)^{\star_{9.10}} = (x^m)$, $(y^n)^{\star_{9.10}} = (y^n)$, $(z^r)^{\star_{9.10}} = (x^m+az^r)^{\star_{9.10}} = (x^m, z^r)^{\star_{9.10}} = (x, z^r)$, $(x^m + ay^n)^{\star_{9.10}} = (x^m, y^n)^{\star_{9.10}} = (x, y^n)$, $(y^n + az^r)^{\star_{9.10}} = (x^m + ay^n + bz^r)^{\star_{9.10}} = (y^n, z^r)^{\star_{9.10}} = (x^m + az^r, y^n + bz^r)^{\star_{9.10}} = (x^m, y^n+az^r)^{\star_{9.10}} = (x^m+ay^n, z^r)^{\star_{9.10}} = (x^m+az^r, y^n)^{\star_{9.10}} = (x^m, y^n, z^r)^{\star_{9.10}} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1269. $\star_{9.11}$:$(0)^{\star_{9.11}} = (0)$, $(x^m)^{\star_{9.11}} = (x^m)$, $(y^n)^{\star_{9.11}} = (y^n)$, $(z^r)^{\star_{9.11}} = (z^r)$, $(x^m+ay^n)^{\star_{9.11}} = (x^m, y^n)^{\star_{9.11}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.11}} = (x^m, z^r)^{\star_{9.11}} = (x, z^r)$, $(y^n+az^r)^{\star_{9.11}} = (x^m+ay^n+bz^r)^{\star_{9.11}} = (y^n, z^r)^{\star_{9.11}} = (x^m+az^r, y^n+bz^r)^{\star_{9.11}} = (x^m, y^n+az^r)^{\star_{9.11}} = (x^m+ay^n, z^r)^{\star_{9.11}} = (x^m+az^r, y^n)^{\star_{9.11}} = (x^m, y^n, z^r)^{\star_{9.11}} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1270. $\star_{9.12}$:$(0)^{\star_{9.12}} = (0)$, $(x^m)^{\star_{9.12}} = (x^m)$, $(y^n)^{\star_{9.12}} = (y^n)$, $(z^r)^{\star_{9.12}} = (z^r)$, $(x^m+ay^n)^{\star_{9.12}} = (x^m, y^n)^{\star_{9.12}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.12}} = (x^m, z^r)^{\star_{9.12}} = (x, z^r)$,

$(y^n+az^r)^{\star_{9.12}} = (y^n,z^r)^{\star_{9.12}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.12}} = (x^m+az^r,y^n+bz^r)^{\star_{9.12}} = (x^m,y^n+az^r)^{\star_{9.12}} = (x^m+ay^n,z^r)^{\star_{9.12}} = (x^m+az^r,y^n)^{\star_{9.12}} = (x^m,y^n,z^r)^{\star_{9.12}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1271. $\star_{9.13}$:$(0)^{\star_{9.13}} = (0)$, $(x^m)^{\star_{9.13}} = (x^m)$, $(y^n)^{\star_{9.13}} = (y^n)$, $(z^r)^{\star_{9.13}} = (z^r)$, $(x^m+ay^n)^{\star_{9.13}} = (x^m,y^n)^{\star_{9.13}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.13}} = (x^m,z^r)^{\star_{9.13}} = (x,z^r)$, $(y^n+az^r)^{\star_{9.13}} = (y^n+az^r)$, $(y^n,z^r)^{\star_{9.13}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.13}} = (x^m+az^r,y^n+bz^r)^{\star_{9.13}} = (x^m,y^n+az^r)^{\star_{9.13}} = (x^m+ay^n,z^r)^{\star_{9.13}} = (x^m+az^r,y^n)^{\star_{9.13}} = (x^m,y^n,z^r)^{\star_{9.13}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1272. $\star_{9.14}$:$(0)^{\star_{9.14}} = (0)$, $(x^m)^{\star_{9.14}} = (x^m)$, $(y^n)^{\star_{9.14}} = (x^m+ay^n)^{\star_{9.14}} = (x^m,y^n)^{\star_{9.14}} = (x,y^n)$, $(z^r)^{\star_{9.14}} = (z^r)$, $(x^m+az^r)^{\star_{9.14}} = (x^m,z^r)^{\star_{9.14}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.14}} = (x^m+ay^n+bz^r)^{\star_{9.14}} = (y^n,z^r)^{\star_{9.14}} = (x^m+az^r,y^n+bz^r)^{\star_{9.14}} = (x^m,y^n+az^r)^{\star_{9.14}} = (x^m+ay^n,z^r)^{\star_{9.14}} = (x^m+az^r,y^n)^{\star_{9.14}} = (x^m,y^n,z^r)^{\star_{9.14}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1273. $\star_{9.15}$:$(0)^{\star_{9.15}} = (0)$, $(x^m)^{\star_{9.15}} = (x^m)$, $(y^n)^{\star_{9.15}} = (y^n)$, $(z^r)^{\star_{9.15}} = (z^r)$, $(x^m+ay^n)^{\star_{9.15}} = (x^m,y^n)^{\star_{9.15}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.15}} = (x^m,z^r)^{\star_{9.15}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.15}} = (x^m+ay^n+bz^r)^{\star_{9.15}} = (y^n,z^r)^{\star_{9.15}} = (x^m+az^r,y^n+bz^r)^{\star_{9.15}} = (x^m,y^n+az^r)^{\star_{9.15}} = (x^m+ay^n,z^r)^{\star_{9.15}} = (x^m+az^r,y^n)^{\star_{9.15}} = (x^m,y^n,z^r)^{\star_{9.15}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1274. $\star_{9.16}$:$(0)^{\star_{9.16}} = (0)$, $(x^m)^{\star_{9.16}} = (x^m)$, $(y^n)^{\star_{9.16}} = (y^n)$, $(z^r)^{\star_{9.16}} = (z^r)$, $(x^m+ay^n)^{\star_{9.16}} = (x^m,y^n)^{\star_{9.16}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.16}} = (x^m,z^r)^{\star_{9.16}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.16}} = (y^n,z^r)^{\star_{9.16}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.16}} = (x^m+az^r,y^n+bz^r)^{\star_{9.16}} = (x^m,y^n+az^r)^{\star_{9.16}} = (x^m+ay^n,z^r)^{\star_{9.16}} = (x^m+az^r,y^n)^{\star_{9.16}} = (x^m,y^n,z^r)^{\star_{9.16}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1275. $\star_{9.17}$:$(0)^{\star_{9.17}} = (0)$, $(x^m)^{\star_{9.17}} = (x^m)$, $(y^n)^{\star_{9.17}} = (y^n)$, $(z^r)^{\star_{9.17}} = (z^r)$, $(x^m+ay^n)^{\star_{9.17}} = (x^m,y^n)^{\star_{9.17}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.17}} = (x^m,z^r)^{\star_{9.17}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.17}} = (y^n+az^r)$, $(y^n,z^r)^{\star_{9.17}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.17}} = (x^m+az^r,y^n+bz^r)^{\star_{9.17}} = (x^m,y^n+az^r)^{\star_{9.17}} = (x^m+ay^n,z^r)^{\star_{9.17}} = (x^m+az^r,y^n)^{\star_{9.17}} = (x^m,y^n,z^r)^{\star_{9.17}} = (x,y^n,z^r)$ $\forall\ m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1276. $\star_{9.18}$:$(0)^{\star_{9.18}} = (0)$, $(x^m)^{\star_{9.18}} = (x^m)$, $(y^n)^{\star_{9.18}} = (x^m+ay^n)^{\star_{9.18}} = (x^m,y^n)^{\star_{9.18}} = (x,y^n)$, $(z^r)^{\star_{9.18}} = (z^r)$, $(x^m+az^r)^{\star_{9.18}} = (x^m+az^r)$, $(x^m,z^r)^{\star_{9.18}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.18}} = (x^m+ay^n+bz^r)^{\star_{9.18}} = (y^n,z^r)^{\star_{9.18}} = (x^m+az^r,y^n+bz^r)^{\star_{9.18}} = (x^m,y^n+az^r)^{\star_{9.18}} = (x^m+ay^n,z^r)^{\star_{9.18}} = (x^m+az^r,y^n)^{\star_{9.18}} = (x^m,y^n,z^r)^{\star_{9.18}} = (x,y^n,z^r)$ $\forall\, m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1277. $\star_{9.19}$:$(0)^{\star_{9.19}} = (0)$, $(x^m)^{\star_{9.19}} = (x^m)$, $(y^n)^{\star_{9.19}} = (y^n)$, $(z^r)^{\star_{9.19}} = (z^r)$, $(x^m+ay^n)^{\star_{9.19}} = (x^m,y^n)^{\star_{9.19}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.19}} = (x^m+az^r)$, $(x^m,z^r)^{\star_{9.19}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.19}} = (x^m+ay^n+bz^r)^{\star_{9.19}} = (y^n,z^r)^{\star_{9.19}} = (x^m+az^r,y^n+bz^r)^{\star_{9.19}} = (x^m,y^n+az^r)^{\star_{9.19}} = (x^m+ay^n,z^r)^{\star_{9.19}} = (x^m+az^r,y^n)^{\star_{9.19}} = (x^m,y^n,z^r)^{\star_{9.19}} = (x,y^n,z^r)$ $\forall\, m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1278. $\star_{9.20}$:$(0)^{\star_{9.20}} = (0)$, $(x^m)^{\star_{9.20}} = (x^m)$, $(y^n)^{\star_{9.20}} = (y^n)$, $(z^r)^{\star_{9.20}} = (z^r)$, $(x^m+ay^n)^{\star_{9.20}} = (x^m,y^n)^{\star_{9.20}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.20}} = (x^m+az^r)$, $(x^m,z^r)^{\star_{9.20}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.20}} = (y^n,z^r)^{\star_{9.20}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.20}} = (x^m+az^r,y^n+bz^r)^{\star_{9.20}} = (x^m,y^n+az^r)^{\star_{9.20}} = (x^m+ay^n,z^r)^{\star_{9.20}} = (x^m+az^r,y^n)^{\star_{9.20}} = (x^m,y^n,z^r)^{\star_{9.20}} = (x,y^n,z^r)$ $\forall\, m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1279. $\star_{9.21}$:$(0)^{\star_{9.21}} = (0)$, $(x^m)^{\star_{9.21}} = (x^m)$, $(y^n)^{\star_{9.21}} = (y^n)$, $(z^r)^{\star_{9.21}} = (z^r)$, $(x^m+ay^n)^{\star_{9.21}} = (x^m,y^n)^{\star_{9.21}} = (x,y^n)$, $(x^m+az^r)^{\star_{9.21}} = (x^m+az^r)$, $(x^m,z^r)^{\star_{9.21}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.21}} = (y^n+az^r)$, $(y^n,z^r)^{\star_{9.21}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.21}} = (x^m+az^r,y^n+bz^r)^{\star_{9.21}} = (x^m,y^n+az^r)^{\star_{9.21}} = (x^m+ay^n,z^r)^{\star_{9.21}} = (x^m+az^r,y^n)^{\star_{9.21}} = (x^m,y^n,z^r)^{\star_{9.21}} = (x,y^n,z^r)$ $\forall\, m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1280. $\star_{9.22}$:$(0)^{\star_{9.22}} = (0)$, $(x^m)^{\star_{9.22}} = (x^m)$, $(y^n)^{\star_{9.22}} = (y^n)$, $(z^r)^{\star_{9.22}} = (x^m+az^r)^{\star_{9.22}} = (x^m,z^r)^{\star_{9.22}} = (x,z^r)$, $(x^m+ay^n)^{\star_{9.22}} = (x^m,y^n)^{\star_{9.22}} = (x^m,y^n)$, $(y^n+az^r)^{\star_{9.22}} = (x^m+ay^n+bz^r)^{\star_{9.22}} = (y^n,z^r)^{\star_{9.22}} = (x^m+az^r,y^n+bz^r)^{\star_{9.22}} = (x^m,y^n+az^r)^{\star_{9.22}} = (x^m+ay^n,z^r)^{\star_{9.22}} = (x^m+az^r,y^n)^{\star_{9.22}} = (x^m,y^n,z^r)^{\star_{9.22}} = (x,y^n,z^r)$ $\forall\, m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1281. $\star_{9.23}$:$(0)^{\star_{9.23}} = (0)$, $(x^m)^{\star_{9.23}} = (x^m)$, $(y^n)^{\star_{9.23}} = (y^n)$, $(z^r)^{\star_{9.23}} = (z^r)$, $(x^m+ay^n)^{\star_{9.23}} = (x^m,y^n)^{\star_{9.23}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.23}} = (x^m,z^r)^{\star_{9.23}} = (x,z^r)$,

$(y^n+az^r)^{\star_{9.23}} = (x^m+ay^n+bz^r)^{\star_{9.23}} = (y^n,z^r)^{\star_{9.23}} = (x^m+az^r,y^n+bz^r)^{\star_{9.23}} = (x^m,y^n+az^r)^{\star_{9.23}} = (x^m+ay^n,z^r)^{\star_{9.23}} = (x^m+az^r,y^n)^{\star_{9.23}} = (x^m,y^n,z^r)^{\star_{9.23}} = (x,y^n,z^r)$ $\forall$ $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1282. $\star_{9.24}$:$(0)^{\star_{9.24}} = (0)$, $(x^m)^{\star_{9.24}} = (x^m)$, $(y^n)^{\star_{9.24}} = (y^n)$, $(z^r)^{\star_{9.24}} = (z^r)$, $(x^m+ay^n)^{\star_{9.24}} = (x^m,y^n)^{\star_{9.24}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.24}} = (x^m,z^r)^{\star_{9.24}} = (x,z^r)$, $(y^n+az^r)^{\star_{9.24}} = (y^n,z^r)^{\star_{9.24}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.24}} = (x^m+az^r,y^n+bz^r)^{\star_{9.24}} = (x^m,y^n+az^r)^{\star_{9.24}} = (x^m+ay^n,z^r)^{\star_{9.24}} = (x^m+az^r,y^n)^{\star_{9.24}} = (x^m,y^n,z^r)^{\star_{9.24}} = (x,y^n,z^r)$ $\forall$ $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1283. $\star_{9.25}$:$(0)^{\star_{9.25}} = (0)$, $(x^m)^{\star_{9.25}} = (x^m)$, $(y^n)^{\star_{9.25}} = (y^n)$, $(z^r)^{\star_{9.25}} = (z^r)$, $(x^m+ay^n)^{\star_{9.25}} = (x^m,y^n)^{\star_{9.25}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.25}} = (x^m,z^r)^{\star_{9.25}} = (x,z^r)$, $(y^n+az^r)^{\star_{9.25}} = (y^n+az^r)$, $(x^m+ay^n+bz^r)^{\star_{9.25}} = (x^m+az^r,y^n+bz^r)^{\star_{9.25}} = (x^m,y^n+az^r)^{\star_{9.25}} = (x^m+ay^n,z^r)^{\star_{9.25}} = (x^m+az^r,y^n)^{\star_{9.25}} = (x^m,y^n,z^r)^{\star_{9.25}} = (x,y^n,z^r)$, $(y^n,z^r)^{\star_{9.25}} = (y^n,z^r)$ $\forall$ $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1284. $\star_{9.26}$:$(0)^{\star_{9.26}} = (0)$, $(x^m)^{\star_{9.26}} = (x^m)$, $(y^n)^{\star_{9.26}} = (y^n)$, $(z^r)^{\star_{9.26}} = (z^r)$, $(x^m+ay^n)^{\star_{9.26}} = (x^m,y^n)^{\star_{9.26}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.26}} = (x^m,z^r)^{\star_{9.26}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.26}} = (x^m+ay^n+bz^r)^{\star_{9.26}} = (y^n,z^r)^{\star_{9.26}} = (x^m+az^r,y^n+bz^r)^{\star_{9.26}} = (x^m,y^n+az^r)^{\star_{9.26}} = (x^m+ay^n,z^r)^{\star_{9.26}} = (x^m+az^r,y^n)^{\star_{9.26}} = (x^m,y^n,z^r)^{\star_{9.26}} = (x,y^n,z^r)$ $\forall$ $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1285. $\star_{9.27}$:$(0)^{\star_{9.27}} = (0)$, $(x^m)^{\star_{9.27}} = (x^m)$, $(y^n)^{\star_{9.27}} = (y^n)$, $(z^r)^{\star_{9.27}} = (z^r)$, $(x^m+ay^n)^{\star_{9.27}} = (x^m,y^n)^{\star_{9.27}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.27}} = (x^m,z^r)^{\star_{9.27}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.27}} = (y^n,z^r)^{\star_{9.27}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.27}} = (x^m+az^r,y^n+bz^r)^{\star_{9.27}} = (x^m,y^n+az^r)^{\star_{9.27}} = (x^m+ay^n,z^r)^{\star_{9.27}} = (x^m+az^r,y^n)^{\star_{9.27}} = (x^m,y^n,z^r)^{\star_{9.27}} = (x,y^n,z^r)$ $\forall$ $m,n,r \in \mathbb{N}$ and $a,b \in k^\times$

1286. $\star_{9.28}$:$(0)^{\star_{9.28}} = (0)$, $(x^m)^{\star_{9.28}} = (x^m)$, $(y^n)^{\star_{9.28}} = (y^n)$, $(z^r)^{\star_{9.28}} = (z^r)$, $(x^m+ay^n)^{\star_{9.28}} = (x^m,y^n)^{\star_{9.28}} = (x^m,y^n)$, $(x^m+az^r)^{\star_{9.28}} = (x^m,z^r)^{\star_{9.28}} = (x^m,z^r)$, $(y^n+az^r)^{\star_{9.28}} = (y^n+az^r)$, $(y^n,z^r)^{\star_{9.28}} = (y^n,z^r)$, $(x^m+ay^n+bz^r)^{\star_{9.28}} = (x^m+az^r,y^n+bz^r)^{\star_{9.28}} = (x^m,y^n+az^r)^{\star_{9.28}} = (x^m+ay^n,z^r)^{\star_{9.28}} = (x^m+az^r,y^n)^{\star_{9.28}} = (x^m,y^n,z^r)^{\star_{9.28}} = (x,y^n,z^r)$ $\forall$ $m,n,r \in$

$\mathbb{N}$ and $a, b \in k^\times$

1287. $\star_{9.29}$:$(0)^{\star 9.29} = (0)$, $(x^m)^{\star 9.29} = (x^m)$, $(y^n)^{\star 9.29} = (y^n)$, $(z^r)^{\star 9.29} = (z^r)$, $(x^m + ay^n)^{\star 9.29} = (x^m, y^n)^{\star 9.29} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.29} = (x^m + az^r)$, $(y^n + az^r)^{\star 9.29} = (x^m + ay^n + bz^r)^{\star 9.29} = (y^n, z^r)^{\star 9.29} = (x^m + az^r, y^n + bz^r)^{\star 9.29} = (x^m, y^n + az^r)^{\star 9.29} = (x^m + ay^n, z^r)^{\star 9.29} = (x^m + az^r, y^n)^{\star 9.29} = (x^m, y^n, z^r)^{\star 9.29} = (x, y^n, z^r)$, $(x^m, z^r)^{\star 9.29} = (x^m, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1288. $\star_{9.30}$:$(0)^{\star 9.30} = (0)$, $(x^m)^{\star 9.30} = (x^m)$, $(y^n)^{\star 9.30} = (y^n)$, $(z^r)^{\star 9.30} = (z^r)$, $(x^m + ay^n)^{\star 9.30} = (x^m, y^n)^{\star 9.30} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.30} = (x^m + az^r)$, $(y^n + az^r)^{\star 9.30} = (y^n, z^r)^{\star 9.30} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star 9.30} = (x^m + az^r, y^n + bz^r)^{\star 9.30} = (x^m, y^n + az^r)^{\star 9.30} = (x^m + ay^n, z^r)^{\star 9.30} = (x^m + az^r, y^n)^{\star 9.30} = (x^m, y^n, z^r)^{\star 9.30} = (x, y^n, z^r)$, $(x^m, z^r)^{\star 9.30} = (x^m, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1289. $\star_{9.31}$:$(0)^{\star 9.31} = (0)$, $(x^m)^{\star 9.31} = (x^m)$, $(y^n)^{\star 9.31} = (y^n)$, $(z^r)^{\star 9.31} = (z^r)$, $(x^m + ay^n)^{\star 9.31} = (x^m, y^n)^{\star 9.31} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.31} = (x^m + az^r)$, $(y^n + az^r)^{\star 9.31} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.31} = (x^m + az^r, y^n + bz^r)^{\star 9.31} = (x^m, y^n + az^r)^{\star 9.31} = (x^m + ay^n, z^r)^{\star 9.31} = (x^m + az^r, y^n)^{\star 9.31} = (x^m, y^n, z^r)^{\star 9.31} = (x, y^n, z^r)$, $(x^m, z^r)^{\star 9.31} = (x^m, z^r)$, $(y^n, z^r)^{\star 9.31} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1290. $\star_{9.32}$:$(0)^{\star 9.32} = (0)$, $(x^m)^{\star 9.32} = (x^m)$, $(y^n)^{\star 9.32} = (y^n)$, $(z^r)^{\star 9.32} = (x^m + az^r)^{\star 9.32} = (x^m, z^r)^{\star 9.32} = (x, z^r)$, $(x^m + ay^n)^{\star 9.32} = (x^m + ay^n)$, $(y^n + az^r)^{\star 9.32} = (x^m + ay^n + bz^r)^{\star 9.32} = (y^n, z^r)^{\star 9.32} = (x^m + az^r, y^n + bz^r)^{\star 9.32} = (x^m, y^n + az^r)^{\star 9.32} = (x^m + ay^n, z^r)^{\star 9.32} = (x^m + az^r, y^n)^{\star 9.32} = (x^m, y^n, z^r)^{\star 9.32} = (x, y^n, z^r)$, $(x^m, y^n)^{\star 9.32} = (x^m, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1291. $\star_{9.33}$:$(0)^{\star 9.33} = (0)$, $(x^m)^{\star 9.33} = (x^m)$, $(y^n)^{\star 9.33} = (y^n)$, $(z^r)^{\star 9.33} = (z^r)$, $(x^m + ay^n)^{\star 9.33} = (x^m + ay^n)$, $(x^m + az^r)^{\star 9.33} = (x^m, z^r)^{\star 9.33} = (x, z^r)$, $(y^n + az^r)^{\star 9.33} = (x^m + ay^n + bz^r)^{\star 9.33} = (y^n, z^r)^{\star 9.33} = (x^m + az^r, y^n + bz^r)^{\star 9.33} = (x^m, y^n + az^r)^{\star 9.33} = (x^m + ay^n, z^r)^{\star 9.33} = (x^m + az^r, y^n)^{\star 9.33} = (x^m, y^n, z^r)^{\star 9.33} = (x, y^n, z^r)$, $(x^m, y^n)^{\star 9.33} = (x^m, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1292. $\star_{9.34}$:$(0)^{\star_{9.34}} = (0)$, $(x^m)^{\star_{9.34}} = (x^m)$, $(y^n)^{\star_{9.34}} = (y^n)$, $(z^r)^{\star_{9.34}} = (z^r)$, $(x^m + ay^n)^{\star_{9.34}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.34}} = (x^m, z^r)^{\star_{9.34}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.34}} = (y^n, z^r)^{\star_{9.34}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{9.34}} = (x^m + az^r, y^n + bz^r)^{\star_{9.34}} = (x^m, y^n + az^r)^{\star_{9.34}} = (x^m + ay^n, z^r)^{\star_{9.34}} = (x^m + az^r, y^n)^{\star_{9.34}} = (x^m, y^n, z^r)^{\star_{9.34}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.34}} = (x^m, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
1293. $\star_{9.35}$:$(0)^{\star_{9.35}} = (0)$, $(x^m)^{\star_{9.35}} = (x^m)$, $(y^n)^{\star_{9.35}} = (y^n)$, $(z^r)^{\star_{9.35}} = (z^r)$, $(x^m + ay^n)^{\star_{9.35}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.35}} = (x^m, z^r)^{\star_{9.35}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.35}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.35}} = (x^m + az^r, y^n + bz^r)^{\star_{9.35}} = (x^m, y^n + az^r)^{\star_{9.35}} = (x^m + ay^n, z^r)^{\star_{9.35}} = (x^m + az^r, y^n)^{\star_{9.35}} = (x^m, y^n, z^r)^{\star_{9.35}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.35}} = (x^m, y^n)$, $(y^n, z^r)^{\star_{9.35}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
1294. $\star_{9.36}$:$(0)^{\star_{9.36}} = (0)$, $(x^m)^{\star_{9.36}} = (x^m)$, $(y^n)^{\star_{9.36}} = (y^n)$, $(z^r)^{\star_{9.36}} = (z^r)$, $(x^m + ay^n)^{\star_{9.36}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.36}} = (x^m, z^r)^{\star_{9.36}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.36}} = (x^m + ay^n + bz^r)^{\star_{9.36}} = (y^n, z^r)^{\star_{9.36}} = (x^m + az^r, y^n + bz^r)^{\star_{9.36}} = (x^m, y^n + az^r)^{\star_{9.36}} = (x^m + ay^n, z^r)^{\star_{9.36}} = (x^m + az^r, y^n)^{\star_{9.36}} = (x^m, y^n, z^r)^{\star_{9.36}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.36}} = (x^m, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
1295. $\star_{9.37}$:$(0)^{\star_{9.37}} = (0)$, $(x^m)^{\star_{9.37}} = (x^m)$, $(y^n)^{\star_{9.37}} = (y^n)$, $(z^r)^{\star_{9.37}} = (z^r)$, $(x^m + ay^n)^{\star_{9.37}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.37}} = (x^m, z^r)^{\star_{9.37}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.37}} = (y^n, z^r)^{\star_{9.37}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{9.37}} = (x^m + az^r, y^n + bz^r)^{\star_{9.37}} = (x^m, y^n + az^r)^{\star_{9.37}} = (x^m + ay^n, z^r)^{\star_{9.37}} = (x^m + az^r, y^n)^{\star_{9.37}} = (x^m, y^n, z^r)^{\star_{9.37}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.37}} = (x^m, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$
1296. $\star_{9.38}$:$(0)^{\star_{9.38}} = (0)$, $(x^m)^{\star_{9.38}} = (x^m)$, $(y^n)^{\star_{9.38}} = (y^n)$, $(z^r)^{\star_{9.38}} = (z^r)$, $(x^m + ay^n)^{\star_{9.38}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.38}} = (x^m, z^r)^{\star_{9.38}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.38}} = (y^n + az^r)$,$(x^m + ay^n + bz^r)^{\star_{9.38}} = (x^m + az^r, y^n + bz^r)^{\star_{9.38}} = (x^m, y^n + az^r)^{\star_{9.38}} = (x^m + ay^n, z^r)^{\star_{9.38}} = (x^m + az^r, y^n)^{\star_{9.38}} = (x^m, y^n, z^r)^{\star_{9.38}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.38}} = (x^m, y^n)$, $(y^n, z^r)^{\star_{9.38}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and

$a, b \in k^\times$

1297. $\star_{9.39}$:$(0)^{\star_{9.39}} = (0)$, $(x^m)^{\star_{9.39}} = (x^m)$, $(y^n)^{\star_{9.39}} = (y^n)$, $(z^r)^{\star_{9.39}} = (z^r)$, $(x^m + ay^n)^{\star_{9.39}} = (x^m + ay^n)$,$(x^m + az^r)^{\star_{9.39}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{9.39}} = (x^m + ay^n + bz^r)^{\star_{9.39}} = (y^n, z^r)^{\star_{9.39}} = (x^m + az^r, y^n + bz^r)^{\star_{9.39}} = (x^m, y^n + az^r)^{\star_{9.39}} = (x^m + ay^n, z^r)^{\star_{9.39}} = (x^m + az^r, y^n)^{\star_{9.39}} = (x^m, y^n, z^r)^{\star_{9.39}} = (x, y^n, z^r)$, $(x^m, z^r)^{\star_{9.39}} = (x^m, z^r)$, ,$(x^m, y^n)^{\star_{9.39}} = (x^m, y^n)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1298. $\star_{9.40}$:$(0)^{\star_{9.40}} = (0)$, $(x^m)^{\star_{9.40}} = (x^m)$, $(y^n)^{\star_{9.40}} = (y^n)$, $(z^r)^{\star_{9.40}} = (z^r)$, $(x^m + ay^n)^{\star_{9.40}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.40}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{9.40}} = (y^n, z^r)^{\star_{9.40}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{9.40}} = (x^m + az^r, y^n + bz^r)^{\star_{9.40}} = (x^m, y^n + az^r)^{\star_{9.40}} = (x^m + ay^n, z^r)^{\star_{9.40}} = (x^m + az^r, y^n)^{\star_{9.40}} = (x^m, y^n, z^r)^{\star_{9.40}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.40}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{9.40}} = (x^m, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1299. $\star_{9.41}$:$(0)^{\star_{9.41}} = (0)$, $(x^m)^{\star_{9.41}} = (x^m)$, $(y^n)^{\star_{9.41}} = (y^n)$, $(z^r)^{\star_{9.41}} = (z^r)$, $(x^m + ay^n)^{\star_{9.41}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{9.41}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{9.41}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.41}} = (x^m + az^r, y^n + bz^r)^{\star_{9.41}} = (x^m, y^n + az^r)^{\star_{9.41}} = (x^m + ay^n, z^r)^{\star_{9.41}} = (x^m + az^r, y^n)^{\star_{9.41}} = (x^m, y^n, z^r)^{\star_{9.41}} = (x, y^n, z^r)$, $(x^m, y^n)^{\star_{9.41}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{9.41}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{9.41}} = (y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1300. $\star_{9.42}$:$(0)^{\star_{9.42}} = (x^m)^{\star_{9.42}} = (x)$, $(y^n)^{\star_{9.42}} = (x^m + ay^n)^{\star_{9.42}} = (x^m, y^n)^{\star_{9.42}} = (x, y^n)$, $(z^r)^{\star_{9.42}} = (x^m + az^r)^{\star_{9.42}} = (x^m, z^r)^{\star_{9.42}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.42}} = (x^m, y^n + az^r)^{\star_{9.42}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.42}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.42}} = (x^m + az^r, y^n + bz^r)^{\star_{9.42}} = (x^m + ay^n, z^r)^{\star_{9.42}} = (x^m + az^r, y^n)^{\star_{9.42}} = (x^m, y^n, z^r)^{\star_{9.42}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1301. $\star_{9.43}$:$(0)^{\star_{9.43}} = (0)$, $(x^m)^{\star_{9.43}} = (x)$, $(y^n)^{\star_{9.43}} = (x^m + ay^n)^{\star_{9.43}} = (x^m, y^n)^{\star_{9.43}} = (x, y^n)$, $(z^r)^{\star_{9.43}} = (x^m + az^r)^{\star_{9.43}} = (x^m, z^r)^{\star_{9.43}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.43}} = (x^m, y^n + az^r)^{\star_{9.43}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.43}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.43}} = (x^m + az^r, y^n + bz^r)^{\star_{9.43}} = (x^m + ay^n, z^r)^{\star_{9.43}} = (x^m + az^r, y^n)^{\star_{9.43}} =$

$(x^m, y^n, z^r)^{\star_{9.43}} = (x, y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1302. $\star_{9.44}$:$(0)^{\star_{9.44}} = (0)$, $(x^m)^{\star_{9.44}} = (x)$, $(y^n)^{\star_{9.44}} = (x^m + ay^n)^{\star_{9.44}} = (x^m, y^n)^{\star_{9.44}} = (x, y^n)$, $(z^r)^{\star_{9.44}} = (z^r)$, $(x^m + az^r)^{\star_{9.44}} = (x^m, z^r)^{\star_{9.44}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.44}} = (x^m, y^n + az^r)^{\star_{9.44}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.44}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.44}} = (x^m + az^r, y^n + bz^r)^{\star_{9.44}} = (x^m + ay^n, z^r)^{\star_{9.44}} = (x^m + az^r, y^n)^{\star_{9.44}} = (x^m, y^n, z^r)^{\star_{9.44}} = (x, y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1303. $\star_{9.45}$:$(0)^{\star_{9.45}} = (x^m)^{\star_{9.45}} = (x)$, $(y^n)^{\star_{9.45}} = (y^n)$, $(z^r)^{\star_{9.45}} = (x^m + az^r)^{\star_{9.45}} = (x^m, z^r)^{\star_{9.45}} = (x, z^r)$, $(x^m + ay^n)^{\star_{9.45}} = (x^m, y^n)^{\star_{9.45}} = (x, y^n)$, $(y^n + az^r)^{\star_{9.45}} = (x^m, y^n + az^r)^{\star_{9.45}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.45}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.45}} = (x^m + az^r, y^n + bz^r)^{\star_{9.45}} = (x^m + ay^n, z^r)^{\star_{9.45}} = (x^m + az^r, y^n)^{\star_{9.45}} = (x^m, y^n, z^r)^{\star_{9.45}} = (x, y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1304. $\star_{9.46}$:$(0)^{\star_{9.46}} = (0)$, $(x^m)^{\star_{9.46}} = (x)$, $(y^n)^{\star_{9.46}} = (y^n)$, $(z^r)^{\star_{9.46}} = (x^m + az^r)^{\star_{9.46}} = (x^m, z^r)^{\star_{9.46}} = (x, z^r)$, $(x^m + ay^n)^{\star_{9.46}} = (x^m, y^n)^{\star_{9.46}} = (x, y^n)$, $(y^n + az^r)^{\star_{9.46}} = (x^m, y^n + az^r)^{\star_{9.46}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.46}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.46}} = (x^m + az^r, y^n + bz^r)^{\star_{9.46}} = (x^m + ay^n, z^r)^{\star_{9.46}} = (x^m + az^r, y^n)^{\star_{9.46}} = (x^m, y^n, z^r)^{\star_{9.46}} = (x, y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1305. $\star_{9.47}$:$(0)^{\star_{9.47}} = (0)$, $(x^m)^{\star_{9.47}} = (x)$, $(y^n)^{\star_{9.47}} = (y^n)$, $(z^r)^{\star_{9.47}} = (z^r)$, $(x^m + ay^n)^{\star_{9.47}} = (x^m, y^n)^{\star_{9.47}} = (x, y^n)$, $(x^m + az^r)^{\star_{9.47}} = (x^m, z^r)^{\star_{9.47}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.47}} = (x^m, y^n + az^r)^{\star_{9.47}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.47}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.47}} = (x^m + az^r, y^n + bz^r)^{\star_{9.47}} = (x^m + ay^n, z^r)^{\star_{9.47}} = (x^m + az^r, y^n)^{\star_{9.47}} = (x^m, y^n, z^r)^{\star_{9.47}} = (x, y^n, z^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1306. $\star_{9.48}$:$(0)^{\star_{9.48}} = (0)$, $(x^m)^{\star_{9.48}} = (x)$, $(y^n)^{\star_{9.48}} = (y^n)$, $(z^r)^{\star_{9.48}} = (z^r)$, $(x^m + ay^n)^{\star_{9.48}} = (x^m, y^n)^{\star_{9.48}} = (x, y^n)$, $(x^m + az^r)^{\star_{9.48}} = (x^m, z^r)^{\star_{9.48}} = (x, z^r)$, $(y^n + az^r)^{\star_{9.48}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.48}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.48}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{9.48}} = (x^m + ay^n, z^r)^{\star_{9.48}} = (x^m + az^r, y^n)^{\star_{9.48}} = (x^m, y^n, z^r)^{\star_{9.48}} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{9.48}} = (x, y^n + az^r)\ \forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1307. $\star_{9.49}$:$(0)^{\star_{9.49}} = (0)$, $(x^m)^{\star_{9.49}} = (x^m)$, $(y^n)^{\star_{9.49}} = (x^m + ay^n)^{\star_{9.49}} = (x^m, y^n)^{\star_{9.49}} =$

(x, y^n), $(z^r)^{\star 9.49} = (x^m + az^r)^{\star 9.49} = (x^m, z^r)^{\star 9.49} = (x, z^r)$, $(y^n + az^r)^{\star 9.49} = (x^m, y^n + az^r)^{\star 9.49} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.49} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.49} = (x^m + az^r, y^n + bz^r)^{\star 9.49} = (x^m + ay^n, z^r)^{\star 9.49} = (x^m + az^r, y^n)^{\star 9.49} = (x^m, y^n, z^r)^{\star 9.49} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1308. $\star_{9.50}$:$(0)^{\star 9.50} = (0)$, $(x^m)^{\star 9.50} = (x^m)$, $(y^n)^{\star 9.50} = (x^m + ay^n)^{\star 9.50} = (x^m, y^n)^{\star 9.50} = (x, y^n)$, $(z^r)^{\star 9.50} = (z^r)$, $(x^m + az^r)^{\star 9.50} = (x^m, z^r)^{\star 9.50} = (x, z^r)$, $(y^n + az^r)^{\star 9.50} = (x^m, y^n + az^r)^{\star 9.50} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.50} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.50} = (x^m + az^r, y^n + bz^r)^{\star 9.50} = (x^m + ay^n, z^r)^{\star 9.50} = (x^m + az^r, y^n)^{\star 9.50} = (x^m, y^n, z^r)^{\star 9.50} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1309. $\star_{9.51}$:$(0)^{\star 9.51} = (0)$, $(x^m)^{\star 9.51} = (x^m)$, $(y^n)^{\star 9.51} = (y^n)$, $(z^r)^{\star 9.51} = (x^m + az^r)^{\star 9.51} = (x^m, z^r)^{\star 9.51} = (x, z^r)$, $(x^m + ay^n)^{\star 9.51} = (x^m, y^n)^{\star 9.51} = (x, y^n)$, $(y^n + az^r)^{\star 9.51} = (x^m, y^n + az^r)^{\star 9.51} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.51} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.51} = (x^m + az^r, y^n + bz^r)^{\star 9.51} = (x^m + ay^n, z^r)^{\star 9.51} = (x^m + az^r, y^n)^{\star 9.51} = (x^m, y^n, z^r)^{\star 9.51} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1310. $\star_{9.52}$:$(0)^{\star 9.52} = (0)$, $(x^m)^{\star 9.52} = (x^m)$, $(y^n)^{\star 9.52} = (y^n)$, $(z^r)^{\star 9.52} = (z^r)$, $(x^m + ay^n)^{\star 9.52} = (x^m, y^n)^{\star 9.52} = (x, y^n)$, $(x^m + az^r)^{\star 9.52} = (x^m, z^r)^{\star 9.52} = (x, z^r)$, $(y^n + az^r)^{\star 9.52} = (x^m, y^n + az^r)^{\star 9.52} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.52} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.52} = (x^m + az^r, y^n + bz^r)^{\star 9.52} = (x^m + ay^n, z^r)^{\star 9.52} = (x^m + az^r, y^n)^{\star 9.52} = (x^m, y^n, z^r)^{\star 9.52} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1311. $\star_{9.53}$:$(0)^{\star 9.53} = (0)$, $(x^m)^{\star 9.53} = (x^m)$, $(y^n)^{\star 9.53} = (y^n)$, $(z^r)^{\star 9.53} = (z^r)$, $(x^m + ay^n)^{\star 9.53} = (x^m, y^n)^{\star 9.53} = (x, y^n)$, $(x^m + az^r)^{\star 9.53} = (x^m, z^r)^{\star 9.53} = (x, z^r)$, $(y^n + az^r)^{\star 9.53} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.53} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.53} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star 9.53} = (x^m + ay^n, z^r)^{\star 9.53} = (x^m + az^r, y^n)^{\star 9.53} = (x^m, y^n, z^r)^{\star 9.53} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star 9.53} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1312. $\star_{9.54}$:$(0)^{\star 9.54} = (0)$, $(x^m)^{\star 9.54} = (x^m)$, $(y^n)^{\star 9.54} = (x^m + ay^n)^{\star 9.54} = (x^m, y^n)^{\star 9.54} = (x, y^n)$, $(z^r)^{\star 9.54} = (z^r)$, $(x^m + az^r)^{\star 9.54} = (x^m, z^r)^{\star 9.54} = (x^m, z^r)$, $(y^n + az^r)^{\star 9.54} = (x^m, y^n + az^r)^{\star 9.54} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.54} = (x, y^n + \frac{b}{a}z^r)$,

$(y^n, z^r)^{\star_{9.54}} = (x^m+az^r, y^n+bz^r)^{\star_{9.54}} = (x^m+ay^n, z^r)^{\star_{9.54}} = (x^m+az^r, y^n)^{\star_{9.54}} = (x^m, y^n, z^r)^{\star_{9.54}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1313. $\star_{9.55}$:$(0)^{\star_{9.55}} = (0)$, $(x^m)^{\star_{9.55}} = (x^m)$, $(y^n)^{\star_{9.55}} = (y^n)$,$(z^r)^{\star_{9.55}} = (z^r)$, $(x^m+ay^n)^{\star_{9.55}} = (x^m, y^n)^{\star_{9.55}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.55}} = (x^m, z^r)^{\star_{9.55}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.55}} = (x^m, y^n + az^r)^{\star_{9.55}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.55}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.55}} = (x^m + az^r, y^n + bz^r)^{\star_{9.55}} = (x^m + ay^n, z^r)^{\star_{9.55}} = (x^m + az^r, y^n)^{\star_{9.55}} = (x^m, y^n, z^r)^{\star_{9.55}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1314. $\star_{9.56}$:$(0)^{\star_{9.56}} = (0)$, $(x^m)^{\star_{9.56}} = (x^m)$, $(y^n)^{\star_{9.56}} = (y^n)$,$(z^r)^{\star_{9.56}} = (z^r)$, $(x^m+ay^n)^{\star_{9.56}} = (x^m, y^n)^{\star_{9.56}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.56}} = (x^m, z^r)^{\star_{9.56}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.56}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.56}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.56}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{9.56}} = (x^m + ay^n, z^r)^{\star_{9.56}} = (x^m + az^r, y^n)^{\star_{9.56}} = (x^m, y^n, z^r)^{\star_{9.56}} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{9.56}} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1315. $\star_{9.57}$:$(0)^{\star_{9.57}} = (0)$, $(x^m)^{\star_{9.57}} = (x^m)$, $(y^n)^{\star_{9.57}} = (x^m + ay^n)^{\star_{9.57}} = (x^m, y^n)^{\star_{9.57}} = (x, y^n)$, $(z^r)^{\star_{9.57}} = (z^r)$, $(x^m + az^r)^{\star_{9.57}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{9.57}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.57}} = (x^m, y^n + az^r)^{\star_{9.57}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.57}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.57}} = (x^m + az^r, y^n + bz^r)^{\star_{9.57}} = (x^m + ay^n, z^r)^{\star_{9.57}} = (x^m + az^r, y^n)^{\star_{9.57}} = (x^m, y^n, z^r)^{\star_{9.57}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1316. $\star_{9.58}$:$(0)^{\star_{9.58}} = (0)$, $(x^m)^{\star_{9.58}} = (x^m)$, $(y^n)^{\star_{9.58}} = (y^n)$,$(z^r)^{\star_{9.58}} = (z^r)$, $(x^m+ay^n)^{\star_{9.58}} = (x^m, y^n)^{\star_{9.58}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.58}} = (x^m+az^r)$, $(x^m, z^r)^{\star_{9.58}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.58}} = (x^m, y^n + az^r)^{\star_{9.58}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.58}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.58}} = (x^m + az^r, y^n + bz^r)^{\star_{9.58}} = (x^m+ay^n, z^r)^{\star_{9.58}} = (x^m+az^r, y^n)^{\star_{9.58}} = (x^m, y^n, z^r)^{\star_{9.58}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1317. $\star_{9.59}$:$(0)^{\star_{9.59}} = (0)$, $(x^m)^{\star_{9.59}} = (x^m)$, $(y^n)^{\star_{9.59}} = (y^n)$,$(z^r)^{\star_{9.59}} = (z^r)$, $(x^m+ay^n)^{\star_{9.59}} = (x^m, y^n)^{\star_{9.59}} = (x, y^n)$, $(x^m+az^r)^{\star_{9.59}} = (x^m+az^r)$, $(x^m, z^r)^{\star_{9.59}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.59}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.59}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.59}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{9.59}} = (x^m + ay^n, z^r)^{\star_{9.59}} =$

$(x^m + az^r, y^n)^{\star 9.59} = (x^m, y^n, z^r)^{\star 9.59} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star 9.59} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1318. $\star_{9.60}$:$(0)^{\star 9.60} = (0)$, $(x^m)^{\star 9.60} = (x^m)$, $(y^n)^{\star 9.60} = (y^n)$,$(z^r)^{\star 9.60} = (x^m+az^r)^{\star 9.60} = (x^m, z^r)^{\star 9.60} = (x, z^r)$, $(x^m+ay^n)^{\star 9.60} = (x^m, y^n)^{\star 9.60} = (x^m, y^n)$, $(y^n+az^r)^{\star 9.60} = (x^m, y^n + az^r)^{\star 9.60} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.60} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.60} = (x^m+az^r, y^n+bz^r)^{\star 9.60} = (x^m+ay^n, z^r)^{\star 9.60} = (x^m+az^r, y^n)^{\star 9.60} = (x^m, y^n, z^r)^{\star 9.60} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1319. $\star_{9.61}$:$(0)^{\star 9.61} = (0)$, $(x^m)^{\star 9.61} = (x^m)$, $(y^n)^{\star 9.61} = (y^n)$,$(z^r)^{\star 9.61} = (z^r)$, $(x^m+ay^n)^{\star 9.61} = (x^m, y^n)^{\star 9.61} = (x^m, y^n)$, $(x^m+az^r)^{\star 9.61} = (x^m, z^r)^{\star 9.61} = (x, z^r)$, $(y^n + az^r)^{\star 9.61} = (x^m, y^n + az^r)^{\star 9.61} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.61} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.61} = (x^m + az^r, y^n + bz^r)^{\star 9.61} = (x^m + ay^n, z^r)^{\star 9.61} = (x^m + az^r, y^n)^{\star 9.61} = (x^m, y^n, z^r)^{\star 9.61} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1320. $\star_{9.62}$:$(0)^{\star 9.62} = (0)$, $(x^m)^{\star 9.62} = (x^m)$, $(y^n)^{\star 9.62} = (y^n)$,$(z^r)^{\star 9.62} = (z^r)$, $(x^m+ay^n)^{\star 9.62} = (x^m, y^n)^{\star 9.62} = (x^m, y^n)$, $(x^m+az^r)^{\star 9.62} = (x^m, z^r)^{\star 9.62} = (x, z^r)$, $(y^n + az^r)^{\star 9.62} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.62} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.62} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star 9.62} = (x^m + ay^n, z^r)^{\star 9.62} = (x^m + az^r, y^n)^{\star 9.62} = (x^m, y^n, z^r)^{\star 9.62} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star 9.62} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1321. $\star_{9.63}$:$(0)^{\star 9.63} = (0)$, $(x^m)^{\star 9.63} = (x^m)$, $(y^n)^{\star 9.63} = (y^n)$,$(z^r)^{\star 9.63} = (z^r)$, $(x^m + ay^n)^{\star 9.63} = (x^m, y^n)^{\star 9.63} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.63} = (x^m, z^r)^{\star 9.63} = (x^m, z^r)$, $(y^n + az^r)^{\star 9.63} = (x^m, y^n + az^r)^{\star 9.63}(x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.63} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.63} = (x^m + az^r, y^n + bz^r)^{\star 9.63} = (x^m + ay^n, z^r)^{\star 9.63} = (x^m + az^r, y^n)^{\star 9.63} = (x^m, y^n, z^r)^{\star 9.63} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1322. $\star_{9.64}$:$(0)^{\star 9.64} = (0)$, $(x^m)^{\star 9.64} = (x^m)$, $(y^n)^{\star 9.64} = (y^n)$,$(z^r)^{\star 9.64} = (z^r)$, $(x^m + ay^n)^{\star 9.64} = (x^m, y^n)^{\star 9.64} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.64} = (x^m, z^r)^{\star 9.64} = (x^m, z^r)$, $(y^n + az^r)^{\star 9.64} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.64} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.64} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star 9.64} = (x^m + ay^n, z^r)^{\star 9.64} = (x^m + az^r, y^n)^{\star 9.64} = (x^m, y^n, z^r)^{\star 9.64} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star 9.64} =$

$(x, y^n + az^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1323. $\star_{9.65}$:$(0)^{\star 9.65} = (0)$, $(x^m)^{\star 9.65} = (x^m)$, $(y^n)^{\star 9.65} = (y^n)$,$(z^r)^{\star 9.65} = (z^r)$, $(x^m + ay^n)^{\star 9.65} = (x^m, y^n)^{\star 9.65} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.65} = (x^m + az^r)$, $(x^m, z^r)^{\star 9.65} = (x^m, z^r)$, $(y^n + az^r)^{\star 9.65} = (x^m, y^n + az^r)^{\star 9.65} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.65} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.65} = (x^m + az^r, y^n + bz^r)^{\star 9.65} = (x^m + ay^n, z^r)^{\star 9.65} = (x^m + az^r, y^n)^{\star 9.65} = (x^m, y^n, z^r)^{\star 9.65} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1324. $\star_{9.66}$:$(0)^{\star 9.66} = (0)$, $(x^m)^{\star 9.66} = (x^m)$, $(y^n)^{\star 9.66} = (y^n)$,$(z^r)^{\star 9.66} = (z^r)$, $(x^m + ay^n)^{\star 9.66} = (x^m, y^n)^{\star 9.66} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.66} = (x^m + az^r)$, $(x^m, z^r)^{\star 9.66} = (x^m, z^r)$, $(y^n + az^r)^{\star 9.66} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.66} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.66} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star 9.66} = (x^m + ay^n, z^r)^{\star 9.66} = (x^m + az^r, y^n)^{\star 9.66} = (x^m, y^n, z^r)^{\star 9.66} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star 9.66} = (x, y^n + az^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1325. $\star_{9.67}$:$(0)^{\star 9.67} = (0)$, $(x^m)^{\star 9.67} = (x^m)$, $(y^n)^{\star 9.67} = (y^n)$,$(z^r)^{\star 9.67} = (x^m + az^r)^{\star 9.67} = (x^m, z^r)^{\star 9.67} = (x, z^r)$, $(x^m + ay^n)^{\star 9.67} = (x^m + ay^n)$, $(x^m, y^n)^{\star 9.67} = (x^m, y^n)$, $(y^n + az^r)^{\star 9.67} = (x^m, y^n + az^r)^{\star 9.67} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.67} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.67} = (x^m + az^r, y^n + bz^r)^{\star 9.67} = (x^m + ay^n, z^r)^{\star 9.67} = (x^m + az^r, y^n)^{\star 9.67} = (x^m, y^n, z^r)^{\star 9.67} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1326. $\star_{9.68}$:$(0)^{\star 9.68} = (0)$, $(x^m)^{\star 9.68} = (x^m)$, $(y^n)^{\star 9.68} = (y^n)$,$(z^r)^{\star 9.68} = (z^r)$, $(x^m + ay^n)^{\star 9.68} = (x^m + ay^n)$, $(x^m, y^n)^{\star 9.68} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.68} = (x^m, z^r)^{\star 9.68} = (x, z^r)$, $(y^n + az^r)^{\star 9.68} = (x^m, y^n + az^r)^{\star 9.68} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.68} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.68} = (x^m + az^r, y^n + bz^r)^{\star 9.68} = (x^m + ay^n, z^r)^{\star 9.68} = (x^m + az^r, y^n)^{\star 9.68} = (x^m, y^n, z^r)^{\star 9.68} = (x, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1327. $\star_{9.69}$:$(0)^{\star 9.69} = (0)$, $(x^m)^{\star 9.69} = (x^m)$, $(y^n)^{\star 9.69} = (y^n)$, $(z^r)^{\star 9.69} = (z^r)$, $(x^m + ay^n)^{\star 9.69} = (x^m + ay^n)$, $(x^m, y^n)^{\star 9.69} = (x^m, y^n)$, $(x^m + az^r)^{\star 9.69} = (x^m, z^r)^{\star 9.69} = (x, z^r)$, $(y^n + az^r)^{\star 9.69} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star 9.69} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star 9.69} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star 9.69} = (x^m + ay^n, z^r)^{\star 9.69}$

$= (x^m + az^r, y^n)^{\star_{9.69}} = (x^m, y^n, z^r)^{\star_{9.69}} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{9.69}} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1328. $\star_{9.70}$:$(0)^{\star_{9.70}} = (0)$, $(x^m)^{\star_{9.70}} = (x^m)$, $(y^n)^{\star_{9.70}} = (y^n)$,$(z^r)^{\star_{9.70}} = (z^r)$, $(x^m + ay^n)^{\star_{9.70}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{9.70}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{9.70}} = (x^m, z^r)^{\star_{9.70}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.70}} = (x^m, y^n + az^r)^{\star_{9.70}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.70}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.70}} = (x^m + az^r, y^n + bz^r)^{\star_{9.70}} = (x^m+ay^n, z^r)^{\star_{9.70}} = (x^m+az^r, y^n)^{\star_{9.70}} = (x^m, y^n, z^r)^{\star_{9.70}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1329. $\star_{9.71}$:$(0)^{\star_{9.71}} = (0)$, $(x^m)^{\star_{9.71}} = (x^m)$, $(y^n)^{\star_{9.71}} = (y^n)$,$(z^r)^{\star_{9.71}} = (z^r)$, $(x^m + ay^n)^{\star_{9.71}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{9.71}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{9.71}} = (x^m, z^r)^{\star_{9.71}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.71}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.71}} = (x, y^n+\frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.71}} = (y^n, z^r)$, $(x^m+az^r, y^n+bz^r)^{\star_{9.71}} = (x^m+ay^n, z^r)^{\star_{9.71}} = (x^m + az^r, y^n)^{\star_{9.71}} = (x^m, y^n, z^r)^{\star_{9.71}} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{9.71}} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1330. $\star_{9.72}$:$(0)^{\star_{9.72}} = (0)$, $(x^m)^{\star_{9.72}} = (x^m)$, $(y^n)^{\star_{9.72}} = (y^n)$,$(z^r)^{\star_{9.72}} = (z^r)$, $(x^m + ay^n)^{\star_{9.72}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{9.72}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{9.72}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{9.72}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.72}} = (x^m, y^n + az^r)^{\star_{9.72}} = (x, y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.72}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.72}} = (x^m + az^r, y^n + bz^r)^{\star_{9.72}} = (x^m + ay^n, z^r)^{\star_{9.72}} = (x^m + az^r, y^n)^{\star_{9.72}} = (x^m, y^n, z^r)^{\star_{9.72}} = (x, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1331. $\star_{9.73}$:$(0)^{\star_{9.73}} = (0)$, $(x^m)^{\star_{9.73}} = (x^m)$, $(y^n)^{\star_{9.73}} = (y^n)$,$(z^r)^{\star_{9.73}} = (z^r)$, $(x^m + ay^n)^{\star_{9.73}} = (x^m + ay^n)$, $(x^m, y^n)^{\star_{9.73}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{9.73}} = (x^m + az^r)$, $(x^m, z^r)^{\star_{9.73}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{9.73}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{9.73}} = (x, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{9.73}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{9.73}} = (x^m + ay^n, z^r)^{\star_{9.73}} = (x^m + az^r, y^n)^{\star_{9.73}} = (x^m, y^n, z^r)^{\star_{9.73}} = (x, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{9.73}} = (x, y^n + az^r)$ $\forall\ m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1332. $\star_{9.74}$ through $\star_{9.147}$ see proof

1406. $\star_{9.148}$ through $\star_{9.221}$ see proof

1480. $\star_{10}$:$(0)^{\star_{10}} = (0)$, $(x^m)^{\star_{10}} = (x^m)$, $(y^n)^{\star_{10}} = (y^n)$, $(z^r)^{\star_{10}} = (z^r)$, $(x^m + ay^n)^{\star_{10}} = (x^m, y^n)^{\star_{10}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10}} = (x^m, z^r)^{\star_{10}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10}} = (y^n, z^r)^{\star_{10}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{10}} = (x^m + az^r, y^n + bz^r)^{\star_{10}} = (x^m, y^n + az^r)^{\star_{10}} = (x^m + az^r, y^n)^{\star_{10}} = (x^m + ay^n, z^r)^{\star_{10}} = (x^m, y^n, z^r)^{\star_{10}} = (x^m, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1481. $\star_{10.1}$:$(0)^{\star_{10.1}} = (0)$, $(x^m)^{\star_{10.1}} = (x^m)$, $(y^n)^{\star_{10.1}} = (y^n)$, $(z^r)^{\star_{10.1}} = (z^r)$, $(x^m + ay^n)^{\star_{10.1}} = (x^m, y^n)^{\star_{10.1}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.1}} = (x^m, z^r)^{\star_{10.1}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10.1}} = (y^n, z^r)^{\star_{10.1}} = (y^n, z^r)$, $(x^m + ay^n + bz^r)^{\star_{10.1}} = (x^m + ay^n + bz^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.1}} = (x^m, y^n + az^r)^{\star_{10.1}} = (x^m + az^r, y^n)^{\star_{10.1}} = (x^m + ay^n, z^r)^{\star_{10.1}} = (x^m, y^n, z^r)^{\star_{10.1}} = (x^m, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1482. $\star_{10.2}$:$(0)^{\star_{10.2}} = (0)$, $(x^m)^{\star_{10.2}} = (x^m)$, $(y^n)^{\star_{10.2}} = (y^n)$, $(z^r)^{\star_{10.2}} = (z^r)$, $(x^m + ay^n)^{\star_{10.2}} = (x^m, y^n)^{\star_{10.2}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.2}} = (x^m, z^r)^{\star_{10.2}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10.2}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.2}} = (x^m + az^r, y^n + bz^r)^{\star_{10.2}} = (x^m, y^n + az^r)^{\star_{10.2}} = (x^m + az^r, y^n)^{\star_{10.2}} = (x^m + ay^n, z^r)^{\star_{10.2}} = (x^m, y^n, z^r)^{\star_{10.2}} = (x^m, y^n, z^r)$, $(y^n, z^r)^{\star_{10.2}} = (y^n, z^r)$, $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1483. $\star_{10.3}$:$(0)^{\star_{10.3}} = (0)$, $(x^m)^{\star_{10.3}} = (x^m)$, $(y^n)^{\star_{10.3}} = (y^n)$, $(z^r)^{\star_{10.3}} = (z^r)$, $(x^m + ay^n)^{\star_{10.3}} = (x^m, y^n)^{\star_{10.3}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.3}} = (x^m, z^r)^{\star_{10.3}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10.3}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.3}} = (x^m + ay^n + bz^r)$, $(y^n, z^r)^{\star_{10.3}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.3}} = (x^m, y^n + az^r)^{\star_{10.3}} = (x^m + az^r, y^n)^{\star_{10.3}} = (x^m + ay^n, z^r)^{\star_{10.3}} = (x^m, y^n, z^r)^{\star_{10.3}} = (x^m, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1484. $\star_{10.4}$:$(0)^{\star_{10.4}} = (0)$, $(x^m)^{\star_{10.4}} = (x^m)$, $(y^n)^{\star_{10.4}} = (y^n)$, $(z^r)^{\star_{10.4}} = (z^r)$, $(x^m + ay^n)^{\star_{10.4}} = (x^m, y^n)^{\star_{10.4}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.4}} = (x^m, z^r)^{\star_{10.4}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10.4}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.4}} = (x^m, y^n + \frac{b}{a}z^r)$, $(y^n, z^r)^{\star_{10.4}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.4}} = (x^m, y^n + az^r)^{\star_{10.4}} =$

$(x^m + az^r, y^n)^{\star_{10.4}} = (x^m + ay^n, z^r)^{\star_{10.4}} = (x^m, y^n, z^r)^{\star_{10.4}} = (x^m, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1485. $\star_{10.5}$:$(0)^{\star_{10.5}} = (0)$, $(x^m)^{\star_{10.5}} = (x^m)$, $(y^n)^{\star_{10.5}} = (y^n)$, $(z^r)^{\star_{10.5}} = (z^r)$, $(x^m + ay^n)^{\star_{10.5}} = (x^m, y^n)^{\star_{10.5}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.5}} = (x^m, z^r)^{\star_{10.5}} = (x^m, z^r)$, $(y^n + az^r)^{\star_{10.5}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.5}} = (x^m + ay^n + bz^r)$, $(y^n, z^r)^{\star_{10.5}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.5}} = (x^m, y^n + az^r)^{\star_{10.5}} = (x^m + az^r, y^n)^{\star_{10.5}} = (x^m + ay^n, z^r)^{\star_{10.5}} = (x^m, y^n, z^r)^{\star_{10.5}} = (x^m, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1486. $\star_{10.6}$ through $\star_{10.9}$ see proof

1490. $\star_{10.10}$ through $\star_{10.13}$ see proof

1494. $\star_{10.14}$:$(0)^{\star_{10.14}} = (0)$, $(x^m)^{\star_{10.14}} = (x^m)$, $(y^n)^{\star_{10.14}} = (y^n)$, $(z^r)^{\star_{10.14}} = (z^r)$, $(x^m + ay^n)^{\star_{10.14}} = (x^m, y^n)^{\star_{10.14}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.14}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.14}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.14}} = (x^m + az^r, y^n + bz^r)^{\star_{10.14}} = (x^m, y^n + az^r)^{\star_{10.14}} = (x^m + az^r, y^n)^{\star_{10.14}} = (x^m + ay^n, z^r)^{\star_{10.14}} = (x^m, y^n, z^r)^{\star_{10.14}} = (x^m, y^n, z^r)$, $(x^m, z^r)^{\star_{10.14}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.14}} = (y^n, z^r)$, $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1495. $\star_{10.15}$:$(0)^{\star_{10.15}} = (0)$, $(x^m)^{\star_{10.15}} = (x^m)$, $(y^n)^{\star_{10.15}} = (y^n)$, $(z^r)^{\star_{10.15}} = (z^r)$, $(x^m + ay^n)^{\star_{10.15}} = (x^m, y^n)^{\star_{10.15}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.15}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.15}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.15}} = (x^m + bz^r, y^n)$, $(x^m, z^r)^{\star_{10.15}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.15}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.15}} = (x^m, y^n + az^r)^{\star_{10.15}} = (x^m + ay^n, z^r)^{\star_{10.15}} = (x^m, y^n, z^r)^{\star_{10.15}} = (x^m, y^n, z^r)$, $(x^m + az^r, y^n)^{\star_{10.15}} = (x^m + az^r, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1496. $\star_{10.16}$:$(0)^{\star_{10.16}} = (0)$, $(x^m)^{\star_{10.16}} = (x^m)$, $(y^n)^{\star_{10.16}} = (y^n)$, $(z^r)^{\star_{10.16}} = (z^r)$, $(x^m + ay^n)^{\star_{10.16}} = (x^m, y^n)^{\star_{10.16}} = (x^m, y^n)$, $(x^m + az^r)^{\star_{10.16}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.16}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.16}} = (x^m, y^n + \frac{b}{a}z^r)$, $(x^m, z^r)^{\star_{10.16}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.16}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.16}} = (x^m + ay^n, z^r)^{\star_{10.16}} = (x^m + az^r, y^n)^{\star_{10.16}} = (x^m, y^n, z^r)^{\star_{10.16}} = (x^m, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{10.16}} = (x^m, y^n + az^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1497. $\star_{10.17}$ through $\star_{10.19}$ see proof

1500. $\star_{10.20}$ through $\star_{10.22}$ see proof

1503. $\star_{10.23}$:$(0)^{\star_{10.23}} = (0)$, $(x^m)^{\star_{10.23}} = (x^m)$, $(y^n)^{\star_{10.23}} = (y^n)$, $(z^r)^{\star_{10.23}} = (z^r)$, $(x^m + ay^n)^{\star_{10.23}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{10.23}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.23}} = (y^n+az^r)$, $(x^m+ay^n+bz^r)^{\star_{10.23}} = (x^m+az^r, y^n+bz^r)^{\star_{10.23}} = (x^m, y^n+az^r)^{\star_{10.23}} = (x^m + az^r, y^n)^{\star_{10.23}} = (x^m + ay^n, z^r)^{\star_{10.23}} = (x^m, y^n, z^r)^{\star_{10.23}} = (x^m, y^n, z^r)$, $(x^m, y^n)^{\star_{10.23}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.23}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.23}} = (y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1504. $\star_{10.24}$:$(0)^{\star_{10.24}} = (0)$, $(x^m)^{\star_{10.24}} = (x^m)$, $(y^n)^{\star_{10.24}} = (y^n)$, $(z^r)^{\star_{10.24}} = (z^r)$, $(x^m + ay^n)^{\star_{10.24}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{10.24}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.24}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.24}} = (x^m, y^n + \frac{b}{a}z^r)$, $(x^m, y^n)^{\star_{10.24}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.24}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.24}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.24}} = (x^m + az^r, y^n)^{\star_{10.24}} = (x^m + ay^n, z^r)^{\star_{10.24}} = (x^m, y^n, z^r)^{\star_{10.24}} = (x^m, y^n, z^r)$, $(x^m, y^n + az^r)^{\star_{10.24}} = (x^m, y^n + az^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1505. $\star_{10.25}$:$(0)^{\star_{10.25}} = (0)$, $(x^m)^{\star_{10.25}} = (x^m)$, $(y^n)^{\star_{10.25}} = (y^n)$, $(z^r)^{\star_{10.25}} = (z^r)$, $(x^m + ay^n)^{\star_{10.25}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{10.25}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.25}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.25}} = (x^m + bz^r, y^n)$, $(x^m, y^n)^{\star_{10.25}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.25}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.25}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.25}} = (x^m, y^n + az^r)^{\star_{10.25}} = (x^m + ay^n, z^r)^{\star_{10.25}} = (x^m, y^n, z^r)^{\star_{10.25}} = (x^m, y^n, z^r)$, $(x^m + az^r, y^n)^{\star_{10.25}} = (x^m + az^r, y^n)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1506. $\star_{10.26}$:$(0)^{\star_{10.26}} = (0)$, $(x^m)^{\star_{10.26}} = (x^m)$, $(y^n)^{\star_{10.26}} = (y^n)$, $(z^r)^{\star_{10.26}} = (z^r)$, $(x^m + ay^n)^{\star_{10.26}} = (x^m + ay^n)$, $(x^m + az^r)^{\star_{10.26}} = (x^m + az^r)$, $(y^n + az^r)^{\star_{10.26}} = (y^n + az^r)$, $(x^m + ay^n + bz^r)^{\star_{10.26}} = (x^m + ay^n, z^r)^{\star_{10.26}} = (x^m + ay^n, z^r)$, $(x^m, y^n)^{\star_{10.26}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.26}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.26}} = (y^n, z^r)$, $(x^m + az^r, y^n + bz^r)^{\star_{10.26}} = (x^m, y^n + az^r)^{\star_{10.26}} = (x^m + ay^n, z^r)^{\star_{10.26}} = (x^m, y^n, z^r)^{\star_{10.26}} = (x^m, y^n, z^r)$ $\forall$ $m, n, r \in \mathbb{N}$ and $a, b \in k^\times$

1507. $\star_{10.27}$: see proof

1508. $\star_{10.28}$ through $\star_{10.31}$ see proof

1512. $\star_{10.32}$ through $\star_{10.37}$ see proof

1518. $\star_{10.38}$ through $\star_{10.41}$ see proof

1522. $\star_{10.42}$:$I^{\star_{10.42}}$ $\forall$ ideals I

Appendix B

Standard closure operations from Theorem 3.4.4 for $k = \mathbb{Z}/2\mathbb{Z}$

1. $\star_{10.24b}$:$(0)^{\star_{10.24b}} = (0)$, $(x^m)^{\star_{10.24b}} = (x^m)$, $(y^n)^{\star_{10.24b}} = (y^n)$, $(z^r)^{\star_{10.24b}} = (z^r)$, $(x^m+y^n)^{\star_{10.24b}} = (x^m+y^n)$, $(x^m+z^r)^{\star_{10.24b}} = (x^m+z^r)$, $(y^n+z^r)^{\star_{10.24b}} = (y^n+z^r)$, $(x^m + y^n + z^r)^{\star_{10.24b}} = (x^m, y^n + z^r)^{\star_{10.24b}} = (x^m, y^n + z^r)$, $(x^m, y^n)^{\star_{10.24b}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.24b}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.24b}} = (y^n, z^r)$,
$(x^m + z^r, y^n + z^r)^{\star_{10.24b}} = (x^m + z^r, y^n + z^r)$, $(x^m + z^r, y^n)^{\star_{10.24b}} =$
$(x^m + y^n, z^r)^{\star_{10.24b}} = (x^m, y^n, z^r)^{\star_{10.24b}} = (x^m, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$
2. $\star_{10.25b}$:$(0)^{\star_{10.25b}} = (0)$, $(x^m)^{\star_{10.25b}} = (x^m)$, $(y^n)^{\star_{10.25b}} = (y^n)$, $(z^r)^{\star_{10.25b}} = (z^r)$, $(x^m+y^n)^{\star_{10.25b}} = (x^m+y^n)$, $(x^m+z^r)^{\star_{10.25b}} = (x^m+z^r)$, $(y^n+z^r)^{\star_{10.25b}} = (y^n+z^r)$, $(x^m + y^n + z^r)^{\star_{10.25b}} = (x^m + z^r, y^n)^{\star_{10.25b}} = (x^m + z^r, y^n)$, $(x^m, y^n)^{\star_{10.25b}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.25b}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.25b}} = (y^n, z^r)$,
$(x^m+z^r, y^n+z^r)^{\star_{10.25b}} = (x^m+z^r, y^n+z^r)$, $(x^m, y^n+z^r)^{\star_{10.25b}} = (x^m+y^n, z^r)^{\star_{10.25b}}$
$= (x^m, y^n, z^r)^{\star_{10.25b}} = (x^m, y^n, z^r)$ $\forall\, m, n, r \in \mathbb{N}$
3. $\star_{10.26b}$:$(0)^{\star_{10.26b}} = (0)$, $(x^m)^{\star_{10.26b}} = (x^m)$, $(y^n)^{\star_{10.26b}} = (y^n)$, $(z^r)^{\star_{10.26b}} = (z^r)$,

$(x^m+y^n)^{\star_{10.26b}} = (x^m+y^n)$, $(x^m+z^r)^{\star_{10.26b}} = (x^m+z^r)$, $(y^n+z^r)^{\star_{10.26b}} = (y^n+z^r)$, $(x^m + y^n + z^r)^{\star_{10.26b}} = (x^m + y^n, z^r)^{\star_{10.26b}} = (x^m + y^n, z^r)$, $(x^m, y^n)^{\star_{10.26b}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{10.26b}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{10.26b}} = (y^n, z^r)$,
$(x^m + z^r, y^n + z^r)^{\star_{10.26b}} = (x^m + z^r, y^n + z^r)$, $(x^m, y^n + z^r)^{\star_{10.26b}} = (x^m + y^n, z^r)^{\star_{10.26b}} = (x^m, y^n, z^r)^{\star_{10.26b}} = (x^m, y^n, z^r)$ $\forall\ m, n, r \in \mathbb{N}$

Appendix C

Standard closure operations from Theorem 3.4.4 for $k = \mathbb{Z}/3\mathbb{Z}$

1. $\star_{\mathbb{Z}/3\mathbb{Z}}$:$(0)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (0)$, $(x^m)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m)$, $(y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (y^n)$, $(z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (z^r)$, $(x^m + y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + y^n)$, $(x^m + 2y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2y^n)$, $(x^m + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r)$, $(x^m + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r)$, $(y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (y^n + z^r)$, $(y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (y^n + 2z^r)$, $(x^m + y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r, y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r, y^n + 2z^r)$, $(x^m + y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r, y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r, y^n + z^r)$, $(x^m + 2y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r, y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r, y^n + z^r)$, $(x^m + 2y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r, y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r, y^n + 2z^r)$, $(x^m, y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m, y^n)$, $(x^m, z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m, z^r)$, $(y^n, z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (y^n, z^r)$, $(x^m, y^n + z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m, y^n + 2z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + z^r, y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2z^r, y^n)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + y^n, z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m + 2y^n, z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m, y^n, z^r)^{\star_{\mathbb{Z}/3\mathbb{Z}}} = (x^m, y^n, z^r)$ for all $m, n, r \in \mathbb{N}$

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