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Lifts of Frobenius on Arithmetic Jet Spaces of Schemes

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BY

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Abstract

According to [Bu05], lifts of Frobenius on formal schemes X over the p-adic completion of the maximal unramified extension of the p-adic integers, $R := \widehat{\mathbb{Z}_p^{ur}}$, may be viewed as arithmetic analogues of vector fields on manifolds. In particular, vector fields on the tangent bundle of a manifold, appearing for instance in Hamiltonian mechanics, have as arithmetic analogues lifts of Frobenius on arithmetic jet spaces $J^1(X)$ of schemes (cf. [BM13]).

In this thesis, we first consider the projective space $X = \mathbb{P}_R^m$ and prove that lifts of Frobenius do not exist on the arithmetic jet spaces $J^n(\mathbb{P}_R^m)$ for $n, m \geq 1$. Exhibiting a contrast in the case $n = m = 1$ between the arithmetic and geometric frameworks, we show on the other hand that the space of vector fields on the tangent bundle $T(\mathbb{P}_k^1)$ lifting vector fields on $\mathbb{P}_k^1$, where k is an algebraically closed field, has dimension 6 over k. Nevertheless, "normalized" vector fields, which play a role in Hamiltonian mechanics, do not exist on $T(\mathbb{P}_k^1)$. We proceed to prove a stronger result for the case $n = m = 1$, that there are no effective Cartier divisors on $J^2(\mathbb{P}^1)$ that are finite-to-one over $J^1(\mathbb{P}^1)$, and discover that an analogous result holds in geometry.

As a final result, we prove the nonexistence of lifts of Frobenius on the first jet space of any smooth quadric hypersurface in projective space.

Contents

1. Background

We start by providing an overview of the mathematical objects, concepts, and facts that form the background and foundation for our subsequent results. Our main reference for the material in this section is [Bu05].

Definition 1.1. For a fixed prime $p \in \mathbb{Z}$, $p \neq 2$, let

$$\mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i : a_i \in \mathbb{Z}, 0 \leq a_i \leq p-1\}$$

denote the p-adic integers.

Definition 1.2. For any ring A, define the p-adic completion of A to be

$$\widehat{A} := \varprojlim A/p^n A.$$

A ring A is p-adically complete if $A \cong \widehat{A}$. Then $\widehat{A}$ is itself p-adically complete.

Remark 1.3. Note that $\mathbb{Z}_p = \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Usually, because of the length of the expressions defining our rings, we will write $A\hat{}$ instead of $\widehat{A}$.

Definition 1.4. The ring

$$R := \widehat{\mathbb{Z}_p^{ur}} = \mathbb{Z}_p[\{\zeta_N\} : p \nmid N]\hat{},$$

where ζ_N are primitive Nth roots of unity, is the p-adic completion of the maximal unramified extension of the of the p-adic integers; it the unique local, complete ring with maximal ideal generated by p and residue field $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. In the general theory of "differential calculus with integers" developed by Buium [Bu05], adjoining the roots of unity above gives our subsequent rings more "constants", in a sense to be clarified after Proposition 1.8. In $\mathbb{Z}$ itself, only $0, 1$, and -1 are "constants".

Definition 1.5. Let $u : A \to B$ be a ring homomorphism. Then B has an A-module structure given by $a \cdot b := u(a)b$. A map $\delta : A \to B$ is called a *p-derivation* if $\delta(1) = 0$ and

(i) $\delta(a_1 + a_2) = \delta a_1 + \delta a_2 - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} a_1^i a_2^{p-i} \cdot 1_B$.

(ii) $\delta(a_1 a_2) = a_1^p \delta a_2 + a_2^p \delta a_1 + p(\delta a_1)(\delta a_2)$.

Definition 1.6. A *lift of Frobenius* on a ring A is a ring homomorphism $\phi : A \to A$ such that $\forall a \in A$, $\phi(a) \equiv a^p \mod p$.

Proposition 1.7. If A is p torsion-free, then there is a bijection

$$\{p\text{-derivations } \delta : A \to A\} \simeq \{\text{lifts of Frobenius } \phi : A \to A\}.$$

Proof. Given $\delta : A \to A$ a p-derivation, $\phi : A \to A$ defined by $\phi(a) = a^p + p\delta a$ is a lift of Frobenius (one checks it is a ring homomorphism), and given a lift of Frobenius $\phi : A \to A$, one checks by a computation that $\delta : A \to A$ defined by $\delta a = \frac{\phi(a) - a^p}{p}$ satisfies the properties of a p-derivation. The hypothesis that A is p torsion-free is necessary so that the second map is well-defined. These maps from p-derivations to lifts of Frobenius and vice versa are inverses. □

Proposition 1.8. There is a unique lift of Frobenius ϕ_R on our ring R given by $\phi_R(a) = a$, $a \in \mathbb{Z}_p$, $\phi_R(\zeta_N) = \zeta_N^p$ for all N, $p \nmid N$. Since R is an integral domain, hence p-torsion free, there is thus a unique p-derivation $\delta_R : R \to R$ given by $\delta_R(r) = \frac{\phi_R(r) - r^p}{p}$.

[Note that for $a \in \mathbb{Z}$, $\phi_R(a) = a \equiv a^p \pmod p$ by Fermat's Little Theorem, and this fact extends to $\mathbb{Z}_p$.]

The "constants" of R, alluded to after Definition 1.4, are by definition

$$\begin{aligned}\{a \in R : \delta(a) = 0\} = \{a \in R : \frac{\phi_R(a) - a^p}{p} = 0\} &= \{a \in R : \phi_R(a) = a^p\} \\ &= \{\zeta_N : p \nmid N\} \cup \{0\}.\end{aligned}$$

Definition/Proposition 1.9. Let R be as before. Let $x = \{x_1, \dots, x_m\}$ and $R\{x\} := R[x, x', x'', \dots, x^{(n)}, \dots]$, where $x^{(i)} = \{x_1^{(i)}, \dots, x_m^{(i)}\}$. Let $\phi : R\{x\} \to R\{x\}$ be the ring homomorphism such that $\phi|_R = \phi_R$, $\phi(x_i) = x_i^p + px_i'$, $\phi(x_i') = (x_i')^p + px_i''$, etc. Then ϕ is a lift of Frobenius which hence induces a p-derivation $\delta : R\{x\} \to R\{x\}$ defined by

$$\delta(F) = \frac{\phi(F) - F^p}{p}.$$

Remark 1.10. By construction, we have

$$\delta(x) = \frac{\phi(x) - x^p}{p} = \frac{x^p + px' - x^p}{p} = x'.$$

Likewise, $\delta(x') = x''$, and in general, $\delta(x^{(i)}) = x^{(i+1)}$.

Definition 1.11. Let $A = \frac{R[x_1,\ldots,x_m]}{(f_1,\ldots,f_r)} =: \frac{R[x]}{(f)}$. (Any finitely generated R-algebra is of this form.) For each $n \geq 1$, define

$$J^n(A) := \frac{R[x, x', \ldots, x^{(n)}]\hat{}}{(f, \delta f, \ldots, \delta^n f)}.$$

We let $J^{-1}(A) = R$ and $J^0(A) = A$. We call $J^n(A)$ the *nth p-jet algebra of A*. For each $i \in \mathbb{Z}$, $-1 \leq i < \infty$, we have ring homomorphisms $\hat{\iota}_i : J^i(A) \to J^{i+1}(A)$ induced by inclusion of numerators. Also, the p-derivation $\delta : R\{x\} \to R\{x\}$ of Definition/Proposition 1.9 induces p-derivations $\delta_i : J^i(A) \to J^{i+1}(A)$ for each i by continuity. (A power series that converges p-adically is mapped by the lift of Frobenius ϕ of Definition/Proposition 1.9 to a power series that converges p-adically, since ϕ maps p to itself.)

Remark 1.12. For future computational purposes, it is worth noting what the elements of $J^n(A)$ look like in more down-to-earth terms. It is a fact that can be proven from the definition of p-adic completion and its universal property that for any ring A, $A[x_1, \ldots, x_m]\hat{} =: A[x]\hat{}$ equals

$$\{\sum_\alpha a_\alpha x^\alpha \in \hat{A}\,[[x]] : a_\alpha \to 0 \ p\text{-adically as } |\alpha| \to 0\},$$

where α is a multi-index and $x = \{x_1, \ldots, x_m\}$. So the elements of $J^n(A) := \frac{R[x,x',\ldots,x^{(n)}]\hat{}}{(f,\delta f,\ldots,\delta^n f)}$ are represented by elements of the power series ring $R[[x, x', \ldots, x^{(n)}]]$ such that the monomial summands become more and more divisible by p as their powers increase. This also means that reducing any element of $J^n(A)$ mod p^s for any s results in a polynomial.

Definition 1.13. Let A be as before. Given an affine scheme $X = \text{Spec } A$, define the *nth arithmetic p-jet space of* X by

$$J^n(X) := \text{ Spf } J^n(A) := \text{ Spf } \frac{R[x, x', \ldots, x^{(n)}]\hat{}}{(f, \delta f, \ldots, \delta^n f)}.$$

Given a scheme $X = \bigcup_{\text{finite}} \text{Spec } A_i$ of finite type over R, the *nth arithmetic p-jet space of* X is defined to be

$$J^n(X) := \bigcup J^n(X_i) = \bigcup_{\text{finite}} \text{Spf } (J^n(A_i)).$$

Example 1.14. Let $X = \mathbb{P}^1_R = \text{Spec } R[x] \cup \text{ Spec } R[y]$, glued via $xy = 1$. Then $J^1(X) = \text{Spf } R[x, x']\hat{} \cup \text{ Spf } R[y, y']\hat{}$. We will return to this example in the results.

Remark 1.15. It is not immediate that the definition of jet space is correct. That is, we must be able to glue the $\text{Spf}(J^n(A_i))$'s together after localizing. That we can do this follows from the compatibility of the functor J^n with localization. ([Bu96], p. 352)

Definition 1.16. Let $X = \bigcup \text{ Spf } \widehat{A_i}$ be a formal scheme over R. A *lift of Frobenius on* X is a morphism of formal schemes $\phi : X \to X$ such that the ring maps $\phi_i^* : \widehat{A_i} \to \widehat{A_i}$ are lifts of Frobenius for each i, with $\phi_i^*|_R = \phi_R$ for all i, and for all i, j, $\phi_i|_{Spf\widehat{A_i} \cap Spf\widehat{A_j}} = \phi_j|_{Spf\widehat{A_i} \cap Spf\widehat{A_j}}$. We also say that the induced map of sheaves $\delta : \mathcal{O}_X \to \mathcal{O}_X$ is a *p-derivation on* X.

Definition 1.17. Let k be an algebraically closed field, and let $X = \text{Spec}(\frac{k[x]}{(f)})$ be a smooth affine scheme over k. The *geometric tangent bundle of* X is defined to be $T(X) := \textbf{Spec}\, S(\Omega_{X/k})$, where $\Omega_{X/k}$ is the sheaf of relative differentials of X over k. One can extend this definition to an arbitrary smooth scheme $X = \bigcup_i X_i$ to get

$$T(X) = \bigcup_i \textbf{Spec}\, S(\Omega_{X_i/k}).$$

(See [Ha77], p. 128.)

Remark 1.18. For each $X_i = \text{Spec}(\frac{k[x]}{(f)})$ affine, we have

$$\mathcal{O}(T(X_i)) = \mathcal{O}(\text{Spec}\, S(\Omega_{X_i/k})) = S(\Omega_{X_i/k}) = \frac{k[x, x']}{(f, df)},$$

where $d : \frac{k[x]}{(f)} \to \Omega_{\frac{k[x]}{(f)}/k}$ and $x' := dx$. Also, it is a fact that k-derivations $\mathcal{O}_X \to \mathcal{O}_X$ are in bijection with sections $s : X \to T(X)$ of the projection $\pi : T(X) \to X$, i.e., maps s such that $\pi \circ s = id_X$, which are by definition the vector fields on X. Similarly, in differential geometry, given a smooth manifold M, derivations on the ring of smooth functions $C^\infty(M)$ are in bijection with smooth sections $s : M \to TM$, i.e., the smooth vector fields on M. It turns out that lifts of Frobenius (equivalently, p-derivations) on a p-adic formal scheme X are in bijection with sections $s : X \to J^1(X)$. We will conclude this as a consequence of the following universal property.

Proposition 1.19. ([Bu96], p. 352) Let R be as above, and let A be as in Definition 1.11. For each $i \in \mathbb{Z}$, $-1 \leq i < \infty$, let $\hat{\iota}_i : J^i(A) \to J^{i+1}(A)$ be the "structure" ring homomorphisms and $\delta_i : J^i(A) \to J^{i+1}(A)$ the p-derivations also described in Definition 1.11. For any R-algebra homomorphism $g : J^{n-1}(A) \to C$ into a p-adically complete ring C and for any p-derivation $\partial : J^{n-1}(A) \to C$ such that $\partial \circ \hat{\imath}^{n-2} = g \circ \delta_{n-2}$, there exists a unique R-algebra homomorphism $h : J^n(A) \to C$ such that $h \circ \hat{\imath}^{n-1} = g$ and $h \circ \delta_{n-1} = \partial$:

$$\begin{tikzcd}
\cdots \arrow[r, "\hat{\iota}_{n-3}", "\delta_{n-3}"'] & J^{n-2}(A) \arrow[r, "\hat{\iota}_{n-2}", "\delta_{n-2}"'] & J^{n-1}(A) \arrow[r, "\hat{\iota}_{n-1}", "\delta_{n-1}"'] \arrow[dr, "g", "\partial"'] & J^n(A) \arrow[d, dotted, "h"] \\
 & & & C
\end{tikzcd} .$$

(Here the notation $\xrightarrow[\delta_i]{\hat{\iota}_i}$ is shorthand for two maps $J^i(A) \xrightarrow{\hat{\iota}_i} J^{i+1}(A)$ and $J^i(A) \xrightarrow{\delta_i} J^{i+1}(A)$.)

Proof. Define $h(r) = g(r)$ for all $r \in R$; $h(x^{(i)}) = g(x^{(i)})$ for $i \neq n$; and

$$h(x^{(n)}) = \partial(x^{(n-1)}).$$

(For notational simplicity, we leave off equivalence class symbols in the proof, though all elements are understood to be classes in their respective quotients.) Note that

$$h \circ \hat{\iota}^{n-1}(r) = h(r) = g(r)$$

for $r \in R$ and

$$h \circ \hat{\iota}^{n-1}(x^{(i)}) = h(x^{(i)}) = g(x^{(i)})$$

for $i \leq n-1$, so that $h \circ \hat{\imath}^{n-1} = g$. Similarly, for all $r \in R$ we have

$$h \circ \delta_{n-1}(r) = h \circ \delta_R(r) = h \circ \hat{\imath}^{n-1} \circ \delta_R(r) = g \circ \delta_R(r) = g \circ \delta_{n-2}(r) = \partial \circ \hat{\imath}^{n-2}(r) = \partial(r);$$

for $i \leq n-2$ we have

$$h \circ \delta_{n-1}(x^{(i)}) = h(x^{(i+1)}) = g(x^{(i+1)}) = g \circ \delta_{n-2}(x^{(i)}) = \partial \circ \hat{\imath}^{n-2}(x^{(i)}) = \partial(x^{(i)});$$

and

$$h \circ \delta_{n-1}(x^{(n-1)}) = h(x^{(n)}) = \partial(x^{(n-1)}),$$

which shows that $h \circ \delta_{n-1} = \partial$. Since C is p-adically complete, we get that for any $P = \sum_\alpha a_\alpha x^{\alpha_0} (x')^{\alpha_1} \cdots (x^{(n)})^{\alpha_n} \in J^n(A)$, $h(P) = \sum_\alpha h(a_\alpha) h(x)^{\alpha_0} h(x')^{\alpha_1} \cdots h(x^{(n)})^{\alpha_n}$

$$= \sum_\alpha a_\alpha g(x)^{\alpha_0} g(x')^{\alpha_1} \cdots g(x^{(n-1)})^{\alpha_n} \partial(x^{(n-1)})^{\alpha_n} \in C,$$

since $a_\alpha \to 0$ p-adically as $\alpha \to \infty$. (See Remark 1.12.)

To see uniqueness, suppose there exists $\tilde{h}$ with the same properties. We have $\tilde{h}(r) = \tilde{h} \circ \hat{\imath}^{n-1}(r) = g(r) = h(r)$; for $i \leq n-1$, $\tilde{h}(x^{(i)}) = \tilde{h} \circ \hat{\imath}^{n-1}(x^{(i)}) = g(x^{(i)}) = h(x^{(i)})$; and $\tilde{h}(x^{(n)}) = \tilde{h} \circ \delta_{n-1}(x^{(n-1)}) = \partial(x^{(n-1)}) = h(x^{(n)})$. Thus, $\tilde{h} = h$, proving uniqueness and concluding the proof.

□

Corollary 1.20. There exists a natural bijection between the p-derivations on a formal scheme X and the sections of the projection $\pi : J^1(X) \to X$, that is, the morphisms of formal schemes $s : X \to J^1(X)$ such that $\pi \circ s = id_X$.

Proof. Write $X = \bigcup \operatorname{Spf} \widehat{A_i}$. Let $\partial : X \to X$ be a p-derivation defined locally by p-derivations $\partial_i : \widehat{A_i} \to \widehat{A_i}$. We also have the identity map $id : X \to X$ given locally by identity maps $id_i : \widehat{A_i} \to \widehat{A_i}$. The projection $\pi : J^1(X) \to X$ is given locally on rings by $\hat{\imath}_{0,i} : \widehat{A_i} \to J^1(A_i)$. This gives the following diagram:

$$\begin{tikzcd} R \arrow[r, "\hat{\imath}_{-1,i}", "\delta_R"'] & \widehat{A_i} \arrow[r, "\hat{\imath}_{0,i}", "\delta"'] \arrow[dr, "id_i", "\partial_i"'] & J^1(A_i) \arrow[d, dotted, "s_i^*"] \\ & & \widehat{A_i} \end{tikzcd} .$$

One checks that the hypotheses of the universal property Proposition 1.19 are satisfied. To give a section $s : X \to J^1(X)$ is equivalent to giving ring maps $s_i^* : J^1(A_i) \to \widehat{A_i}$ such that $s_i^* \circ \hat{\imath}_{0,i} = id_i$ that glue. The universal property gives morphisms $s_i^* : J^1(A_i) \to \widehat{A_i}$, with $s_i^* \circ \hat{\imath}_{0,i} = id_i$. One gets glueability from the fact that the δ_i's and id_i's glue by assumption. The fact that this map from p-derivations to sections is a bijection follows because the section property requires that $s_i^*|_{\widehat{A_i}} = id_i$, so that any s_i^* is uniquely determined by the value $s_i^*(x_i')$ (where $\widehat{A_i} = \frac{R[x_i]^\wedge}{(f_i)}$), and we can take $\partial_i(x_i') = s_i^*(x_i')$. $\square$

Definition 1.21. Let $X = \text{Spf } \frac{R[x]^\wedge}{(f)}$ be an affine p-adic formal scheme. A *normalized p-derivation* $\mathscr{D} : J^1(X) \to J^1(X)$ is a p-derivation such that

$$\mathscr{D}(x) = \delta(x) := x'.$$

For X a non-affine formal scheme, a normalized p-derivation $\mathscr{D} : J^1(X) \to J^1(X)$ is a p-derivation on $J^1(X)$ (see Definition 1.16) such that on each affine piece, $\mathscr{D}$ is normalized. We call the corresponding lift of Frobenius on $J^1(X)$ a normalized lift of Frobenius. Similarly, for an affine scheme $X = \text{Spec } \frac{k[x]^\wedge}{(f)}$ with tangent bundle $T(X) = \text{Spec } \frac{k[x,x']^\wedge}{(f,df)}$ (recall the notation of Remark 1.18), we call a derivation $D : \frac{k[x,x']^\wedge}{(f,df)} \to \frac{k[x,x']^\wedge}{(f,df)}$ *normalized* if

$$D(x) = x' := dx,$$

and for a non-affine scheme X, we call $D : \mathcal{O}(T(X)) \to \mathcal{O}(T(X))$ normalized if it is normalized on each affine piece.

Definition 1.22. Let $X = \bigcup_i \text{Spf } \widehat{A_i}$ be a formal scheme over R. An *effective Cartier divisor on* X is given by an open cover $\{U_i\}$ of X and a collection of functions $f_i \in \mathcal{O}(U_i)$ such that for each i, j, there exists $u_{ij} \in \mathcal{O}(U_i \cap U_j)^\times$ such that $f_i = u_{ij} f_j$. Such a divisor defines a formal subscheme $\bigcup \text{Spf } \frac{\widehat{A_i}}{(f_i)}$ of X. An effective Cartier divisor on a scheme is defined in a similar way.

Definition 1.23. A morphism of formal schemes $\psi : X \to Y$ is *finite* if it has the form $\psi : X = \bigcup \text{Spf } \widehat{A_i} \to Y = \bigcup \text{Spf } \widehat{B_i}$, with $\widehat{A_i}$ a finitely-generated $\widehat{B_i}$-module for each i.

In the statement of the following proposition we use the fact ([BuSa14], p. 680, also shown in the proof) that there is a natural closed immersion $J^2(X) \subset J^1(J^1(X))$.

Proposition 1.24. A section $s : J^1(X) \to J^1(J^1(X))$ corresponds to a normalized p-derivation $\mathscr{D} : J^1(X) \to J^1(X)$ if and only if $\mathrm{Im}(s) \subset J^2(X) \subset J^1(J^1(X))$.

Remark 1.25. Note that if a section s as above satisfies $\mathrm{Im}(s) \subset J^2(X)$ and X is a smooth curve, then $\mathrm{Im}(s)$ is a Cartier divisor on $J^2(X)$ finite over $J^1(X)$.

Proof. (of Proposition 1.24) For simplicity, we look at the case where $X = \mathbb{A}^1 = \mathrm{Spf}\,(R[x]\hat{\ })$; the general case is proved by reduction to this case via étale coordinates. We have $J^1(X) = \mathrm{Spf}\,R[x,\delta x]\hat{\ }$ and $J^1(J^1(X)) = \mathrm{Spf}\,R[x,\delta x,\delta_1 x,\delta_1(\delta x)]\hat{\ }$, where $\delta_1 : J^1(R[x]\hat{\ }) \to J^1(J^1(R[x]\hat{\ })) = J^1(R[x,\delta x]\hat{\ })$ is the p-derivation obtained as in Definition/Proposition 1.9 and Definition 1.11 (note $\delta_1 \neq \delta : J^1(R[x]\hat{\ }) \to J^2(R[x]\hat{\ })$. We have the following commutative diagram relating $\mathscr{D}$ and s^*, by Corollary 1.20:

$$\begin{tikzcd}
R[x,\delta x]\hat{\ } \arrow[r, "\iota", "\delta_1"'] \arrow[dr, "id", "\mathscr{D}"'] & R[x,\delta x,\delta_1 x,\delta_1(\delta x)]\hat{\ } \arrow[d, dotted, "s^*"] \\
& R[x,\delta x]\hat{\ }
\end{tikzcd},$$

with $s^*(x) = x$, $s^*(\delta x) = \delta x$, $s^*(\delta_1 x) = \mathscr{D}(x)$, $s^*(\delta_1 \delta x) = \mathscr{D}(\delta x)$. Observe that $\mathscr{D}$ being normalized is equivalent to

$$\ker s^* \supset (\delta x - \delta_1 x)$$

since

$$s^*(\delta x - \delta_1 x) = \delta x - \mathscr{D}(x) = 0 \Leftrightarrow \mathscr{D}(x) = \delta x.$$

We next claim that

$$J^2(R[x]\hat{\ }) \cong \frac{R[x,\delta x,\delta_1 x,\delta_1(\delta x)]\hat{\ }}{(\delta x - \delta_1 x)}.$$

From Proposition 1.19, we get the following diagram:

$$\begin{tikzcd}
R[x,\delta x]\hat{\ } \arrow[r, "\iota", "\delta_1"'] \arrow[dr, "id", "\delta"'] & R[x,\delta x,\delta_1 x,\delta_1(\delta x)]\hat{\ } \arrow[d, dotted, "h"] \\
& R[x,\delta x,\delta^2 x]\hat{\ }
\end{tikzcd},$$

where $h(x) = x$, $h(\delta x) = \delta x$, $h(\delta_1 x) = \delta x$, and $h(\delta_1 \delta x) = \delta^2(x)$. This map is surjective and its kernel is $(\delta x - \delta_1 x)$, so

$$\frac{R[x, \delta x, \delta_1 x, \delta_1(\delta x)]\hat{}}{(\delta x - \delta_1 x)} \cong R[x, \delta x, \delta^2 x]\hat{} = J^2(R[x]\hat{}).$$

Now, $\mathrm{Im} s \subset J^2(X) \cong \mathrm{Spf}\ \left(\frac{R[x,\delta x,\delta_1 x,\delta_1(\delta x)]\hat{}}{(\delta x-\delta_1 x)}\right)$ is equivalent to $\ker s^* \supset (\delta x - \delta_1 x)$ (which, as stated before, is equivalent to $\mathscr{D}$ being normalized) by the following fact: given A, B, integral domains and a map $s : \mathrm{Spec} A \to \mathrm{Spec}\ B$ induced by a ring map $s^* : B \to A$, we have that $\mathrm{Im} s \subset \mathrm{Spec} B/I \Leftrightarrow \ker s^* \supset I$.

□

In the appendix, we explain the significance in the context of arithmetic differential equations of a p-derivation (equivalently, lift of Frobenius) being normalized.

We now state and prove the quotient rule for elements in any ring A for which there is a p-derivation $\delta : A \to B$; we will use this in the proof of Theorem B.

Proposition 1.26. Let $\delta : A \to B$ be a p-derivation and $\phi : A \to B$ the corresponding lift of Frobenius. Given $f \in A$, $g \in A^\times$, we have

$$\delta(f/g) = \frac{g^p \delta f - f^p \delta g}{g^p \phi(g)}.$$

Proof. We have

$$\delta(f) = \delta\big(\frac{f}{g} \cdot g\big) = (f/g)^p \delta g + g^p \delta(f/g) + p\delta(f/g)\delta g,$$

hence

$$\delta(f/g)(g^p + p\delta g) = \delta f - (f/g)^p \delta g,$$

so

$$\begin{aligned} \delta(f/g) &= \frac{\delta f - (f/g)^p \delta g}{g^p + p\delta g} \\ &= \frac{g^p \delta f - f^p \delta g}{g^p(g^p + p\delta g)} = \frac{g^p \delta f - f^p \delta g}{g^p \phi(g)}. \end{aligned}$$

□

Definition 1.27. *Projective m-space over R* is defined to be

$$\mathbb{P}^m_R := \bigcup_{i=0}^{m} \text{Spec } R[y_{i0}, \ldots, y_{im}],$$

where $y_{ij} = \frac{x_j}{x_i}$. In other words, we take $m+1$ affine n-dimensional planes and glue any pair Spec $(R[y_{i0}, \ldots, y_{in}])_{y_{ij}}$ and Spec $(R[y_{j0}, \ldots, y_{jn}])_{y_{ji}}$ via the identity isomorphism. We obtain a corresponding p-adic formal scheme

$$\widehat{\mathbb{P}^m_R} := \bigcup_{i=0}^{m} \text{Spf } R[y_{i0}, \ldots, y_{im}]\hat{},$$

and the nth jet space of $\mathbb{P}^m$, $J^n(\mathbb{P}^m)$, is by definition

$$J^n(\mathbb{P}^m) := \bigcup_{i=0}^{m} \text{Spf } R[y_{i0}, \ldots, y_{im}, y'_{i0}, \ldots, y'_{im}, \ldots, y^{(n)}_{i0}, \ldots, y^{(n)}_{im}]\hat{}.$$

Definition 1.28. A *smooth quadric* $Q \subset \mathbb{P}^m_R$ is a hypersurface

$$Q = \bigcup_{k=0}^{m} \text{Spec } \frac{R[y_{k0}, \ldots, y_{km}]}{(\sum_{i,j} a_{ij} y_{ki} y_{kj})},$$

such that $A = (a_{ij}) \in GL_{m+1}(R)$ and is symmetric.

Definition 1.29. Fix the notation of Remark 1.18. Given a scheme $X = \bigcup_i \text{Spec } A_i$, where each $A_i = \frac{k[x_i]}{(f_i)}$ and gluing on each pair i, j is given by $x_i \mapsto g_j(x_j)$ for some $g_j \in A_j$, the *geometric jet space* $J^2(X)$ is defined to be

$$\bigcup_i \text{Spec } \frac{k[x_i, x'_i, x''_i]}{(f_i, df_i, d^2 f_i)},$$

glued via $x_i \mapsto g_j(x_j)$, $x'_i \mapsto d(g_j(x_j))$, $x''_i \mapsto d^2(g_j(x_j))$, where

$$d : k[x, x', x'', \ldots] \to k[x, x', x'', \ldots]$$

is the derivation such that $dx = x'$, $dx' = x''$, etc.

2. Results

We now state our results; we give the proofs in section 3. Before we begin, we recall the analogy between the first jet space of a p-adic formal scheme, $J^1(X)$, and the tangent bundle of a manifold, TM, and between lifts of Frobenius on p-adic formal schemes and vector fields on a manifold. We also recall the analogy between the p-jet space $J^2(X)$ and the geometric jet space $J^2(X)$. (See Definitions 1.17 and 1.29 and Remark 1.18.) Because of this, we state corresponding results on $T(\mathbb{P}^1)$ and the geometric $J^2(\mathbb{P}^1)$ and point out where results carry over and where they do not.

We include the first three propositions in this section for completeness of our discussion. Proposition 2.1 is an immediate consequence of Grothendieck's existence theorem in formal geometry. [Gr61]

Proposition 2.1. The lifts of Frobenius on $\widehat{\mathbb{P}^n_R}$ are given on the x_i's by $\phi(x_i) = x_i^p + pF_i := G_i$, $i = 0, 1, \ldots, n$, where the G_i's are homogeneous polynomials in $x_0, x_1, \ldots, x_n$ of degree p.

The above proposition can be viewed as an analogue of the following well-known result in the geometric case.

Proposition 2.2. Let $k = \bar{k}$ be an algebraically closed field. The vector fields on $\mathbb{P}^n_k$ are of the form $\sum_{i=0}^n L_i \frac{\partial}{\partial x_i}$, where each $L_i = \sum_{j=0}^n a_{ij} x_j \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$.

Here is our first main result.

Theorem A. There are no lifts of Frobenius on the p-jet spaces $J^n(\mathbb{P}^m)$ for $n, m \geq 1$.

It is interesting to compare Theorem A with the following result in the geometric case.

Proposition A'. Let k be an algebraically closed field. The space of vector fields on the geometric tangent bundle $T(\mathbb{P}^1_k)$ lifting vector fields on $\mathbb{P}^1_k$ has dimension 6 over k. However, there are no normalized vector fields on $T(\mathbb{P}^1_k)$.

Here is our second main result:

Theorem B. There are no effective Cartier divisors D on the p-jet space $J^2(\mathbb{P}^1)$ such that $D \to J^1(\mathbb{P}^1)$ is finite.

Compare Theorem B with the following.

Proposition B'. There are no effective Cartier divisors D on the geometric jet space $J^2(\mathbb{P}^1)$, Spec $k[x, x', x''] \cup$ Spec $k[y, y', y'']$, such that $D \to T(\mathbb{P}^1)$ is finite.

We conclude with the following results:

Proposition C. Let $Q \subset \mathbb{P}^m_R$ be a smooth quadric hypersurface. There are no lifts of Frobenius on $J^1(Q)$.

Proposition D. Let $x = (x_{ij})$ be an n-by-n matrix of indeterminates. There exists no lift of Frobenius on $M_n =$ Spec $R[x]$ inducing a lift of Frobenius on
$GL_n =$ Spec $R[x, \det x^{-1}]$.

3. Proofs of Results

Proposition 2.1. The lifts of Frobenius on $\widehat{\mathbb{P}^n_R}$ are given on the x_i's by $\phi(x_i) = x_i^p + pF_i := G_i$, $i = 0, 1, \ldots, n$, where the F_i's are homogeneous polynomials in $x_0, x_1, \ldots, x_n$.

Proof. Any morphism of formal schemes $\widehat{\phi} : \widehat{\mathbb{P}^n_R} \to \widehat{\mathbb{P}^n_R}$ is induced by a morphism of schemes $\phi : \mathbb{P}^n_R \to \mathbb{P}^n_R$, by a GAGA theorem of Grothendieck [Gr61]. Thus, it is enough to find the morphisms $\mathbb{P}^n_R \to \mathbb{P}^n_R$ satisfying the additional property required to be lifts of Frobenius. Morphisms from any scheme X to $\mathbb{P}^n_R$ are well-known and are described, for example, in [Ha77], Thm. II.7.1; any such morphism ϕ is induced by an invertible sheaf $\mathscr{L}$ on X and global sections $s_0, \ldots, s_n \in \Gamma(X, \mathscr{L})$ which generate $\mathscr{L}$, with $s_i = \phi^*(x_i)$ on rings under this isomorphism.

Since R is a UFD, Pic $R = 0$, so $\mathrm{Pic}(\mathbb{P}^n_R) \cong \mathbb{Z}$, and every invertible sheaf on $\mathbb{P}^n_R$ is of the form $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$. Also, $\Gamma(\mathbb{P}^n_R, \mathcal{O}(d))$ identifies with the R-module of homogeneous polynomials of degree d in $n+1$ variables. We know that for each $i = 0, 1, \ldots, n$, $\phi_i(x_i) = x_i^p + pF_i$ because ϕ is a lift of Frobenius. Therefore, our sections must come from homogeneous polynomials of degree p. It remains to show that that the $G_i := x_i^p + pF_i$'s generate $\mathcal{O}(p)$, i.e. that the zero scheme $Z := Z(G_0, \ldots, G_n)$ is empty. We know that the G_i's have no common zeros mod p since $Z(x_0^p, x_1^p, \ldots, x_n^p) = \emptyset$, so $Z(\overline{G_0}, \ldots, \overline{G_n})(k) = (Z \cap \mathbb{P}^n_k)(k) = \emptyset$. By Hilbert's Nullstellensatz, since k is algebraically closed, $Z \cap \mathbb{P}^n_k = \emptyset$. Now, we know that $\mathbb{P}^n_R \xrightarrow{\pi} \mathrm{Spec}R$ is proper ([Ha77], Thm II.4.9), so $\pi(Z)$ is closed in Spec $R = \{(0), (p)\}$. Assume $Z \neq \emptyset$. Then $\pi(Z) \neq \emptyset$, so $(p) \in \pi(Z)$, a contradiction.

□

Proposition 2.2. Let $k = \bar{k}$. The vector fields on $\mathbb{P}^n_k$ are of the form $\sum_{i=0}^n L_i \frac{\partial}{\partial x_i}$, where each $L_i = \sum_{j=0}^n a_{ij} x_j \in H^0(\mathbb{P}^n_k, \mathcal{O}(1))$.

Proof. The vector fields on $X = \mathbb{P}^n_k$ are by definition $H^0(\mathbb{P}^n_k, \mathscr{T}_X)$, where $\mathscr{T}_X$ is the tangent sheaf on X. Recall the Euler exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to \mathscr{T}_X \to 0,$$

[Ha77], p. 182. This induces an exact sequence

$$0 \to H^0(X, \mathcal{O}_X) = k \to H^0(X, \mathcal{O}_X(1))^{n+1} \xrightarrow{\psi} H^0(X, \mathscr{T}_X) \to H^1(X, \mathcal{O}_X) = 0,$$

the last equality by [Ha77], Thm. III.5.1(b). The map ψ is given by $(L_0, \ldots, L_n) \mapsto \sum_{i=0}^n L_i \frac{\partial}{\partial x_i}$. Thus, $H^0(X, \mathscr{T}_X) = \mathrm{Im}\,\psi = \{\sum_{i=0}^n L_i \frac{\partial}{\partial x_i} : L_i \in H^0(\mathbb{P}_k^n, \mathcal{O}(1))\}$.

□

Theorem A. There are no lifts of Frobenius on the p-jet spaces $J^n(\mathbb{P}^m)$ for $n, m \geq 1$.

Proof. Step 1: As a warm-up, we first prove the statement for $n = m = 1$.

Recall that

$$J^1(\mathbb{P}^1) := \text{ Spf } R[x, x']\hat{} \cup \text{ Spf } R[y, y']\hat{},$$

where $xy = 1$. Assume we have a lift of Frobenius on $J^1(\mathbb{P}^1)$. Then it is given by ring homomorphisms $\phi_1 : R[x, x']\hat{} \to R[x, x']\hat{}$ and $\phi_2 : R[y, y']\hat{} \to R[y, y']\hat{}$ satisfying Definition 1.16. We have

$$\phi_1(x') = (x')^p + p h_1(x, x'), \quad \phi_1(y') = (y')^p + p h_2(y, y')$$

for some $h_1 \in R[x, x']\hat{}$, $h_2 \in R[y, y']\hat{}$. For computational purposes, we rewrite

$$\phi_1(x') = \sum_{i=0}^{\infty} \alpha_i(x)(x')^i, \quad \phi_2(y') = \sum_{i=0}^{\infty} \beta_i(y)(y')^i,$$

where for each i, $\alpha_i \in R[x]\hat{}$, $\beta_i \in R[y]\hat{}$. Observe that

$$\alpha_p(x) \equiv \beta_p(y) \equiv 1 \pmod{p},$$

while

$$\alpha_i(x) \equiv \beta_i(y) \equiv 0 \pmod{p}$$

for $i \neq p$. The intersection compatibility condition requires that $\phi_1(y') = \phi_2(y')$, where on the LHS, $y' := \delta(\frac{1}{x})$ via the isomorphism $R[x, x', x^{-1}]\hat{} \to R[y, y', y^{-1}]\hat{}$. Note that $y' \equiv= \frac{x'}{x^{2p}}(1 - p\frac{x'}{x^p}) \pmod{p^2}$. We compute toward a contradiction:

$$\begin{aligned}
\phi_1(y') &\equiv \phi_1(-\frac{x'}{x^{2p}}(1-p\frac{x'}{x^p})) \pmod{p^2} \\
&\equiv -\frac{\phi_1(x')}{(x^p+pg_1(x,x'))^{2p}}(1-p\frac{\phi_1(x')}{(x^p+pg_1(x,x'))^p}) \\
&\equiv -\frac{\phi_1(x')}{x^{2p^2}}(1-p\frac{\phi_1(x')}{x^{p^2}}) \\
&\equiv -\frac{\sum_{i=0}^\infty \alpha_i(x)(x')^i}{x^{2p^2}}(1-p\frac{\sum_{i=0}^\infty \alpha_i(x)(x')^i}{x^{p^2}}) \\
&\equiv -\frac{\sum_{i=0}^\infty \alpha_i(x)(x')^i}{x^{2p^2}}(1-p\frac{\alpha_p(x)(x')^p}{x^{p^2}}) \\
&\equiv -\frac{\sum_{i=0}^\infty \alpha_i(x)(x')^i}{x^{2p^2}}+\frac{p\alpha_p^2(x)(x')^{2p}}{x^{3p^2}}.
\end{aligned}$$

In the last two steps, we used that $\alpha_i(x) \equiv 0 \pmod p$ for $i \neq p$ and that $\alpha_p(x) \equiv 1 \pmod p$. Moreover, we have

$$\begin{aligned}
\phi_2(y') = \sum_{i=0}^\infty \beta_i(y)(y')^i &\equiv \sum_{i=0}^\infty \beta_i(\frac{1}{x})(-\frac{x'}{x^{2p}}(1-p\frac{x'}{x^p}))^i \pmod{p^2} \\
&\equiv \sum_{i=0}^\infty \beta_i(\frac{1}{x})\frac{(-1)^i(x')^i}{x^{2pi}}(1-ip\frac{x'}{x^p}) \\
&\equiv \sum_{i=0}^\infty (\beta_i(\frac{1}{x})\frac{(-1)^i(x')^i}{x^{2pi}}+\frac{ip\beta_i(\frac{1}{x})(-1)^{i+1}(x')^{i+1}}{x^{p(2i+1)}}) \\
&\equiv \sum_{i=0}^\infty (\beta_i(\frac{1}{x})\frac{(-1)^i(x')^i}{x^{2pi}}).
\end{aligned}$$

The last step follows from $\beta_i(\frac{1}{x}) \equiv 0 \pmod p$, $i \neq p$, and that if $i = p$ we have $ip = p^2$. Since we need $\phi_1(y') \equiv \phi_2(y') \pmod{p^2}$, we compare the above two expressions. It will be enough to examine only the degree $2p$ terms in x'. We have

$$-\frac{\alpha_{2p}(x)(x')^{2p}}{x^{2p^2}}+\frac{p\alpha_p^2(x)(x')^{2p}}{x^{3p^2}} \equiv \beta_{2p}(\frac{1}{x})(-1)^{2p}\frac{(x')^{2p}}{x^{2p(2p)}} \pmod{p^2}.$$

Clearing denominators, this yields an equality in the ring of Laurent polynomials

$$-\tilde{\alpha}_{2p}(x)x^{2p^2}+p\tilde{\alpha}_p^2(x)x^{p^2} = \tilde{\beta}_{2p}(\frac{1}{x}), \tag{3.1}$$

where $\sim$ means image in $(R/p^2R)[x,x^{-1}]$. Recall that the order of a Laurent polynomial $P(x)$ is the smallest exponent of its monomials. Now the LHS of (3.1) has order $\geq p^2$, while the RHS has order ≤ 0, unless both sides are 0. Hence,

$$-\alpha_{2p}(x)x^{2p^2} + p\alpha_p^2(x)x^{p^2} \equiv 0 \pmod{p^2},$$

so

$$-\alpha_{2p}(x)x^{p^2} + p\alpha_p^2(x) \equiv 0 \mod p^2.$$

Using that $\alpha_{2p}(x) \equiv 0 \pmod{p}$ and $\alpha_p(x) \equiv 1 \pmod{p}$, we obtain, for some $g_1, g_2, g_3 \in R[x]\hat{}$,

$$-pg_1(x)x^{p^2} + p(pg_2(x)+1)^2 = p^2 g_3(x);$$

hence

$$-g_1(x)x^{p^2} + p^2 g_2^2(x) + 2pg_2(x) + 1 = pg_3(x);$$

so

$$1 - g_1(x)x^{p^2} \equiv 0 \pmod{p}.$$

Setting $x = 0$, we get $1 \equiv 0 \pmod{p}$, a contradiction. This ends the proof of the fact that there is no lift of Frobenius on $J^1(\mathbb{P}^1)$.

Step 2: Extend argument to $J^1(\mathbb{P}^n)$.

We recall by Definition 1.27 that

$$J^1(\mathbb{P}^n) = \bigcup_{i=0}^{n} \text{Spf } R[y_{i0}, \dots, y_{in}, y'_{i0}, \dots, y'_{in}]\hat{},$$

where $y_{ij} = \frac{x_j}{x_i}$. Assume there exists a lift of Frobenius $J^1(\mathbb{P}^n) \to J^1(\mathbb{P}^n)$. This translates to having ring endomorphisms $\phi_i : A_i := R[y_{i0}, \dots, y_{in}, y'_{i0}, \dots, y'_{in}]\hat{} \to A_i$ gluing on localizations such that

$$\phi_i(y_{ij}) = y_{ij}^p + pg_{ij}(y_{i0}, \dots, y_{in}, y'_{i0}, \dots, y'_{in})$$

and

$$\phi_i(y'_{ij}) = (y'_{ij})^p + ph_{ij}(y_{i0}, \dots, y_{in}, y'_{i0}, \dots, y'_{in}).$$

It is necessary that for each i, i', j, $\phi_i(y'_{ij}) = \phi_{i'}(y'_{ij})$. Letting $i' = j = 0$, $i = 1$, we require $\phi_0(y'_{10}) = \phi_1(y'_{10})$. We may rewrite

$$\phi_1(y'_{10}) = \sum_{k=0}^{\infty} \alpha_{1,0_k}(y_{10}, \ldots, y_{1n}, y'_{10}, \ldots, \widehat{y'_{10}}, \ldots, y'_{1n})(y'_{10})^k.$$

(In the last expression, $\widehat{y'_{10}}$ means that y'_{10} is omitted.) Observe that $\alpha_{1,0_k} \equiv 0 \pmod p$ for $k \neq p$ and $\alpha_{1,0_p} \equiv 1 \pmod p$.

Since $y_{10}y_{01} = 1$, a calculation shows that

$$y'_{10} = -\frac{y'_{01}}{y_{01}^{2p}}(1 - p\frac{y'_{01}}{y_{01}^{p}}) \pmod{p^2}.$$

Also, tracing through the calculation for $J^1(\mathbb{P}^1)$ of $\phi_1(y')$ and $\phi_2(y')$, we find that

$$\phi_0(y'_{10}) \equiv -\frac{\sum_{k=0}^{\infty} \alpha_{0,1_k}(y'_{01})^i}{y_{01}^{2p^2}} + \frac{p\alpha_{0,1_p}^{\,2}(y'_{01})^{2p}}{y_{01}^{3p^2}}$$

and

$$\phi_1(y'_{10}) \equiv \sum_{k=0}^{\infty} \alpha_{1,0_k} \frac{(-1)^k(y'_{01})^k}{y_{01}^{2pk}}.$$

Comparing the coefficients of the degree $2p$ terms in y'_{01} of the above two expressions, we obtain, in the same way as for $J^1(\mathbb{P}^1)$,

$$\begin{aligned}-\alpha_{0,1_{2p}}(y_{01}, \ldots, y_{0n}, \widehat{y'_{01}}, \ldots, y'_{0n})y_{01}^{2p^2} + p\alpha_{0,1_p}^{\,2}(y_{01}, \ldots, y_{0n}, \widehat{y'_{01}}, \ldots, y'_{0n})y_{01}^{p^2} \\ \equiv \alpha_{1,0_{2p}}(y_{10}, \ldots, y_{1n}, \widehat{y'_{10}}, \ldots, y'_{1n}) \pmod{p^2}.\end{aligned}$$

In order to compare both sides of this congruence, we rewrite the expressions in terms of one common set of variables. Note that $y_{1s} = \frac{y_{0s}}{y_{01}}$. Also, $y_{1s}y_{01} = y_{0s}$, so that

$$\delta(y_{1s}y_{01}) = \delta(y_{0s});$$

hence

$$y_{1s}^p y_{01}' + y_{01}^p y_{1s}' + p y_{1s}' y_{01}' = y_{0s}';$$

so

$$\begin{aligned} y_{1s}' &= \frac{y_{0s}' - y_{1s}^p y_{01}'}{y_{01}^p + p y_{01}'} \\ &= \frac{y_{0s}' - (\frac{y_{0s}}{y_{01}})^p y_{01}'}{y_{01}^p + p y_{01}'} \\ &= \frac{y_{0s}' y_{01}^p - y_{0s}^p y_{01}'}{y_{01}^{2p} + p y_{01}' y_{01}^p} \\ &\equiv \frac{y_{0s}' y_{01}^p - y_{0s}^p y_{01}'}{y_{01}^{2p}} (1 - \frac{p y_{01}'}{y_{01}^p}) \pmod{p^2}. \end{aligned}$$

Substituting for each y_{1s}, we obtain $A \equiv B \pmod{p^2}$, where

$$A = -\alpha_{0,1_{2p}}(y_{01}, \ldots, y_{0n}, y_{02}', \ldots, y_{0n}') y_{01}^{2p^2} + p \alpha_{0,1_p}^2 (y_{01}, \ldots, y_{0n}, y_{02}', \ldots, y_{0n}') y_{01}^{p^2}$$

and

$$B = \alpha_{1,0_{2p}}(\frac{1}{y_{01}}, \ldots, \frac{y_{0n}}{y_{01}}, \frac{y_{02}' y_{01}^p - y_{02}^p y_{01}'}{y_{01}^{2p}} (1 - \frac{p y_{01}'}{y_{01}^p}), \ldots, \frac{y_{0n}' y_{01}^p - y_{0n}^p y_{01}'}{y_{01}^{2p}} (1 - \frac{p y_{01}'}{y_{01}^p})).$$

By the same argument as in the $J^1(\mathbb{P}^1)$ proof, observing that the y_{01}'s in the denominators of the RHS can only decrease the RHS's order in y_{01},

$$-\alpha_{0,1_{2p}}(y_{01}, \ldots, y_{0n}, y_{02}', \ldots, y_{0n}') y_{01}^{2p^2} + p \alpha_{0,1_p}^2 (y_{01}, \ldots, y_{0n}, y_{02}', \ldots, y_{0n}') y_{01}^{p^2} \equiv 0 \pmod{p^2}.$$

Since $\alpha_{0,1_2}p \equiv 0 \pmod{p}$ and $\alpha_{0,1_p} \equiv 1 \pmod{p}$, we get, for some $\gamma_1, \gamma_2, \gamma_3 \in R[y_{01}, \ldots, y_{0n}, y_{02}', \ldots, y_{0n}']\hat{}$,

$$-p y_{01}^{p^2} \gamma_1 + p(p\gamma_2 + 1)^2 = p^2 \gamma_3;$$

hence

$$-y_{01}^{p^2} \gamma_1 + (p\gamma_2 + 1)^2 = p\gamma_3;$$

so

$$1 - y_{01}^{p^2}\gamma_1 \equiv 0 \pmod{p}.$$

Letting $y_{01} = 0$, we get $1 \equiv 0 \pmod{p}$, a contradiction. Thus, there are no lifts of Frobenius on $J^1(\mathbb{P}^n)$.

Step 3: Generalize for $J^n(\mathbb{P}^m)$.

Recall that

$$J^n(\mathbb{P}^m) = \bigcup_{i=0}^{m} \operatorname{Spf} R[y_{i0}, \ldots, y_{im}, y'_{i0}, \ldots, y'_{im}, \ldots, y_{i0}^{(n)}, \ldots, y_{im}^{(n)}]\hat{}\,,$$

where as before $y_{ij} = \frac{x_j}{x_i}$. A lift of Frobenius on $J^n(\mathbb{P}^m)$ is given by ring endomorphisms $\phi_i : A_i = R[y_{i0}, \ldots, y_{im}, y'_{i0}, \ldots, y'_{im}, \ldots, y_{i0}^{(n)}, \ldots, y_{im}^{(n)}]\hat{} \to A_i$ which are lifts of Frobenius and which glue on localizations. As before, the compatibility condition requires

$$\phi_i(y'_{ij}) = (y'_{ij})^p + pf_{ij}(y_{i0}, \ldots, y_{im}, y'_{i0}, \ldots, y'_{im}, \ldots, y_{i0}^{(n)}, \ldots, y_{im}^{(n)}).$$

We may rewrite as before

$$\phi_1(y'_{10}) = \sum_{k=0}^{\infty} \alpha_{1,0_k}(y_{01}, \ldots, y_{0m}, \widehat{y'_{01}}, y'_{02}, \ldots, y'_{0m}, \ldots, y_{01}^{(n)}, \ldots, y_{0m}^{(n)})(y'_{10})^k.$$

Once again, $\alpha_{1,0_k} \equiv 0 \pmod{p}$ for $k \neq p$ and $\alpha_{1,0_p} \equiv 1 \pmod{p}$.

Imitating the steps for the $J^1(\mathbb{P}^n)$ proof, we obtain

$$\begin{aligned} -\alpha_{0,1_{2p}}(y_{01}, \ldots, y_{0m}, \widehat{y'_{01}}, y'_{02}, \ldots, y'_{0m}, \ldots, y_{01}^{(n)}, \ldots, y_{0m}^{(n)})y_{01}^{2p^2} + \\ p\alpha_{0,1_p}^2(y_{01}, \ldots, y_{0m}, \widehat{y'_{01}}, y'_{02}, \ldots, y'_{0m}, \ldots, y_{01}^{(n)}, \ldots, y_{0m}^{(n)})y_{01}^{p^2} \\ \equiv \alpha_{1,0_{2p}}(y_{10}, \ldots, y_{1m}, \widehat{y'_{10}}, y'_{12}, \ldots, y'_{1m}, \ldots, y_{10}^{(n)}, \ldots, y_{1m}^{(n)}) \pmod{p^2}. \end{aligned}$$

Note that $y_{1s} = \frac{y_{0s}}{y_{01}}$, which has order -1 in y_{01}. We claim that $\widetilde{y_{1s}^{(r)}} := y_{1s}^{(r)} \pmod{p^2}$ has negative order in y_{01} for all $s \in \{0, \ldots, m\}$, $r \in \{0, \ldots, n\}$. The base case of $r = 0$ was just stated, so assume we have $\widetilde{y_{1s}^{(r-1)}}$ with negative order in y_{01}. Write $\widetilde{y_{1s}^{(r-1)}} := y_{01}^{-t} f$, where $-t$ is the order in y_{01} of $y_{1s}^{(r-1)}$ and

$$f \in R[y_{10}, \ldots, y_{1m}, \widehat{y'_{10}}, y'_{12}, \ldots, y'_{1m}, \ldots, y_{10}^{(r-1)}, \ldots, y_{1m}^{(r-1)}]$$

has order 0 in y_{01}. In the following, all elements are understood to be their images mod p^2; we leave off $\sim$ for simplicity of notation. We have

$$\begin{aligned}
y_{1s}^{(r)} =& \delta(y_{1s}^{(r-1)}) = \delta(y_{01}^{-t} f) \\
=& (y_{01}^{-t})^p \delta(f) + f^p \delta(y_{01}^{-t}) + p\delta(y_{01}^{-t})\delta(f) \\
=& y_{01}^{-tp}\Big(\frac{\phi_p(f) - f^p}{p}\Big) + f^p\Big(\frac{(y_{01}^p + py'_{01})^{-t} - y_{01}^{-tp}}{p}\Big) + \\
& p\Big(\frac{(y_{01}^p + py'_{01})^{-t} - y_{01}^{-tp}}{p}\Big)\Big(\frac{\phi_p(f) - f^p}{p}\Big),
\end{aligned}$$

where $\phi_p : \mathcal{O}(J^{r-1}(\mathbb{P}^m)) \to \mathcal{O}(J^r(\mathbb{P}^m))$ is the lift of Frobenius corresponding to the universal p-derivation of Definition 1.11. But since the order of f in y_{01} is 0, the same holds for $\phi_p(f)$ and f^p. Therefore, the above expression has order ≤ 0 in y_{01}. Hence, the presence of the extra "prime variables" $y_{1s}^{(r)}$ can only decrease the order of the RHS. This leads to the conclusion, as in the proof for $J^n(\mathbb{P}^1)$, that

$$\begin{aligned}
&-\alpha_{0,1_{2p}}(y_{01}, \ldots, y_{0m}, \widehat{y'_{01}}, y'_{02}, \ldots, y'_{0m}, \ldots, y_{01}^{(n)}, \ldots, y_{0m}^{(n)}) y_{01}^{p^2} \\
&\qquad + p\alpha_{0,1_p}^2(y_{01}, \ldots, y_{0m}, \widehat{y'_{01}}, y'_{02}, \ldots, y'_{0m}, \ldots, y_{01}^{(n)}, \ldots, y_{0m}^{(n)}) \equiv 0 \pmod{p^2}
\end{aligned}$$

and we deduce a contradiction as before.

□

Proposition A'. Let k be an algebraically closed field. The space of vector fields on the geometric tangent bundle $T(\mathbb{P}^1_k)$ lifting vector fields on $\mathbb{P}^1_k$ has dimension 6 over k. However, there are no normalized vector fields on $T(\mathbb{P}^1_k)$.

Proof. A vector field on $T(\mathbb{P}^1_k)$ is given by a k-derivation D on the ring of global functions of $T(\mathbb{P}^1_k)$, that is, by $D_1 : k[x, x'] \to k[x, x']$ and $D_2 : k[y, y'] \to k[y, y']$ such that $D_1(x) = D_2(x)$, $D_1(y) = D_2(y)$, $D_1(x') = D_2(x')$, and $D_1(y') = D_2(y')$. To simplify matters, we look at the case where D extends a derivation on $\mathbb{P}^1$, i.e, where $D_1 : k[x] \to k[x]$ and $D_2 : k[y] \to k[y]$ are derivations. We have:

$$
\begin{aligned}
D_1(y) = D_1(\frac{1}{x}) &= \frac{xD_1(1) - 1D_1(x)}{x^2} \\
&= -\frac{D_1(x)}{x^2}.
\end{aligned}
$$

Write $D_1(x) = a_0 + a_1x + \cdots + a_nx^n$ and $D_2(y) = b_0 + b_1y + \cdots + b_my^m$. Since we must have $D_1(y) = D_2(y)$, this gives

$$
-\frac{a_0 + a_1x + \cdots + a_nx^n}{x^2} = b_0 + b_1y + \cdots + b_my^m,
$$

hence

$$
-a_0y^2 - a_1y - a_2 - a_3\frac{1}{y} - \cdots - a_n\frac{1}{y^{n-2}} = b_0 + b_1y + \cdots + b_my^m.
$$

Thus, $n, m \leq 2$, and so $D_1(x) = a_0 + a_1x + a_2x^2$ and $D_2(y) = -a_2 - a_1y - a_0y^2$.

Now, we have

$$
\begin{aligned}
D_1(y') = D_1(-\frac{x'}{x^2}) &= \frac{x^2D_1(-x') - (-x')2xD_1(x)}{x^4} \\
&= \frac{-x^2D_1(x') + 2xx'D_1(x)}{x^4} \\
&= -y^2D_1(x') + 2y^3(-\frac{y'}{y^2})(a_0 + a_1x + a_2x^2) \\
&= -y^2D_1(x') - 2yy'(a_0 + a_1x + a_2x^2) \\
&= -y^2D_1(x') - 2a_0yy' - 2a_1y' - \frac{2a_2y'}{y}.
\end{aligned}
$$

We require

$$
-y^2D_1(x') - 2a_0yy' - 2a_1y' - \frac{2a_2y'}{y} = D_1(y') = D_2(y') \in k[y, y'].
$$

Also, $D_1(x') \in k[x, x']$; write $D_1(x') = \sum_{n,m\geq 0} a_{n,m}x^n(x')^m$. Then

$$
\begin{aligned}
-y^2 D_1(x') - 2a_0 yy' - 2a_1 y' - \frac{2a_2 y'}{y} &= -y^2\Big(\sum_{n,m\geq 0} a_{n,m} x^n (x')^m\Big) - 2a_0 yy' \\
&\quad - 2a_1 y' - \frac{2a_2 y'}{y} \\
&= -y^2\Big(\sum_{n,m\geq 0} a_{n,m} y^{-n} (-\frac{y'}{y^2})^m\Big) - 2a_0 yy' \\
&\quad - 2a_1 y' - \frac{2a_2 y'}{y} \\
&= -\sum_{n,m\geq 0} a_{n,m} y^{-n+2-2m} (-y')^m - 2a_0 yy' \\
&\quad - 2a_1 y' - \frac{2a_2 y'}{y}.
\end{aligned}
$$

If $a_{n,m} \neq 0$ for any n, m pair such that $m \geq 2$, then we would have a monomial of order ≥ 2 in y' and order $-n+2-2m \leq -n+2-2(2) = -n-2 \leq -2$ in y, which can't happen because none of the other monomials will cancel this. Hence, $a_{n,m} = 0$ for $m \geq 2$. Splitting into the $m = 0$ and $m = 1$ cases, we are left with

$$
\begin{aligned}
&-\sum_{n\geq 0, m\in\{0,1\}} a_{n,m} y^{-n+2-2m} (-y')^m - 2a_0 yy' - 2a_1 y' - \frac{2a_2 y'}{y} \\
&= -\sum_{n_0\geq 0} a_{n_0,0} y^{-n_0+2} - \sum_{n_1 \geq 0} a_{n_1,1} y^{-n_1} y' - 2a_0 yy' - 2a_1 y' - \frac{2a_2 y'}{y}.
\end{aligned}
$$

From this, we see that $n_0 \leq 2$ and $n_1 \leq 1$; otherwise, we get, respectively, monomials y^r with $r \leq -1$, and $y^s y'$ with $s \leq -2$. Hence,

$$
D_1(y') = -a_{0,0} y^2 - a_{1,0} y - a_{2,0} + a_{0,1} y' + \frac{a_{1,1} y'}{y} - 2a_0 yy' - 2a_1 y' - \frac{2a_2 y'}{y}.
$$

This implies that $a_{1,1} = 2a_2$, and so, with the surviving coefficients, we get

$$
D_1(x') = 2a_2 xx' + a_{0,1} x' + a_{2,0} x^2 + a_{1,0} x + a_{0,0}
$$

and

$$
D_2(y') = D_1(y') = -2a_0 yy' + (a_{0,1} - 2a_1) y' - a_{0,0} y^2 - a_{1,0} y - a_{2,0}.
$$

for any $a_0, a_2, a_{0,0}, a_{1,0}, a_{2,0}, a_{0,1} \in k$. Therefore, the space of vector fields on $T(\mathbb{P}^1)$ lifting vector fields on $\mathbb{P}^1$ has dimension 6 over k.

However, we show that there are no normalized vector fields on $T(\mathbb{P}^1)$. To see this, assume there is a normalized derivation on $T(\mathbb{P}^1) =$ Spec $k[x, x'] \cup$ Spec $k[y, y']$, $xy = 1$. Then we have $D_1 : k[x, x'] \to k[x, x']$, $D_2 : k[y, y'] \to k[y, y']$ such that $D_1(x) = x'$, $D_2(y) = y'$. We have

$$\begin{aligned} D_1(y') = D_1(-\frac{x'}{x^2}) &= -D_1(\frac{x'}{x^2}) \\ &= -\frac{x^2 D_1(x') - x' D_1(x^2)}{x^4} \\ &= -\frac{x^2 D_1(x') - x'(2xx')}{x^4} \\ &= -\frac{D_1(x')}{x^2} + \frac{2(x')^2}{x^3} \\ &= -y^2 D_1(x') + 2(-\frac{y'}{y^2})^2 y^3 \\ &= -y^2 D_1(x') + \frac{2(y')^2}{y}. \end{aligned}$$

Thus, $-y^2 D_1(x') + \frac{2(y')^2}{y}$ equals $D_2(y') \in k[y, y']$. Write

$$D_1(x') = \sum_{n,m \geq 0} a_{n,m} x^n (x')^m = \sum_{n,m \geq 0} a_{n,m} (\frac{1}{y})^n (\frac{y'}{y^2})^m.$$

Then

$$\begin{aligned} -y^2 D_1(x') + \frac{2(y')^2}{y} &= -y^2 \big(\sum_{n,m \geq 0} a_{n,m} y^{-n} (\frac{y'}{y^2})^m \big) + \frac{2(y')^2}{y} \\ &= -\big(\sum_{n,m \geq 0} a_{n,m} y^{-n-2m+2} (y')^m \big) + \frac{2(y')^2}{y}. \end{aligned}$$

Let us extract the coefficient of $(y')^2$: it is $-\big(\sum_{n \geq 0} a_{n,2} y^{-n-2}\big) + 2y^{-1}$, which cannot not be in $k[y]$ since $-n-2 < -1$ for all $n \geq 0$. This contradicts that $D_2(y') \in k[y, y']$. $\square$

Theorem B. There are no effective Cartier divisors D on the p-jet space $J^2(\mathbb{P}^1)$ such that $D \to J^1(\mathbb{P}^1)$ is finite.

We need the following:

Lemma: Let A be an integral domain and and suppose $A[x]/(f)$ is finite over A. Then f is monic in x, up to multiplication by an invertible element of A.

Proof: Suppose $A[x]/(f)$ is finite over A but f not monic in x. By assumption $A[x]/(f) = A\overline{x^n} + A\overline{x^{n-1}} + \cdots + A\overline{x} + A$ for some $n \in \mathbb{N}$, where upper bar means class mod f. Since f is not monic up to multiplication by an element of $A^\times$, we can write $f = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_0$, where $a_i \in A$, $a_r \notin A^\times$. We have

$$\overline{x^{r+n}} = \overline{b_n x^n + \cdots + b_1 x + b_0}$$

for some $b_i \in A$. Hence, $x^{r+n} - b_n x^n - \cdots - b_1 x - b_0 = f \cdot h$, so $\deg_x h = n$. So $h = c_n x^n + \cdots + c_1 x + c_0$, $c_i \in A$. Thus $a_r c_n = 1$, so $a_r \in A^\times$, a contradiction. This concludes the proof of the lemma.

Proof. (of Theorem) Recall that

$$J^1(\mathbb{P}^1) = \text{Spf } R[x,x']\hat{} \cup \text{Spf } R[y,y']\hat{}$$

and

$$J^2(\mathbb{P}^1) = \text{Spf } R[x,x',x'']\hat{} \cup \text{Spf } R[y,y',y'']\hat{}.$$

Toward a contradiction, assume there exists a $D \subset J^2(\mathbb{P}^1)$ finite over $J^1(\mathbb{P}^1)$. Write $D = \text{Spf } \frac{R[x,x',x'']\hat{}}{(f)} \cup \text{Spf } \frac{R[y,y',y'']\hat{}}{(g)}$, where $f = U \cdot g$ for some

$$U \in (R[y,y',y'',y^{-1}]\hat{})^\times;$$

f and g in this last equality are considered in $R[y,y',y'',y^{-1}]\hat{}$ via the isomorphism $R[x,x',x'',x^{-1}]\hat{} \to R[y,y',y'',y^{-1}]\hat{}$ sending $x \mapsto y^{-1}$, $x' \mapsto \delta(y^{-1})$, $x'' \mapsto \delta^2(y^{-1})$. Then by definition, $\frac{R[x,x',x'']\hat{}}{(f)}$ and $\frac{R[y,y',y'']\hat{}}{(g)}$ are finitely generated $R[x,x']\hat{}$- and $R[y,y']\hat{}$-modules, respectively. Denoting by the bar of a ring its quotient mod the ideal generated by p, we hence have that

$$\frac{\overline{R[x,x',x'']\hat{}}}{(f)} = \frac{(R/pR)[x,x'][x'']}{(f \mod p)}, \quad \frac{\overline{R[y,y',y'']\hat{}}}{(f)} = \frac{(R/pR)[y,y'][y'']}{(g \mod p)}$$

are finitely generated over $\overline{R[x,x']\hat{}} = (R/pR)[x,x']$ and $\overline{R[y,y']\hat{}} = (R/pR)[y,y']$. By the lemma above, the leading coefficients of $f \mod p$ in x'' and $g \mod p$ in y'' are invertible in $(R/pR)[x,x']$ and $(R/pR)[y,y']$, respectively. But $\big((R/pR)[x,x']\big)^{\times} = (R/pR)^{\times} = \{r + pR : r \in R^{\times}\} = \big((R/pR)[y,y']\big)^{\times}$. Write

$$f \equiv r_1(x'')^n + A_{n-1}(x,x')(x'')^{n-1} + \cdots + A_1(x,x')x'' + A_0(x,x') \pmod p, \quad (3.2)$$

where $r_1 \in R^{\times}$ and for each i, $A_i(x,x') \in R[x,x']\hat{}$. Now,

$$(R[y,y',y'',y^{-1}]\hat{})^{\times} = \{\lambda y^N + ph : \lambda \in R^{\times}, N \in \mathbb{Z}, h \in R[y,y',y'',y^{-1}]\hat{}\}.$$

Indeed, the elements of the LHS must have p-adic valuation 1 and their reduction mod p must be invertible in $R[y,y',y'',y^{-1}]$. Conversely, for any $\lambda y^N + ph$ of the RHS, we have that $\frac{1}{\lambda y^N + ph} = \frac{1}{\lambda y^N(1 + p\frac{h}{\lambda y^N})} = \frac{1}{\lambda y^N}\sum_{i=0}^{\infty}(-p)^i(\frac{h}{\lambda y^N})^i$ is in $R[y,y',y'',y^{-1}]\hat{}$ and is $(\lambda y^N + pH)^{-1}$. So we have

$$f = (\lambda y^N + ph)g \quad (3.3)$$

for some $\lambda \in R^{\times}$, $N \in \mathbb{Z}$, $h \in R[y,y',y'',y^{-1}]\hat{}$. Let us compute f of equation (3.2) in terms of y, y', y'' in order to compare to the form of f of equation (3.3). By the quotient rule (Proposition 1.26)

$$\begin{aligned}
x'' = \delta^2(1/y) = \delta\Big(\frac{-y'}{y^p\phi(y)}\Big) &= -\delta\Big(\frac{y'}{y^p\phi(y)}\Big)\\
&= -\frac{(y^p\phi(y))^p y'' - (y')^p\delta(y^p\phi(y))}{(y^p\phi(y))^p\phi(y^p\phi(y))}\\
&= -\frac{(y^p(y^p+py'))^p y'' - (y')^p\delta(y^p\phi(y))}{(y^p(y^p+py'))^p\phi(y^p(y^p+py'))}\\
&\equiv -\frac{y^{2p^2}y'' - (y')^p\delta(y^p\phi(y))}{y^{2p^2}\phi(y^{2p})} \pmod p\\
&\equiv -\frac{y^{2p^2}y'' - (y')^p\delta(y^p\phi(y))}{y^{2p^2}(y^p + p(y'))^{2p}}\\
&\equiv -\frac{y^{2p^2}y'' - (y')^p\delta(y^p\phi(y))}{y^{4p^2}}.
\end{aligned}$$

Note that for any k,

$$\delta(y^k) = \frac{(y^p + px')^k - y^{kp}}{p} = \frac{\sum_{i=1}^k \binom{k}{i}(y^p)^{k-i}(py')^i}{p} = \sum_{i=1}^k \binom{k}{i} y^{p(k-i)} p^{i-1} (y')^i.$$

Also, from the definitions it follows that δ and ϕ commute, so that

$$\begin{aligned}
\delta(y^p\phi(y)) =& (y^p)^p \delta(\phi(y)) + \phi(y)^p \delta(y^p) + p\delta(y^p)\delta(\phi(y)) \\
=& y^{p^2}\phi(y') + (y^p + py')^p \Big(\sum_{i=1}^p \binom{p}{i} y^{p(p-i)} p^{i-1} (y')^i\Big) + \\
& p\Big(\sum_{i=1}^p \binom{p}{i} y^{p(p-i)} p^{i-1} (y')^i\Big)\phi(y') \\
=& y^{p^2}((y')^p + py'') + (y^p + py')^p \Big(\sum_{i=1}^p \binom{p}{i} y^{p(p-i)} p^{i-1} (y')^i\Big) + \\
& p\Big(\sum_{i=1}^p \binom{p}{i} y^{p(p-i)} p^{i-1} (y')^i\Big)((y')^p + py'') \\
\equiv& y^{p^2}(y')^p \pmod{p}.
\end{aligned}$$

Thus,

$$x'' \equiv -\frac{y^{2p^2}y'' - (y')^p(y^{p^2}(y')^p)}{y^{2p^2}\phi(y^{2p})} \equiv \frac{-y^{2p^2}y'' + y^{p^2}(y')^{2p}}{y^{4p^2}} \pmod{p}.$$

Therefore,

$$\begin{aligned}
f \equiv r_1\Big(\frac{-y^{2p^2}y'' + y^{p^2}(y')^{2p}}{y^{4p^2}}\Big)^n + A_{n-1}(\frac{1}{y}, \frac{y'}{y^{2p}})\Big(\frac{-y^{2p^2}y'' + y^{p^2}(y')^{2p}}{y^{4p^2}}\Big)^{n-1} + \cdots \\
+ A_1(\frac{1}{y}, \frac{y'}{y^{2p}})\Big(\frac{-y^{2p^2}y'' + y^{p^2}(y')^{2p}}{y^{4p^2}}\Big) + A_0(\frac{1}{y}, \frac{y'}{y^{2p}}) \pmod{p}
\end{aligned}$$

$$\begin{aligned}
\equiv [r_1(-y^{2p^2}y'' + y^{p^2}(y')^{2p})^n + A_{n-1}(\frac{1}{y}, \frac{y'}{y^{2p}})(-y^{2p^2}y'' + y^{p^2}(y')^{2p})^{n-1}y^{4p^2} + \cdots \quad (3.4) \\
+ A_1(\frac{1}{y}, \frac{y'}{y^{2p}})(-y^{2p^2}y'' + y^{p^2}(y')^{2p})y^{4p^2(n-1)} + A_0(\frac{1}{y}, \frac{y'}{y^{2p}})y^{4p^2 n}]/y^{4p^2 n}.
\end{aligned}$$

According to equation (3.3), expression (3.4) is $\equiv \lambda y^N \cdot g \pmod p$, so by clearing the denominator y^{4p^2n}, we find that the numerator of (4) is $\equiv \lambda y^{4p^2n+N} g \pmod p$. Writing

$$g \equiv r_2(y'')^n + B_{n-1}(y,y')(y'')^{n-1} + \cdots + B_1(y,y')y'' + B_0(y,y'),$$

this means that

$$r_1(-y^{2p^2}y'' + y^{p^2}(y')^{2p})^n + A_{n-1}(\frac{1}{y}, \frac{y'}{y^{2p}})(-y^{2p^2}y'' + y^{p^2}(y')^{2p})^{n-1}y^{4p^2} + \cdots \tag{3.5}$$

$$+A_1(\frac{1}{y}, \frac{y'}{y^{2p}})(-y^{2p^2}y'' + y^{p^2}(y')^{2p})y^{4p^2(n-1)} + A_0(\frac{1}{y}, \frac{y'}{y^{2p}})y^{4p^2n}$$

$$\equiv \lambda y^{4p^2n+N}[r_2(y'')^n + B_{n-1}(y,y')(y'')^{n-1} + \cdots + B_1(y,y')y'' + B_0(y,y')] \pmod p.$$

Expanding out the LHS of (3.5) and comparing with the RHS, we see that we must have

$$r_1(-1)^n y^{2p^2n}(y'')^n \equiv r_2\lambda y^{4p^2n+N}(y'')^n \pmod p,$$

so that $N = -2p^2n$. Now, according to the RHS, the $(y'')^0$ coefficient must have order $\geq 4p^2n - 2p^2n = 2p^2n$ in y. The LHS of (3.5) has the monomial $r_1 y^{p^2n}(y')^{2pn}$ with order $< 2p^2n$ in y, so it suffices for $n > 0$ to show that no other term of the LHS can cancel said monomial. Note that the only terms of the LHS with order 0 in y'' come from the last terms of each binomial expansion. Thus, if there did exist such other term, there would exist $l, m \geq 0$ and $k \geq 1$ such that

$$(\frac{1}{y^l})(\frac{y'}{y^{2p}})^m y^{p^2(n-k)}(y')^{2p(n-k)}y^{4p^2k} = y^{p^2n}(y')^{2pn}.$$

By looking at the power of y', this tells us that $m = 2pk$, and so we compare the order of y on each side: we must have $-l - 2p(2pk) + p^2(n-k) + 4p^2k = p^2n \Leftrightarrow -l = p^2k$. But $-l \leq 0$ and $p^2k \geq p^2(1)$, so this is impossible.

$\square$

Proposition B'. There are no effective Cartier divisors D on the geometric jet space $J^2(\mathbb{P}^1)$, Spec $k[x,x',x'']\cup$ Spec $k[y,y',y'']$, such that $D\to T(\mathbb{P}^1)$ is finite.

Proof. We prove the proposition by exactly the same approach as we did Theorem B. We again assume there exists such a D; we have $D=$ Spec $\frac{k[x,x',x'']}{(f)}\cup$ Spec $\frac{k[y,y',y'']}{(g)}$, where $f=U\cdot g$ for some $U\in k[y,y',y'',y^{-1}]^\times$, with $\frac{k[x,x',x'']}{(f)}$ and $\frac{k[y,y',y'']}{(g)}$ finitely-generated $k[x,x']$- and $k[y,y']$-modules, respectively. By the lemma in Theorem B's proof, f and g are monic, up to multiplication by nonzero element $\lambda_1,\lambda_2\in k$, in x'' and y'', respectively. We also know that U can be written $U=\lambda y^N$ for some $\lambda\in k$, $\lambda\neq 0$, and $N\in\mathbb{Z}$. Hence, we can write

$$f=\lambda_1(x'')^n+A_{n-1}(x,x')(x'')^{n-1}+\cdots+A_1(x,x')x''+A_0(x,x')=\lambda y^N g. \qquad (3.6)$$

Note that $x'=d(\frac{1}{y})=\frac{y(0)-1(y')}{y^2}=-\frac{y'}{y^2}$, while

$$\begin{aligned}x''=d(x')=d(-\frac{y'}{y^2})&=-d(\frac{y'}{y^2})\\&=-\frac{y^2(y'')-y'(2yy')}{y^4}\\&=\frac{-y^2y''+2y(y')^2}{y^4}.\end{aligned}$$

Thus,

$$\begin{aligned}f=\lambda_1\big(\frac{-y^2y''+2y(y')^2}{y^4}\big)^n+A_{n-1}(\frac{1}{y},\frac{y'}{y^2})\big(\frac{-y^2y''+2y(y')^2}{y^4}\big)^{n-1}+\cdots\\+A_1(\frac{1}{y},\frac{y'}{y^2})\big(\frac{-y^2y''+2y(y')^2}{y^4}\big)+A_0(\frac{1}{y},\frac{y'}{y^2})\end{aligned}$$

$$=[\lambda_1(-y^2y''+2y(y')^2)^n+A_{n-1}(\frac{1}{y},\frac{y'}{y^2})(-y^2y''+2y(y')^2)^{n-1}y^4+\cdots \qquad (3.7)$$

$$+A_1(\frac{1}{y},\frac{y'}{y^2})(-y^2y''+2y(y')^2)y^{4(n-1)}+A_0(\frac{1}{y},\frac{y'}{y^2})y^{4n}]/y^{4n}.$$

By equation (3.6), expression (3.7) equals $\lambda y^N \cdot g$. We get by clearing the denominator of (3.7) that the numerator of (3.7) equals $\lambda y^{4n+N} g$. Writing $g = \lambda_2 (y'')^n + B_{n-1}(y, y')(y'')^{n-1} + \cdots + B_1(y, y')y'' + B_0(y, y')$, we have

$$\lambda_1(-y^2y'' + 2y(y')^2)^n + A_{n-1}(\frac{1}{y}, \frac{y'}{y^2})(-y^2y'' + 2y(y')^2)^{n-1}y^4 + \cdots \tag{3.8}$$

$$+A_1(\frac{1}{y}, \frac{y'}{y^2})(-y^2y'' + 2y(y')^2)y^{4(n-1)} + A_0(\frac{1}{y}, \frac{y'}{y^2})y^{4n}$$
$$= \lambda y^{4n+N}[\lambda_2(y'')^n + B_{n-1}(y, y')(y'')^{n-1} + \cdots + B_1(y, y')y'' + B_0(y, y')].$$

From the LHS of (3.8) we get

$$\lambda_1(-1)^n y^{2n}(y'')^n = \lambda\lambda_2 y^{4n+N}(y'')^n,$$

so $N = -2n$. The RHS of (3.8) says that the $(y'')^0$ coefficient must have order $\geq 4n + N = 4n - 2n = 2n$ in y, and since the LHS has the monomial $\lambda_1 2^n y^n (y')^{2n}$ with order $< 2n$ in y (assuming $n > 0$), it is enough to show that no other term of the LHS can cancel this monomial. The only terms of the LHS with order 0 in y'' are from the last terms of each binomial expansion, so if there did exist such other term, there would exist $l, m \geq 0$ and $k \geq 1$ such that

$$(\frac{1}{y^l})(\frac{y'}{y^2})^m y^{n-k}(y')^{2(n-k)}y^{4k} = y^n(y')^{2n}.$$

Looking at the power of y', we get that $m = 2k$, and so we compare the order of y on each side: we must have $-l - 2(2k) + (n-k) + 4k = n \Leftrightarrow -l = k$. Since $-l \leq 0$ and $k \geq 1$, this is impossible.

$\square$

Proposition C. Let $Q \subset \mathbb{P}^m_R$ be a smooth quadric hypersurface. There are no lifts of Frobenius on $J^1(Q)$.

Proof. Let $Q = V(\sum_{i,j=0}^m a_{ij}x_ix_j)$, where $(a_{ij}) \in GL_{m+1}(R)$, i.e., $\det(a_{ij}) \not\equiv 0 \mod p$. We first show that through an automorphism of $R[x_0, \ldots, x_m]$, Q can be rewritten as $V(\sum_i z_i^2)$, where z_i the image of x_i. Consider $q = \sum_{i,j} a_{ij}x_ix_j$. There are two cases:

Case 1: q has square monomials. WLOG, suppose one of the monomials of q is $a_{00}x_0^2 = ax_0^2$.

Case 1a: $a \in R^\times$. Write

$$\begin{aligned} q &= ax_0^2 + 2ax_0L_0(x_1,\dots,x_m) + Q_0(x_1,\dots,x_m) \\ &= a(x_0^2 + 2x_0L_0 + L_0^2) - aL_0^2 + Q_0 \\ &= a(x_0+L_0)^2 + \widetilde{Q_0}(x_1,\dots,x_m). \end{aligned}$$

The automorphism will send $x_0 \mapsto x_0 + L_0' := \widetilde{z_0}$. For $\widetilde{Q_0}(x_1,\dots,x_m)$, we are placed into either Case 1 or Case 2 and we proceed inductively to rewrite q as a sum of squares.

Case 1b: $a \notin R^\times$. Then the p-adic valuation $v(a)$ is > 0.

If $v(L_0) > 0$ also (see above), then we look at Q_0 and place ourselves into either Case 1 or Case 2. It is not possible that $v(a_{ii}), v(L_i) > 0$ for all i, i.e., to remain in this case indefinitely, because this would mean that all entries of (a_{ij}) are divisible by p, so that $\det(a_{ij}) \equiv 0 \mod p$.

If $v(L_0) = 0$, then the coefficient of some monomial of L_0 has valuation 0. WLOG, say $v(a_{01}) = 0$. Set $x_0 = \widetilde{x_0} - \widetilde{x_1}$, $x_1 = \widetilde{x_0} + \widetilde{x_1}$ to get

$$\begin{aligned} q &= a(\widetilde{x_0} - \widetilde{x_1})^2 + a_{01}(\widetilde{x_0}^2 - \widetilde{x_1}^2) + \cdots \\ &= (a + a_{01})\widetilde{x_0}^2 + \cdots. \end{aligned}$$

Since $a_{01} \in R^\times$, $v(a+a_{01}) = 0$, and we go to Case 1a, with $a+a_{01}$ taking the place of a.

Case 2: Q has no square monomials. Some coefficient a_{ij} is invertible; otherwise, $\det(a_{ij}) \equiv 0 \mod p$. WLOG, say $a_{01} := a \in R^\times$. We have $Q = ax_0x_1 + \cdots$. Substitute $x_0 = \widetilde{x_0} - \widetilde{x_1}$, $x_1 = \widetilde{x_0} + \widetilde{x_1}$, $x_i = \widetilde{x_i}$ for $i \geq 2$. Then

$$Q = a(\widetilde{x_0}^2 - \widetilde{x_1}^2) + \cdots = a\widetilde{x_0}^2 - a\widetilde{x_1}^2 + \cdots$$

and we are reduced to Case 1a.

We see through the above process that the first step of the induction goes through, i.e., that a first change of variables removes the presence of one of the x_i's and gives a single $\widetilde{z_i}^2$ monomial. Assuming WLOG that $x_i = x_0$, the matrix representing q in the new set of variables has the form

$$\begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b_{11} & & b_{1m} \\ \vdots & \vdots & & \dots \\ 0 & b_{1m} & & b_{mm} \end{pmatrix},$$

where $a \in R^{\times}$ and $(b_{ij}) \in GL_m(R)$. We can perform the same process on (b_{ij}) (note that $\det(b_{ij}) \not\equiv 0 \mod p$; otherwise, $\det(a_{ij}) \equiv 0 \mod p$), and so the inductive step holds.

After the induction, we have $q = \sum_i c_i \widetilde{z_i}^2$, with $c_i \in R^{\times}$ for all i. To make the square monomials of q have coefficients of 1, we can make another change of variables $z_i = \sqrt{c_i}(\widetilde{z_i})$, using the fact that every element of $R^{\times}$ has a square root in R, a consequence of Hensel's Lemma.

Now, once we have rewritten $q = \sum_i z_i^2$, we perform a final change of variables. If m is odd (so that there is an even number of variables), we set

$$\begin{aligned} x_0 &= z_0 + iz_1 \\ x_1 &= z_0 - iz_1 \\ x_2 &= z_2 + iz_3 \\ x_3 &= z_2 - iz_3 \\ &\vdots \\ x_{m-1} &= z_{m-1} + iz_m \\ x_m &= z_{m-1} - iz_m. \end{aligned}$$

This yields

$$q = x_0x_1 + x_2x_3 + \cdots + x_{m-1}x_m.$$

By swapping x_1 and x_3 for the purposes of the upcoming step of the computation, we obtain

$$q = x_0x_3 + x_1x_2 + \cdots + x_{m-1}x_m.$$

We have

$$
\begin{aligned}
J^1(Q) =& J^1\big(\mathrm{Proj}\big(\frac{R[x_0,x_1,\ldots,x_m]}{(q)}\big)\\
&= \mathrm{Spf}\ \frac{R\big[\frac{x_1}{x_0},\ldots,,\frac{x_m}{x_0},(\frac{x_1}{x_0})',\ldots,(\frac{x_m}{x_0})'\big]^{\widehat{}}}{\big(\frac{x_3}{x_0}+(\frac{x_1}{x_0})(\frac{x_2}{x_0})+\cdots(\frac{x_{m-1}}{x_0})(\frac{x_m}{x_0}),\delta(\frac{x_3}{x_0}+(\frac{x_1}{x_0})(\frac{x_2}{x_0})+\cdots(\frac{x_{m-1}}{x_0})(\frac{x_m}{x_0}))\big)}\cup\cdots\\
&= \mathrm{Spf}\ \frac{R\big[y_{01},\ldots,y_{0m},y'_{01},\ldots,y'_{0m}\big]^{\widehat{}}}{\big(y_{03}+y_{01}y_{02}+\cdots+y_{0,m-1}y_{0n},\delta(y_{03}+y_{01}y_{02}+\cdots+y_{0,m-1}y_{0m})\big)}\cup\cdots,
\end{aligned}
$$

where $y_{ij} := \frac{x_j}{x_i}$.

We use the computation from the proof that there are no lifts of Frobenius on $J^1(\mathbb{P}^n)$ (step 2 of proof of Theorem A). Observe that modding an element of a quotient ring A/I by the ideal $(p^2A+I)/I$ is computationally equivalent to modding a representative in A of that element by the ideal p^2A+I, by the Third Isomorphism Theorem. Therefore, letting

$$
A_0/I_0 = \mathrm{Spf}\ \frac{R\big[y_{01},\ldots,y_{0m},y'_{01},\ldots,y'_{0m}\big]^{\widehat{}}}{\big(y_{03}+y_{01}y_{02}+\cdots+y_{0,m-1}y_{0m},\delta(y_{03}+y_{01}y_{02}+\cdots+y_{0,m-1}y_{0m})\big)}
$$

and using the same notation as in the $J^1(\mathbb{P}^n)$ proof (see top of p. 22), we get

$$
\begin{aligned}
&-\alpha_{0,1_{2p}}(y_{01},\ldots,y_{0m},y'_{02},\ldots,y'_{0m})y_{01}^{2p^2}+\\
&p\alpha_{0,1_p}^2(y_{01},\ldots,y_{0m},y'_{02},\ldots,y'_{0m})y_{01}^{p^2}\equiv 0 \mod (p^2A_0+I_0).
\end{aligned}
$$

We know that $\alpha_{0,1_{2p}} \equiv 0 \mod (pA_0+I_0)$ and $\alpha_{0,1_p} \equiv 1 \mod (pA_0+I_0)$, so we have

$$
-(p\gamma_1+h_1)y_{01}^{2p^2}+p(p\gamma_2+h_2+1)^2y_{01}^{p^2} = p^2\gamma_3+h_3
$$

for some $h_1,h_2,h_3 \in I_0$, $\gamma_1,\gamma_2,\gamma_3 \in A_0$. This gives

$$-p\gamma_1 y_{01}^{2p^2} + p y_{01}^{p^2} \equiv 0 \mod (p^2 A_0 + I_0),$$

so

$$-p\gamma_1 y_{01}^{2p^2} + p y_{01}^{p^2} = p^2\gamma_4 + \gamma_5(y_{03} + y_{01}y_{02} + \cdots + y_{0,m-1}y_{0m}) + \gamma_6\delta(y_{03} + y_{01}y_{02} + \cdots + y_{0,m-1}y_{0m})$$

for some $\gamma_4, \gamma_5, \gamma_6 \in A_0$.

Examine the coefficient of $y_{01}^{p^2}$ on both sides of the equation. On the LHS, this coefficient is p. For the RHS, observe that the coefficient of $y_{01}^{p^2}$ in $\gamma_5(y_{03} + y_{01}y_{02} + \cdots + y_{0,n-1}y_{0m})$ is 0. Meanwhile,

$$\begin{aligned} D :&= \delta(y_{03} + y_{01}y_{02} + \cdots + y_{0,m-1}y_{0m}) \\ &= \frac{\phi(y_{03} + y_{01}y_{02} + \cdots + y_{0,m-1}y_{0m}) - (y_{03} + y_{01}y_{02} + \cdots + y_{0,m-1}y_{0m})^p}{p}. \end{aligned}$$

Upon expanding this out [recall that $\phi(y_{0i}) = y_{0i}^p + p y_{0i}'$], we find that no monomial of D is a power of y_{01} only, but is also divisible by some other variable, so the coefficient of $y_{01}^{p^2}$ in D is 0. This leaves us to examine $p^2\gamma_4$. But the coefficient of $y_{01}^{p^2}$ in $p^2\gamma_4$ must be divisible by p^2, while p is not.

This concludes the proof in the case that m is odd. If m is even, then for the first $m-1$ variables, we perform the stated change of variables and set $x_m = z_m$ to get

$$q = x_0x_3 + x_1x_2 + \cdots + x_{m-2}x_{m-1} + x_m^2.$$

Tracing through the above proof for the even case, we see that the presence of this extra variable does not have influence.

We have shown that there exists no lift of Frobenius on $J^1(Q)$. $\square$

Proposition D. Let $x = (x_{ij})$ be an n-by-n matrix of indeterminates. There exists no lift of Frobenius on $M_n = \text{Spec } R[x]$ inducing a lift of Frobenius on GL_n.

Proof. We prove the statement for $n = 2$ first.

Let $\phi : M_2 = \text{Spec } R[a, b, c, d] \to M_2$ be a lift of Frobenius. On rings, this is given by $\phi : R[a, b, c, d] \to R[a, b, c, d]$ such that $\phi(a) = a^p + pA$, $\phi(b) = b^p + pB$, $\phi(c) = c^p + pC$, $\phi(d) = d^p + pD$ for some $A, B, C, D \in R[a, b, c, d]$.

Suppose this induces $\tilde{\phi} : GL_2 = \text{Spec } R[a, b, c, d, \frac{1}{ad-bc}] \to GL_2$. This is given on rings by $\tilde{\phi} : R[a, b, c, d, \frac{1}{ad-bc}] \to R[a, b, c, d, \frac{1}{ad-bc}]$. But we have the following lemma.

Lemma: Let $f \in R[a, b, c, d]$ be irreducible. Then

$$R\big[a, b, c, d, \frac{1}{f}\big]^\times = \{\lambda f^m, \lambda \in R^\times, m \in \mathbb{Z}\}.$$

Proof of lemma: Certainly RHS $\subset$ LHS. For any $u \in$ LHS, there exists $v \in R[a, b, c, d, \frac{1}{f}]$ such that $uv = 1$. Since $R[a, b, c, d, \frac{1}{f}] = R[a, b, c, d]_f$, we can write $u = \frac{g}{f^{m_1}}$, $v = \frac{h}{f^{m_2}}$, where $g, h \in R[a, b, c, d]$ and $m_1, m_2 \in \mathbb{Z}$. We may assume that g and h do not have f as a factor because $R[a, b, c, d]$ is a UFD. Thus, $gh = f^{m_1+m_2}$ and since f is irreducible in $R[a, b, c, d]$, either $g \in R[a, b, c, d]^\times = R^\times$ or $h \in R^\times$. Without loss of generality, suppose $h \in R^\times$. Then $f^{m_1+m_2}$ divides g. But we already said that g does not have f as a factor, so $m_1 + m_2 = 0$, and $f^{m_1+m_2} = 1$. This in turn implies that $g = \lambda \in R^\times$, so $u = \frac{\lambda}{f^{m_1}}$, where $g \in R^\times$ and $m_1 \in \mathbb{Z}$. We can rewrite $u = \lambda f^m$, where $m = -m_1 \in \mathbb{Z}$.

Since ring homomorphisms map invertible elements to invertible elements, we must have that $\tilde{\phi}(ad-bc) \in R[a, b, c, d, \frac{1}{ad-bc}]^\times$. It is well known that the determinant polynomial is irreducible, so we can apply the above lemma to $f = ad - bc$, and we have

$$\tilde{\phi}(ad - bc) = \lambda(ad - bc)^m \tag{3.9}$$

for some $\lambda \in R^\times$, $m \in \mathbb{Z}$. This yields

$$(a^p + pA)(d^p + pD) - (b^p + pB)(c^p + pC) = \lambda(ad - bc)^m.$$

Reducing mod p, we get

$$(ad)^p - (bc)^p \equiv \lambda(ad)^m + \lambda \sum_{k=1}^{m} \binom{m}{k} (ad)^{m-k}(-bc)^k,$$

which implies that $m = p$ and $\lambda \equiv 1 \pmod{p}$.

So we have

$$(a^p + pA)(d^p + pD) - (b^p + pB)(c^p + pC) = \lambda(ad - bc)^p,$$

which gives

$$\begin{aligned} &(ad)^p - (bc)^p + p(Ad^p + Da^p - Bc^p - Cb^p) \equiv \\ \lambda\big[(ad)^p - p(ad)^{p-1}bc + &\sum_{k=2}^{p} \binom{p}{k} (ad)^{p-k}(bc)^k\big] \pmod{p^2}. \end{aligned}$$

But every term of the LHS has order $\geq p$ in at least one of a, b, c, or d, while the RHS has the term $-p\lambda(ad)^{p-1}bc$ whose order is less than p in a, b, c, and d. Thus, we arrive at a contradiction.

Now consider arbitrary n. Let $\phi : M_n = \text{Spec } R[x] \to M_n$, where $x = \{x_{ij}\}_{1 \leq i,j \leq n}$. On rings this is given by $\phi : R[x] \to R[x]$ such that $\phi(x_{ij}) = x_{ij}^p + pA_{ij}$ for $A_{ij} \in R[x]$. Assume this induces $\tilde{\phi} : GL_n = \text{Spec } R[x, \det(x)^{-1}] \to GL_n$. [With $\det(x)^{-1}$, we consider x to be the n by n matrix (x_{ij}).] In exactly the same way as for $n = 2$, one shows that

$$R[x, \det(x)^{-1}]^\times = \{\lambda \det(x)^m, \lambda \in R^\times, m \in \mathbb{Z}\}.$$

Hence, we have

$$\tilde{\phi}(\det(x)) = \lambda \det(x)^m \tag{3.10}$$

for some $\lambda \in R^\times$, $m \in \mathbb{Z}$.

Let us specialize by setting $x_{ij} = \begin{cases} 1 & \text{if } i = j > 2 \\ 0, & \text{if } i \neq j;\ i, j > 2 \end{cases}$. Then we have

$$x = \begin{pmatrix} x_{11} & x_{12} & 0 & \cdots & \cdots & 0 \\ x_{21} & x_{22} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & 1 & & & \vdots \\ \vdots & \vdots & 0 & 1 & & \\ \vdots & & \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & & 0 & \cdots & 1 \end{pmatrix}.$$

Applying Laplace's determinant formula $\det x = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} M_{ij}$, where M_{ij} is the determinant of the $n-1$ by $n-1$ matrix obtained removing the ith row, jth column of x, we get in this specialized case, with $i = 1$, that

$$\begin{aligned} \det(x) &= x_{11}M_{11} - x_{12}M_{12} \\ &= x_{11}(x_{22} \cdot 1 \cdots 1) - x_{12}(x_{21} \cdot 1 \cdots 1) \\ &= x_{11}x_{22} - x_{12}x_{21}. \end{aligned}$$

Thus, equation (3.10) above reduces to equation (3.9), which we showed cannot hold, so we are done.

□

4. Appendix: Motivation and Further Background

We begin this final section by discussing the relevance of lifts of Frobenius in the larger context of "arithmetic differential equations," to be defined shortly. First recall the situation for smooth manifolds in differential geometry ([Le03], p. 435). Let M be an m-dimensional smooth manifold with local coordinates $x = \{x_1, \ldots, x_m\}$. A smooth vector field $V : M \to TM$ is given by $V = \sum_{i=1}^{m} V_i \frac{\partial}{\partial x_i}$ for functions $V_i \in C^\infty(M)$. An integral curve of V is a smooth curve $\gamma : J \to M$, where $J \subset \mathbb{R}$ an open interval is typically considered as a time domain, such that $\gamma'(t) = V_{\gamma(t)}$, i.e. the tangent vector to the curve γ at each point is determined by the value of the vector field V at that point. To find the integral curves $\gamma(t) = (x_1(t), \ldots, x_m(t))$ for some vector field, one solves the system of ordinary differential equations

$$\{x_1'(t) = V_1(x(t)), \ldots, x_m'(t) = V_m(x(t))\}$$

for $x_1(t), \ldots, x_m(t)$. Moreover, the collection of all integral curves forms a flow on M.

There is an analogous situation in arithmetic. For simplicity, we consider an affine p-adic formal scheme X. Any $g \in \mathcal{O}(J^n(X))$ represented by a power series G induces a map $g_* : X(R) \to R$ by $g_*(a) := G(a, \delta a, \ldots, \delta^n a)$. The following definition is found in [BM13], p. 4.

Definition 4.1. (a) A system of arithmetic differential equations of order n on X is a subset $\mathcal{E}$ of $\mathscr{O}(J^n(X))$.

(b) A solution of $\mathcal{E}$ is an R-point $a \in X(R)$ such that $g_*(a) = 0$ for all $g \in \mathcal{E}$. The set of all solutions of $\mathcal{E}$ is denoted $Sol(\mathcal{E})$.

(c) Given a p-derivation $\mathscr{D} : \mathcal{O}(X) \to \mathcal{O}(X)$, the δ-flow of $\mathscr{D}$ is the system of arithmetic differential equations of order 1, denoted by $\mathcal{E}(\mathscr{D})$, which is the ideal in $\mathcal{O}(J^1(X))$ generated by elements of the form $\delta g_j - \mathscr{D} g_j$, where $g_j \in \mathcal{O}(X)$ generate $\mathcal{O}(X)$ as an R-algebra.

Remark 4.2. For a δ-flow $\mathcal{E}(\mathscr{D})$ as in (c) above,

$$Sol(\mathcal{E}(\mathscr{D})) = \{a \in X(R) : \delta g_j(a) = \mathscr{D} g_j(a)\, \forall j\}.$$

Whereas integral curves $\gamma(t)$ satisfying $\gamma'(t) = V_{\gamma(t)}$ are solutions to ODE's determined by a vector field (derivation) V, the solutions to a system of ADE's are integral points (R-points) satisfying an analogous property.

Example 4.3. Let $X = \mathbb{A}^1 = \mathrm{Spf}\ R[x]\hat{}$, and let $\mathscr{D} : \mathcal{O}(X) = R[x]\hat{} \to R[x]\hat{}$ be the p-derivation on X given by $\mathscr{D}(x) = x^2$. Note that x generates $R[x]\hat{}$ as a topological R-algebra; $\mathcal{E}(\mathscr{D}) = (x' - x^2) \subset \mathcal{O}(X)$ is the δ-flow of $\mathscr{D}$. Also,

$$X(R) = \mathrm{Hom}(\mathrm{Spf} R, X) \simeq \mathrm{Hom}_{ring}(R[x]\hat{}, R) = R,$$

so

$$Sol(\mathcal{E}(\mathscr{D})) = \{a \in R : \delta a = a^2\}.$$

The following is some of the motivation from physics for looking at p-derivations on the first jet space $J^1(X)$ of a scheme $X = \mathrm{Spec}\frac{R[x]}{(f)}$. First recall from Definition 1.21 that a normalized p-derivation $\mathscr{D} : \mathcal{O}(J^1(X)) \to \mathcal{O}(J^1(X))$ is a p-derivation satisfying $\mathscr{D}(x) = x'$. The differential geometric counterpart of $J^1(X)$ is the tangent bundle TM of a manifold M. If M has local coordinates $\mathbf{q} = (q_1, \ldots, q_n)$, then the local coordinate functions of TM are $(\mathbf{q}, \mathbf{p})$, where $\mathbf{p} = \dot{\mathbf{q}}$, and we consider a vector field θ on TM to be analogous to $\mathscr{D}$ if

$$\theta(q_i) = \dot{q}_i := p_i$$

for each i.

A special case of a normalized vector field is a Hamiltonian vector field defined by an energy (Hamiltonian) function on the phase space of a mechanical system given by coordinates $(\mathbf{q}, \mathbf{p})$:

$$E(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\mathbf{q}) = K(\mathbf{p}) + V(\mathbf{q}),$$

where K and V are kinetic and potential energies, respectively. Indeed, the Hamiltonian vector field

$$\theta = \sum_{i=1}^{n} \left(\frac{\partial E}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial E}{\partial q_i}\frac{\partial}{\partial p_i}\right)$$

satisfies for each i

$$\theta(q_i) = \frac{\partial E}{\partial p_i} = p_i = \dot{q}_i.$$

Moreover, along the flow determined by θ, the energy function E is constant– that is, energy is conserved. (cf. [Le03], pp. 484-488)

Let us now state some already known facts about lifts of Frobenius. They provide some of the motivation and context for our results.

Proposition 4.4. Let X be an affine, formally smooth p-adic formal scheme over R. Then X has a lift of Frobenius.

Proof. Write $X =$ Spf B. We prove by induction that for all $n \geq 1$, there exists a lift of Frobenius $\phi_n : B/p^nB \to B/p^nB$ compatible with that on R, and by the inverse limit functor we will get a lift of Frobenius $\phi : B \to B$. The base case $n = 1$ holds since we have $\phi_1 : B/pB \to B/pB$ the Frobenius map, as B/pB has characteristic p. Now, assume the above holds for n and show its holds for $n+1$. We have the following diagram:

$$\begin{tikzcd}
 & R/p^{n+1}R \arrow[r, "\iota"] \arrow[dl, "\overline{\phi_R}"] & B/p^{n+1}B \arrow[dl, dotted, "\phi_{n+1}"'] \arrow[dr, "\pi"] & \\
R/p^{n+1}R \arrow[r, "\iota"] & B/p^{n+1}B \arrow[r, "\pi"] & B/p^nB & B/p^nB, \arrow[l, "\phi_n"]
\end{tikzcd}$$

where π and ι are the canonical projection and injection, respectively, and $\overline{\phi_R}$ is induced from $\phi_R : R \to R$ of Proposition 1.8. One checks that $\pi \circ \iota \circ \overline{\phi_R} = \phi_n \circ \pi \circ \iota$, so by definition of smoothness we get a map ϕ_{n+1} making the diagram commute. The commutativity of the right half gives compatibility of ϕ_{n+1} with ϕ_n, and the commutativity of the left half of the diagram says that ϕ_{n+1} extends ϕ_R.

□

We have the following variant of Proposition 4.4. In what follows a formal scheme will be called smooth if it is the completion of a smooth scheme.

Proposition 4.5. Given Y_2 an affine smooth formal scheme over another affine smooth formal scheme Y_1, with maps $\phi_{Y_2,Y_1} : Y_2 \to Y_1$ and $\pi : Y_2 \to Y_1$ such that $\phi_{Y_2,Y_1} \mod p$ is Frobenius composed with π, then there exists a lift of Frobenius $\phi_{Y_2} : Y_2 \to Y_2$ such that $\pi \circ \phi_{Y_2} = \phi_{Y_2,Y_1}$.

Proof. Write $Y_1 = \operatorname{Spf} A$, $Y_2 = \operatorname{Spf} B$. Then the statement can be rephrased as follows: let $\gamma : A \to B$ be a morphism of rings, and assume that for all $n \geq 1$, A/p^nA and B/p^nB are smooth over R/p^nR and B/p^nB smooth over A/p^nA. Assume we have $\phi_{A,B} : A \to B$ such that $\phi_{A,B}(a) \equiv a^p \pmod{p}$ for all $a \in A$. We must find a $\phi_B : B \to B$ such that $\phi_B(b) \equiv b^p \pmod{p}$ for all $b \in B$ and $\phi_B(a) = \phi_{A,B}(a)$ for all $a \in A$. This can be proven using the same basic argument as was done for Proposition 4.4. We have a commutative diagram

$$\begin{tikzcd}
A/p^{n+1}A \arrow[r, "\overline{\gamma}"] \arrow[d, "\overline{\phi_{A,B}}"] & B/p^{n+1}B \arrow[dl, dotted, "\phi_{n+1}"] \arrow[dr, "\pi"] & \\
B/p^{n+1}B \arrow[r, "\pi"] & B/p^nB & B/p^nB. \arrow[l, "\phi_n"]
\end{tikzcd}$$

One checks that $\phi_n \circ \pi \circ \overline{\gamma} = \pi \circ \overline{\phi_{A,B}}$. We get ϕ_{n+1} as in the diagram, and so this means ϕ_{n+1} is a lift of Frobenius on $B/p^{n+1}B$ that extends $\overline{\phi_{A,B}}$. By functoriality, we get ϕ_B as desired. $\square$

Proposition 4.6. Let X be smooth and $J^1(X)$ be affine. Then $J^1(X)$ has a normalized lift of Frobenius.

Proof. Recall that a normalized p-derivation $\delta_{J^1(X)}$ on $J^1(X)$ sends x to x'; equivalently, the corresponding normalized lift of Frobenius sends x to $x^p + px'$.

We have the lift of Frobenius $\phi : \mathcal{O}(X) \to \mathcal{O}(J^1(X))$ induced by the universal p-derivation of Definition 1.11. By the preceding proposition, letting $Y_2 = J^1(X)$ and $Y_1 = X$, as $J^1(X)$ is smooth over X, we get a lift of Frobenius $\phi_{J^1(X)} : \mathcal{O}(J^1(X)) \to \mathcal{O}(J^1(X))$ that extends ϕ. In particular, this means that $\phi_{J^1(X)}(x) = \phi(x) = x^p + p\delta x = x^p + px'$. $\square$

For the sake of completeness we end our discussion by stating some known results about lifts of Frobenius on curves of genus 1 (Theorems 4.7-4.8) and ≥ 2 (Theorems 4.9-4.10), respectively.

For Theorems 4.7 and 4.8, let E be an elliptic curve over R. For the definition of "canonical lift" we refer to [Me72].

Theorem 4.7. [Me72] E is a canonical lift if and only if it has a lift of Frobenius.

Theorem 4.8. [BuSa14] $J^1(E)$ has a lift of Frobenius if and only if E has a lift of Frobenius.

For Theorems 4.9 and 4.10, let X be a smooth projective curve of genus $g \geq 2$.

Theorem 4.9. [Ray83] X has no lifts of Frobenius.

Theorem 4.10. [Bu96] $J^1(X)$ is affine; in particular it has lifts of Frobenius.

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