4-1-1975

Data Pulse Design - Realizability Considerations

Gerald Foschini

Andres C. Salazar

J. Inst. Maths Applics (1975) **15**, 225–237

Data Pulse Design—Intersymbol Interference Aspects

G. J. FOSCHINI AND A. C. SALAZAR
Bell Laboratories, Holmdel, New Jersey 07733, *U.S.A.*

[Received 19 July 1973 and in revised form 1 February 1974]

Past theoretical intersymbol interference studies aimed at finding optimal overall network characteristics have not given much attention to design limitations. In this paper we address the problem of identifying, from an ensemble of filters, that optimal filter design which maximizes a sampling signal to intersymbol interference noise power ratio. Sufficient conditions are given for such non-linear extremal problems to be well posed. Continuing under mild assumptions, attention is then focused on networks of parallel structure. In this case a procedure for constructing a solution is given. The parallel structure solution provides a partial solution to more complex network design problems. Examples of R–C and bandlimited network ensembles are considered in detail. Finally it is shown that these non-linear extremal problems generally lie outside the realm of more standard frequency domain norm minimization problems such as weighted least squares and minimax.

The development is general enough to encompass design parameters having components in function space.

1. Introduction

To REDUCE intersymbol interference (ISI) in synchronous data transmission systems, an objective has been to synthesize a pulse, vanishing at all but one sampling time. Such a pulse gives infinite signal to ISI noise ratio (ρ). In practice, synthesizability limitations and timing inaccuracies result in pulse streams that are not ISI free. Because of the inevitability of ISI the problem becomes one of minimizing it subject to actual design constraints.

Now $t^{-1} \sin t$ requires the least bandwidth for transmitting data at a specified speed without ISI but its discontinuous spectrum must be approximated in practice. The pulse family obtained by convolving the flat spectrum of $t^{-1} \sin t$ with any other pulse spectrum offers many gradual roll-off spectra for which the corresponding time functions vanish at the required sampling times (since convolution in the frequency domain corresponds to multiplication in the time domain). Perhaps the best known of these resultant spectra is the raised cosine which has twice the bandwidth of $t^{-1} \sin t$. While smoothing the edges of the frequency response is an important step towards making the output pulse shape realizable, the solution to the actual pulse design problem is hardly complete.

There is another way to view this gap between the pulse designs suggested by theoretical ISI work and available network structures. Consider an ensemble of network transfer characteristics $\{P(i\omega, \lambda)\}_{\lambda \in \Lambda}$ where λ is an adjustable vector parameter. A network from the family is to be selected and pulsed periodically by a fairly and independently signed sequence of delta functions. We then confront the problem of finding the settings of λ which would maximize ρ. But the maximization of ρ relative to λ can be seen to be a difficult task due to the non-linear nature of the problem. We shall give an abstract mathematical formulation of this problem. Existence of a solution will be demonstrated under fairly general conditions. Then we shall analyse the special case when $P(i\omega, \lambda)$ depends linearly on some components of λ (the optimization problem remains non-linear). For the latter case we shall outline a procedure which leads to the optimal settings for the components of λ. We shall demonstrate the procedure by exercising it for particular network cases.

We stress that the focus in this paper is on intersymbol interference and to avoid obfuscating the presentation, we shall not take into account crosstalk, multi-level transmission, frequency weighted power constraints, or additive noise. In closing we shall show how such considerations can be accommodated for an important class of network structures without changing the essential form from that of a pure intersymbol interference problem.

The ideas in this paper have their roots in Foschini (1970).

We shall need the following notation:

$$\|f\|_p = \left(\int |f|^p\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty$$

$\|f\|_\infty$ = ess sup$|f|$ which for our purposes we can take as sup$|f|$ and, for the most part, max$|f|$

$L_p(a, b)$ the space of all complex functions on (a, b) for which $\|f\|_p < \infty$

$P(i\omega)^*$ the complex conjugate of $P(i\omega)$

$\bar{A}$ closure of set A

$\mathbf{F}$ the Fourier transform operator on $L_2(-\infty, +\infty)$

$$\mathbf{F}(f(t)) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\, dt = F(\omega) \quad \text{for } f(t)$$

sufficiently well behaved. Note capital letters are used for transforms while lower case letters are used in the time domain

$\langle \mathbf{w}, \mathbf{v} \rangle = \sum_i^n w_i v_i^*$ or $\int wv^*$ depending on whether $\mathbf{w}$, $\mathbf{v}$ are Euclidean vectors or square integrable functions

$C^k[a, b]$ set of functions on $[a, b]$ with k continuous derivatives where the topology is defined by the semi-norms

$$\left\{ q_l(c(t)) = \max_{[a,b]} \left| \frac{d^l c(t)}{dt^l} \right| \right\}_{l=0}^{k}$$

T the time domain under consideration

$\Phi_{(a,b)}$ the indicator function of the interval (a, b) which is defined to be one on (a, b) and zero elsewhere on R, the real line

2. Preliminaries

The data signal $s(t)$ is composed of an equally spaced stream of independently signed copies of a single pulse $p(t)$ where both signs are equally probable and

$$s(t) = \sum_{n=-\infty}^{\infty} a_n p(t+\tau+2n\pi), \qquad a_n = \pm 1.$$

For convenience we have normalized the system information rate to $1/2\pi$ bits/s. Thus, the extraction of the information being conveyed by the polarity of each pulse is performed at the sample times $\{\tau+2n\pi\}_{n=-\infty}^{\infty}$. Hence, at time $t = 0$ the information-bearing portion of the signal is $a_0 p(\tau)$ and the ISI noise portion is given by

$$\sum_{n\neq 0} a_n p(\tau+2n\pi).$$

The signal power is then $p^2(\tau)$ while the average ISI noise power is

$$\sum_{n\neq 0} p^2(\tau+2n\pi).$$

It then follows that the signal to ISI noise power ratio function $\rho(\tau)$ is given by

$$\rho(\tau) = \frac{p^2(\tau)}{\sum_{n\neq 0} p^2(\tau+2\pi n)} = \left\{\frac{\sum_{-\infty}^{+\infty} p^2(\tau+2\pi n)}{p^2(\tau)} - 1\right\}^{-1}$$

(so long as $p(\tau) \neq 0$). Next we assume that $p(t)$, the data pulse shape, is parametrized by a vector $\lambda \in \Lambda$, that is $p(t) = p(t, \lambda)$. The parameter λ provides the flexibility needed to represent the impulse responses of a channel (or filter) ensemble. From the above expression for $\rho(\tau)$ we see that the problem of finding the network which yields minimum intersymbol interference can be portrayed as the search for λ^{opt} which maximizes the temporal maximum of $\alpha(\tau, \lambda)$ where

$$\alpha(\tau, \lambda) \triangleq \frac{p^2(\tau, \lambda)}{\sum_{-\infty}^{+\infty} p^2(\tau+2\pi n, \lambda)} \quad \text{since} \quad \frac{d}{d\alpha}(\rho(\alpha) \triangleq \alpha(1-\alpha)^{-1}) > 0.$$

In following sections we present a precise mathematical treatment of this problem (which is referred to as the Problem). The sensitivity function $\rho(\tau, \lambda^{\text{opt}})$ is also of interest to us. A point t^{opt} of global maximum of $\rho(\tau, \lambda^{\text{opt}})$ is an optimal sampling time and $\rho(\tau, \lambda^{\text{opt}})$ describes the sensitivity of ρ to a uniform timing shift. The next section gives conditions which guarantee existence of t^{opt} and λ^{opt}.

3. Existence Theorem

To obtain a result lending insight into the Problem we begin by letting Λ be a compact metric space. (In applications such as those involving time-varying networks or filters employing tapered transmission lines the set Λ could have a function space factor. For this reason we do not restrict Λ to be Euclidean subset.) Let T be $(-\infty, +\infty)$, or T can be taken to be $[0, \infty)$ which is the appropriate domain for causal signals. For any $f\colon T \to R$ we shall write $\sum f(t+2\pi n)$ generically for $\sum_{n=-\infty}^{+\infty} f(t+2\pi n)$ when $T = (-\infty, +\infty)$ or $\sum_{n=0}^{\infty} f(\bar{t}+2\pi n)$, $\bar{t} = t(\text{mod } 2\pi)$ when $T = [0, \infty)$.

Definition. A family of real functions $\{p(t, \lambda)\}_{\lambda\in\Lambda}$ defined and continuous on $T\times\Lambda$ is called *admissible* if there exists

(a) a unimodal $L_2(T)$ function $p_*(t)$ for which $|p(t, \lambda)| \leqslant p^*(t)$ holds uniformly in λ.

(b) a constant $A > 0$ such that $\sum p^2(t+2\pi n, \lambda) \geqslant A$ holds uniformly in t and λ.

THEOREM. *Let* $\beta(\lambda) = \|\alpha(t, \lambda)\|_\infty$. *If* $\{p(t, \lambda)\}_{\lambda \in \Lambda}$ *is admissible then for at least one point in* Λ

$$\sup_{\Lambda} \beta(\lambda)$$

is achieved.

Proof. Since Λ is compact it is enough to show that $\beta(\lambda)$ is continuous. Despite the continuity assumption on $T \times \Lambda$, continuity of $\beta(\lambda)$ is by no means immediate. The difficulty stems primarily from the generality of the notion of admissible family. See Example 1 in Section 5, for a practical family of R–C impulse responses discontinuous at $t = 0$ on $(-\infty, +\infty)$ yet admissible using $T = [0, +\infty)$ (by virtue of the right continuity of the family).

Let $I \subset T$ be compact, we shall freely use elementary results like (a) $I \times \Lambda$ is compact, (b) any continuous $f(t, \lambda)$: $I \times \Lambda \to R$ is uniformly continuous and (c) if $f(t, \lambda)$ is as in (b) then $\lambda_k \to \lambda_\infty$ implies $f(t, \lambda_k) \to f(t, \lambda_\infty)$ uniformly.

If $\lambda_k \to \lambda_\infty$ then $p^2(t, \lambda_k) \to p^2(t, \lambda_\infty)$ uniformly on T. To see this let $\varepsilon > 0$ be given and find $t_\varepsilon \in T$ so that $p_*^2(t) < \varepsilon/2$ for $|t| \geq t_\varepsilon$. On $\{t\colon |t| \leq t_\varepsilon\} \times \Lambda$, $p^2(t, \lambda)$ is uniformly continuous and so we have that for k sufficiently large

$$\max_{|t| < t_\varepsilon} |p^2(t, \lambda_k) - p^2(t, \lambda_\infty)| < \frac{\varepsilon}{2}.$$

Thus for k sufficiently large $\max_{t \in T} |p^2(t, \lambda_k) - p^2(t, \lambda_\infty)| < \varepsilon$ establishing uniform convergence.

Next if $\lambda_k \to \lambda_\infty$ then $\sum p^2(t + 2\pi n, \lambda_k) \to \sum p^2(t + 2\pi n, \lambda_\infty)$ uniformly. First see that by the 2π periodicity of these sums it is enough to establish uniform convergence on $[0, 2\pi)$. We can extend $\sum p^2(t + 2\pi n, \lambda)$ continuously to $[0, 2\pi]$ by defining the 2π value to be $\sum_{-\infty}^{+\infty} p^2(2\pi n, \lambda)$ or $\sum_{1}^{\infty} p^2(2\pi n, \lambda)$ according to whether $T = (-\infty, +\infty)$ or $T = [0, \infty)$ respectively. Now (a) provides $p^2(2\pi n)$ as the constants for a successful Weierstrass–M test to get the continuity of the extension on $[0, 2\pi] \times \Lambda$. By uniform continuity the required uniform convergence follows.

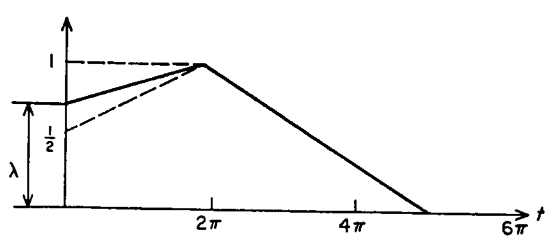

FIG. 1.

From (b) and the triangle inequality, the quotient $\alpha(t, \lambda_k) \to \alpha(t, \lambda_\infty)$ uniformly. Finally, as $k \to \infty$

$$| \|\alpha(t, \lambda_k)\|_\infty - \|\alpha(t, \lambda_\infty)\|_\infty | \leq \|\alpha(t, \lambda_k) - \alpha(t, \lambda_\infty)\|_\infty \to 0$$

we have $\beta(\lambda)$ is continuous and so the proof is completed.

It is not difficult to contrive admissible $\{p(t, \lambda)\}_\Lambda$ on $[0, \infty)$ for which a λ^{opt} guaranteed by the Theorem is such that $\|\alpha(t, \lambda^{opt})\|_\infty$ is not achieved at any t. The following admissible class offers an illustration of this pathology.

$$p(t, \lambda) = \Phi_{[0,5\pi]} \times \left[\min\left(\lambda + \left(\frac{1-\lambda}{2\pi} \right) t, (5 - t\pi^{-1}) \tfrac{1}{3} \right) \right]$$
$$\Lambda = [0{\cdot}5, 1]$$

However, we have the following result.

COROLLARY 1. *If* $T = (-\infty, +\infty)$ *then for each* λ^{opt} *there exists at least one* t^{opt} *at which* $\|\alpha(t, \lambda^{opt})\|_\infty^-$ *is achieved.*

Proof. From the proof of the Theorem we can see that if the admissible family is continuous then so is $\alpha(t, \lambda)$ and $\{\lim_{|t|\to\infty} \alpha(t, \lambda) = 0\}_\Lambda$.

COROLLARY 2. *If* $T = [0, \infty)$ *but* $\{p(0, \lambda) = 0\}_\Lambda$ *the pathology cannot occur.*

Proof. It is equivalent to replace $[0, \infty)$ by $(-\infty, +\infty)$ extending the family so that they each vanish for $t < 0$. So Corollary 2 follows from Corollary 1.

4. Parallel Structures

Henceforth we shall be concerned with certain admissible families which we shall term parallel structures. To introduce the structure, we shall need an underlying Hilbert space $\mathscr{H}$. Let Λ_* be a compact metric space and suppose that for each (t, λ_*), $\mathbf{h}(t, \lambda_*) \in \mathscr{H}$ and satisfies $\sum \|\mathbf{h}(t+2\pi n, \lambda_*)\|^2 < \infty$. We require that for some metric topology (not necessarily the one associated with Hilbert space convergence), $\Lambda_1 \subset \mathscr{H}$ be compact and

$$\{\langle \lambda_1, \mathbf{h}(t, \lambda_*)\rangle\}_{\Lambda \triangleq \Lambda_1 \times \Lambda_*}$$

be admissible. If t^{opt} exists (e.g., the hypothesis of Corollary 2 is satisfied) then this admissible family is called a parallel structure.

The problem for parallel structures becomes

$$\max_{t,\lambda} \frac{\langle \lambda_1, \mathbf{h}(t, \lambda_*)\rangle^2}{\sum \langle \lambda_1, \mathbf{h}(t+2\pi n, \lambda_*)\rangle^2} \qquad \text{where} \quad \lambda = (\lambda_1, \lambda_*).$$

We give a procedure for attacking the Problem. Two consistency conditions sufficient to allow this procedure to continue are listed along the way.

Now for each (t, λ_*) the Schwarz inequality implies

$$\phi(\xi, \eta) \triangleq \sum \langle \xi, \mathbf{h}(t+2\pi n, \lambda_*)\rangle \langle \eta, \mathbf{h}(t+2\pi n, \lambda_*)\rangle$$
$$\leqslant \|\xi\| \cdot \|\eta\| \sum \|\mathbf{h}(t+2\pi n, \lambda_*)\|^2$$

for all ξ and η in $\mathscr{H}$. Since $\phi(\xi, \eta)$ is a bounded bilinear functional there exists (Halmos, 1951) a unique bounded linear operator $\mathbf{B}(t, \lambda_*)$ on $\mathscr{H}$ such that

$$\phi(\xi, \eta) = \langle \xi, \mathbf{B}(t, \lambda_*)\eta \rangle.$$

By symmetry $\mathbf{B}(t, \lambda_*)$ is self adjoint and since $\phi(\xi, \xi) \geqslant 0$ it is positive semi-definite. Let $\mathbf{B}^{\frac{1}{2}}(t, \lambda_*)$ be its non-negative definite self adjoint square root (Treves, 1967; p. 488).

Condition 1. $\mathbf{B}^{-1}$ is well defined on $\{\mathbf{h}(t, \lambda_*)\}_{T \times \Lambda_*}$.

Again we have a self adjoint square root $\mathbf{B}^{-\frac{1}{2}}(t, \lambda_*)$ and so the problem becomes

$$\max_{t,\lambda} \frac{\langle \mathbf{B}^{+\frac{1}{2}}(t, \lambda_*)\lambda_1, \mathbf{B}^{-\frac{1}{2}}(t, \lambda_*)\mathbf{h}(t, \lambda_*)\rangle^2}{\langle \mathbf{B}^{\frac{1}{2}}(t, \lambda_*)\lambda_1, \mathbf{B}^{\frac{1}{2}}(t, \lambda_*)\lambda_1\rangle}.$$

By the Schwarz inequality

$$\sup_{t,\lambda*} \langle \mathbf{B}^{-\frac{1}{2}}(t, \lambda_*)\mathbf{h}(t, \lambda_*), \mathbf{B}^{-\frac{1}{2}}(t, \lambda_*)\mathbf{h}(t, \lambda_*)\rangle$$

bounds the quotient.

Condition 2. Λ_1 contains a multiple of each vector of the form $\{\mathbf{B}^{-1}(t, \lambda_*)\mathbf{h}(t, \lambda_*)\}_{T \times \Lambda_*}$.
Then, for each (t, λ_*),

$$\max_{\lambda_1} \frac{\langle \lambda_1, \mathbf{h}(t, \lambda_*)\rangle^2}{\sum \langle \lambda_1, \mathbf{h}(t+2\pi n, \lambda_*)\rangle^2} = \langle \mathbf{h}(t, \lambda_*), \mathbf{B}^{-1}(t, \lambda_*)\mathbf{h}(t, \lambda_*)\rangle \tag{1}$$

whose sup is achieved at some $t^{\text{opt}}, \lambda_*^{\text{opt}}$.

So if we can find a $t^{\text{opt}}, \lambda_*^{\text{opt}}$, maximizing the right-hand side of (1), then

$$\langle \mathbf{h}(t^{\text{opt}}, \lambda_*^{\text{opt}}), \mathbf{B}^{-1}(t^{\text{opt}}, \lambda_*^{\text{opt}})\mathbf{h}(t^{\text{opt}}, \lambda_*^{\text{opt}})\rangle \tag{2}$$

is the value α^{opt} that we have been seeking, and $\lambda_1^{\text{opt}} = \mathbf{B}^{-1}(t^{\text{opt}}, \lambda_*^{\text{opt}})\mathbf{h}(t^{\text{opt}}, \lambda_*^{\text{opt}})$ is an extremal λ_1. Letting

$$\alpha(\tau, \lambda^{\text{opt}}) = \langle \lambda_1^{\text{opt}}, \mathbf{h}(\tau, \lambda_*^{\text{opt}})\rangle^2 \langle \lambda_1^{\text{opt}}, \mathbf{B}(\tau, \lambda_*^{\text{opt}})\lambda_1^{\text{opt}}\rangle^{-1}$$

we point out that

$$\rho(\tau, \lambda^{\text{opt}}) = \alpha(\tau, \lambda^{\text{opt}})(1-\alpha(\tau, \lambda^{\text{opt}}))^{-1}$$

describes the corresponding sensitivity of ρ to a uniform timing shift when λ is optimally set. Note $\alpha^{\text{opt}}(1-\alpha^{\text{opt}})^{-1}$ is the maximum signal-to-noise ratio, the global maximum of $\rho(\tau, \lambda^{\text{opt}})$.

Recapping, we have shown (under quite general conditions) that the search for a $t^{\text{opt}}, \lambda_1^{\text{opt}}, \lambda_*^{\text{opt}}$ can be accomplished by optimizing a function of $t^{\text{opt}}, \lambda_*^{\text{opt}}$ alone and once this is done λ_1^{opt} is given by locating the peak of the sensitivity function of a single variable t given by (1). In lumped parameter network theory the function of $t^{\text{opt}}, \lambda_*^{\text{opt}}$ may appear to be transcendental. In the first R–C network example in the next section we will find that this transcendentiality is only illusory and it is a rational function optimization that is involved. In many applications Λ_* is a single point and so the only true parameter is λ_1.

In applications the effectiveness with which the above procedure can be implemented depends on the extent to which the special structure of the class $\{\mathbf{h}(t, \lambda_*)\}$ and the set Λ_1 simplify matters. Along these lines, we notice that the compactness constraint on Λ_1 can be relaxed to only requiring that the projection of Λ_1 onto the unit surface be compact since α is invariant under a scale change. If $\mathscr{H}$ is finite dimensional and Λ_1 is the unit surface, then Λ_1 is compact relative to the usual topology on $\mathscr{H}$ and Condition 2 is obviously satisfied. Last, we briefly note that Condition 1 is easily satisfied if $\mathbf{B}$ can be shown to be bounded below on $\mathscr{H}$, i.e., $\exists m > 0$, $\forall \mathbf{x} \in \mathscr{H}$, $\|\mathbf{Bx}\| \geqslant m\|\mathbf{x}\|$. It will become apparent in the next section that certain band-limited and lumped parameter network classes are especially suited to simplification of the procedure.

5. Applications

In this section illustrative examples will be presented of pulse filter configurations of the parallel structure genre which satisfy the hypothesis of the existence theorem discussed earlier. In addition the procedure of the previous section will be exercised to obtain the parameter settings which optimize the signal to ISI noise ratio for two cases.

Example 1. *A Class of R–C Networks*

Let $\{l_i\}_{i=1}^k$ be a finite sequence of positive integers. Let

$$\Lambda = \{\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = (\lambda_{11}, \lambda_{21}, \ldots, \lambda_{k1}, \lambda_{12}, \lambda_{22}, \ldots, \lambda_{k2})\}$$

be a non-empty compact subset of R^{2k} so that $\boldsymbol{\lambda}_1$ is never zero and $\boldsymbol{\lambda}_2$ never has a non-positive component. The set $\{P(i\omega, \boldsymbol{\lambda})\}_{\lambda\in\Lambda}$ represents the family

$$\left\{\sum_{j=1}^{k} \lambda_{j1}(i\omega+\lambda_{j2}^{-1})^{-l_j}\right\}_{\lambda\in\Lambda}$$

which consists of transfer characteristics of R–C networks.

It can be shown that the corresponding impulse responses $\{p(t, \boldsymbol{\lambda})\}_{\lambda\in\Lambda}$ form an admissible family. Conditions (a) and (b) follow from the fact that there exist constants $\hat{\lambda}_2$ and λ_2 and a time t_0 such that

$$|p(t, \boldsymbol{\lambda})| \leqslant \exp(-t/\lambda_2) \quad \text{and} \quad \exp(-t/\hat{\lambda}_2) \leqslant |p(t, \boldsymbol{\lambda})| \quad \text{for } t > t_0.$$

Hence there is at least one optimal $\boldsymbol{\lambda} \in \Lambda$ which minimizes ISI.

We can compute the power of the signal plus ISI by recognizing that the power expression is of the form of linear combinations of differentiated geometric progressions. Thus letting $r_j = e^{-2\pi/\lambda_{j2}}$, $\mathbf{D}_2$ be the diagonal matrix with λ_{j2} in the jjth position and $\bar{t} = t \pmod{2\pi}$ we get

$$P_{S+N} = \langle e^{-\bar{t}\mathbf{D}_2^{-1}}\boldsymbol{\lambda}_1, [R_{ij}(\bar{t}, r_i, r_j)]\, e^{-\bar{t}\mathbf{D}_2^{-1}}\boldsymbol{\lambda}_1\rangle$$

where the matrix entries $R_{ij}(\bar{t}, r_i, r_j)$ are rational functions of the independent variables.

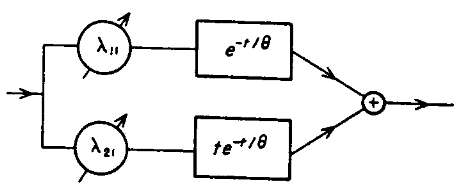

FIG. 2.

On the other hand the signal power on the interval $[(m-1)2\pi, m(2\pi))$ is of the form

$$\langle e^{-\bar{t}\mathbf{D}_2^{-1}}\boldsymbol{\lambda}_1, \mathbf{q}(\bar{t}+2\pi m, r_1^m, \ldots, r_k^m)\rangle^2$$

where $\mathbf{q}$ is a vector with components that are rational functions in their arguments. Notice that the change of variables $\boldsymbol{\mu} = e^{-\bar{t}\mathbf{D}_2^{-1}}\boldsymbol{\lambda}_1$, removes the transcendental nature of the optimization. So the optimization of ρ is observed to reduce to the optimization of a rational function of $(r_1, \ldots, r_k, \bar{t})$ with m as a parameter. If optimal $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k, \bar{t})$ and $\bar{m}$ can be found the solution is complete. A sufficient condition for the existence of these optimal parameters is that $\min\{l_i\}_1^k \geqslant 2$ which implies $p(0, \boldsymbol{\lambda}) = 0$ for $\boldsymbol{\lambda} \in \Lambda$. In some situations this rational function optimization is not required. An important case occurs when $l_i = i$ and λ_2 is a singleton of the form $(\theta, \theta, \ldots, \theta)$. Then the network corresponding to $\{P(i\omega, \lambda)\}_{\lambda \in A}$ is equivalent to the $k-1$ degree Laguerre network of Lee (1963). We determine the optimal first degree Laguerre network by finding the optimal λ_1 in the network depicted by Figure 2.

We compute the value for the power of the signal and ISI noise. Thus

$$P_{S+N} = \sum_{n=0}^{\infty} (\lambda_{11} + \lambda_{21}(t+2\pi n))^2 \, e^{-2(t+2\pi n)\tau/\theta} \qquad (0 \leqslant t < 2\pi)$$
$$= \langle \lambda_1, \mathbf{B}\lambda_1 \rangle$$

where

$$\mathbf{B} = \frac{1}{1-r} \begin{bmatrix} 1 & t+\dfrac{2\pi r}{1-r} \\ t+\dfrac{2\pi r}{1-r} & t^2+\dfrac{4\pi t r}{1-r}+\dfrac{4\pi^2 r(1+r)}{(1-r)^2} \end{bmatrix} e^{-2t/\theta}$$

and where $r = e^{-4\pi/\theta}$ is the progression rate.

From straightforward algebra we see then that the optimal value for α is

$$\alpha^{\text{opt}} = 1-r^2.$$

We note that the value of λ, for which the optimal value of ρ is achieved (viz., $r^{-2}-1$), is

$$\lambda_1^{\text{opt}} = \frac{(1-r)^2}{2\pi} \begin{pmatrix} t^{\text{opt}} + 2\pi\dfrac{1+r}{1-r} \\ -1 \end{pmatrix}$$

where t^{opt} is any point in $[0, 2\pi)$.

For this example we find that α^{opt} and ρ^{opt} do not depend on t^{opt} (on $[0, 2\pi)$), the time of sampled signal, and that sampling the signal at any subsequent interval (the nth interval being $[2\pi n, 2\pi(n+1))$) only detracts from α. The latter result follows by a simple demonstration that the optimal value of α over the nth interval is greater than the optimal value over the $(n+1)$st interval.

Example 2. *Band-limited Networks*

Consider an ensemble of filters with linearly independent transfer characteristics $\{P_j(\omega)\}_1^N$ that are differentiable everywhere, vanish off $|\omega| < \frac{1}{2}$ and satisfy

$$\max_j \int_{-\frac{1}{2}}^{+\frac{1}{2}} |P_j'(\omega)| \, d\omega \triangleq M < \infty.$$

Let $\Lambda = \{(\lambda_1, \lambda_2, \ldots, \lambda_J)\}$ be a compact set in R^J excluding 0, with $a \triangleq \max_A \|\lambda\|$.

Then

$$\mathbf{F}^{-1}\left\{\mathbf{Q} = \left\{\sum_1^J \lambda_j P_j(\omega)\right\}_A\right\}$$

is a parallel structure. For the first admissibility condition notice $aJM/|t|$ bounds any member of $\mathbf{F}^{-1}(\mathbf{Q})$ for $t \geqslant 1$ since $|tp_j(t)| \leqslant M$. Now $p_*(t)$ exists since the $\mathbf{F}^{-1}(\mathbf{Q})$ are uniformly bounded in derivative since for all t

$$|p_j'(t)| \leqslant \int |\omega P_j(\omega)| \, d\omega \leqslant \tfrac{1}{4} \int |P_j(\omega)|^2 \, d\omega < \infty.$$

Now consider

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{j=1}^{N} \lambda_j P_j(\omega)\right|^2 d\omega = \langle \boldsymbol{\lambda}, \mathbf{B}\boldsymbol{\lambda}\rangle \geqslant \|\boldsymbol{\lambda}\|^2 \underline{e}$$

where $\underline{e}$ = minimum eigenvalue of $\mathbf{B} = [b_{ij}]$ and

$$[b_{ij}] = \left[\int_{-\frac{1}{2}}^{\frac{1}{2}} P_i(\omega)P_j(\omega)\, d\omega\right]$$

is a constant non-singular Gram matrix. The $S+N$ power which is invariant under a change in t satisfies

$$\sum_{n=-\infty}^{\infty} \left(\sum_{j=1}^{N} \lambda_j p_j(t+n2\pi)\right)^2 = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sum_{j=1}^{N} \lambda_j P_j(\omega)\right|^2 d\omega \geqslant \frac{\|\boldsymbol{\lambda}\|^2}{\pi}\underline{e} > 0$$

which is the fulfilment of the second condition. We note that the optimization procedure is simplified in this case since $\mathbf{B}$ has no dependence on the time parameter t.

For a numerical illustration of the above consider the band-limited family

$$P(\omega, \boldsymbol{\lambda}_1) = \max(0, (1-(2\omega)^2)^2(\lambda_{11}+\lambda_{21}(2\omega)^2+\lambda_{31}(2\omega)^4))$$

where $\boldsymbol{\Lambda}_1 = \{\boldsymbol{\lambda}_1 \neq 0, P(\omega, \boldsymbol{\lambda}_1) \geqslant 0 \text{ for each } \omega \in (-\frac{1}{2}, +\frac{1}{2})\}$. Note $\boldsymbol{\Lambda} \cup \{0\}$ is closed (and convex) since $\boldsymbol{\Lambda}_1 \cup \{0\} = \bigcap_{|\omega|<\frac{1}{2}} \{\boldsymbol{\lambda} | \langle \boldsymbol{\lambda}_1, (1, \omega^2, \omega^4)\rangle \geqslant 0\}$ an intersection of half spaces.† For all $\gamma > 0$, $\gamma\boldsymbol{\Lambda} = \boldsymbol{\Lambda}$ so the spherical projection of $\boldsymbol{\Lambda}$ is identical to its intersection with the surface of the unit sphere and hence is compact. This condition sets up a well based extremal problem. From $\lambda_1 \neq 0, P(\omega, \boldsymbol{\lambda}_1) \geqslant 0$, and the t-invariance of the $S+N$ power, $t^{\text{opt}} = 0$.

Following the procedure for parallel structures we evaluate the pertinent integrals and obtain

$$\beta(\boldsymbol{\lambda}_1) = \frac{1}{2}\frac{\langle \boldsymbol{\lambda}_1, (1, 7^{-1}, 21^{-1})\rangle^2}{\langle \boldsymbol{\lambda}_1, \boldsymbol{\beta}\boldsymbol{\lambda}_1\rangle}$$

where

$$\mathbf{B} = 5\begin{pmatrix} 7^{-1} & (7.11)^{-1} & 3(7.11.13)^{-1} \\ (7.11)^{-1} & 3(7.11.13)^{-1} & (7.11.13)^{-1} \\ 3(7.11.13)^{-1} & (7.11.13)^{-1} & (11.13.17)^{-1} \end{pmatrix}.$$

We get $\alpha^{\text{opt}} = \alpha(0, \boldsymbol{\lambda}_1^{\text{opt}}) = \frac{572}{630}$ which is achieved at any non-zero multiple of $\boldsymbol{\lambda}_1^{\text{opt}} = (1, -2, 17)$. The positive multiples of $\boldsymbol{\lambda}_1^{\text{opt}}$ are in $\boldsymbol{\Lambda}_1$ since

$$(1-(2\omega)^2)^2(1-2^3\omega^2+2^4.17.\omega^4) = (1-(2\omega)^2)^2[(1-(2\omega)^2)^2+2^8\omega^4]$$

which is positive on $|\omega| < 2^{-1}$. Note

$$\rho^{\text{opt}} = \frac{\alpha^{\text{opt}}}{1-\alpha^{\text{opt}}} = \frac{572}{58} \approx 10\text{ d}B.$$

The sensitivity of ρ to a sampling time shift Δ (for $|\Delta| < 2\pi$) can be analysed from

$$\rho(\Delta) = p^2(\Delta, \boldsymbol{\lambda}_1^{\text{opt}})(2\langle \boldsymbol{\lambda}_1^{\text{opt}}, \boldsymbol{\beta}\boldsymbol{\lambda}_1^{\text{opt}}\rangle - p^2(\Delta, \boldsymbol{\lambda}_1^{\text{opt}}))^{-1}.$$

In the preceding illustration the parallel structures procedure was expedited by the fact that the value of t^{opt} was evident. Let $\{n_k\}_0^\infty$ be a strictly increasing sequence of non-negative integers with $n_0 = 0$. Generally for band-limited families like

$$P_k(\omega, \boldsymbol{\lambda}_1) = (+(i2\omega)^2)^2\langle \boldsymbol{\lambda}_1, ((i\omega)^0, (i\omega)^{n_1}, \ldots, (i\omega)^{n_k})\rangle \qquad (\boldsymbol{\lambda}_1 = (\lambda_{11}, \ldots, \lambda_{k1}) \neq 0)$$

† The envelope technique allows a simple explicit determination of $\boldsymbol{\Lambda}_1$ but we do not need this here.

the bound of $\alpha_k(\tau, \lambda_1)$, given by (1), must be found first to locate t^{opt}. The authors have employed a computer program using Fast Fourier transform and quadrature routines to obtain $\langle \mathbf{h}(t), \mathbf{B}^{-1}\mathbf{h}(t)\rangle$ for such problems. (It is not difficult to argue from Muntz's theorem (Rice, 1964) that if there is a subsequence e of even n_k's such that $\sum_e n_k^{-1} = \infty$ then $\rho_k^{\text{opt}} < \infty$.)

The next example illustrates the occurrence of a non-Euclidean Λ set.

Example 3. *An R–C Filter Continuum*

A filter which can be viewed as a continuous version of a finite and discrete R–C filter we have discussed in Example 1 has output

$$p(t, \boldsymbol{\lambda}) = \int_{\tau_1}^{\tau_2} \lambda(\tau) t\, e^{-t/\tau}\, d\tau,$$

Physically, a model for this filter could be that of an R–C filter with rapidly time-varying lumped element values. In this case $\lambda(\tau)$ denotes the distribution of the time constants.

As an example we choose Λ to be the $\mathbf{C}^0[\tau_1, \tau_2]$ closure of $\{\lambda(\tau)\colon m \leqslant |\lambda(\tau)| \leqslant M_1;\ |\lambda'(\tau)| \leqslant M_2;\ \lambda(\tau) \in \mathbf{C}^1[\tau_1, \tau_2]\}$. It is relatively simple to show that $\{p(t, \boldsymbol{\lambda})\}_{\lambda \in \Lambda}$ forms an admissible family since Λ is uniformly bounded above and below. Now to show Λ is a compact subset of $\mathbf{C}^0[\tau_1, \tau_2]$ we note $A_m = \{\lambda(\tau)\colon |\lambda(\tau)| < m\}$ is an open set in $\mathbf{C}^1[\tau_1, \tau_2]$ since the seminorm $q_0(\boldsymbol{\lambda})$ is continuous. Define

$$\Gamma = \{\lambda(\tau)\colon |\lambda(\tau)| \leqslant M_1;\ |\lambda'(\tau)| \leqslant M_2;\ \lambda(\tau) \in \mathbf{C}^1[\tau_1, \tau_2]\}.$$

It follows that $\bar{\Gamma}$ is $\mathbf{C}^0[\tau_1, \tau_2]$ compact since a bounded subset of $\mathbf{C}^1[\tau_1, \tau_2]$ has compact closure in $\mathbf{C}^0[\tau_1, \tau_2]$ (Treves (1967)). Thus $\bar{\Gamma} - \mathbf{A}_m$ is also closed and compact as well since it is a closed subset of $\bar{\Gamma}$. Clearly, $\overline{\Gamma - \mathbf{A}_m} \subset \bar{\Gamma} - \mathbf{A}_m$ and so $\Lambda = \overline{\Gamma - \mathbf{A}_m}$ is compact, being a closed subset of a compact set.

Evidently $\{p(t, \boldsymbol{\lambda})\}_\Lambda$ satisfies the requirements for being a parallel structure. The Problem is deeper than in the previous examples and it is beyond our scope here to pursue to completion the determination of $\rho^{\text{opt}}, \boldsymbol{\lambda}^{\text{opt}}$, and t^{opt}. Nonetheless, it is interesting to look closer at the form of the problem. The expression for α on $[2\pi m, (m+1)2\pi)$ (m a non-negative integer) is

$$\alpha(t, m, \tau_1, \tau_2) = \frac{\left(\int_{\tau_2}^{\tau_2} \lambda(\theta)(\bar{t} + 2\pi m)\, e^{-(\bar{t}+2\pi m)/\theta}\, d\theta\right)^2}{\sum_{n=0}^{\infty} \left\{\int_{\tau_1}^{\tau_2} \lambda(\theta)(\bar{t} + n2\pi)\, e^{-(\bar{t}+2\pi n)/\theta}\, d\theta\right\}^2}, \qquad (\bar{t} = t(\text{mod } 2\pi)).$$

After summing the denominator and making some simple variable changes we then seek the minimum of $t^{-2}\langle \mathbf{B}_{\bar{t},m}\boldsymbol{\mu}, \boldsymbol{\mu}\rangle$ with the added constraint that $\int \boldsymbol{\mu} = 1$ and where $\mathbf{B}_{\bar{t},m}$ is an integral operator with a rational symmetric kernel

$$\left[\frac{\bar{t}}{1-r} + \frac{4\pi\bar{t}r}{(1-r)^2} + \frac{4\pi^2 r(1+r)}{(1-r)^3}\right] r^{-m}, \qquad r = xy, \quad e^{-2\pi/\tau_1} \leqslant x, \qquad y \leqslant e^{-2\pi/\tau_2}.$$

This example indicates that time-varying filter design problems lead to a challenging class of operator extremal problems. These problems are further enriched by considerations of independent additive noise and a weighted average power constraint on the pulse train.

6. Discussion

First, we inquire as to whether or not within the framework of ordinary lumped parameter networks an ideal (intersymbol interference-free) pulse can be achieved. In other words does there exist a passive lumped parameter network having finite energy in its impulse response, $h(t)$, such that for some $\hat{t}$ the impulse response vanishes at each point in $\{\hat{t}+2\pi n\}_{-\infty}^{+\infty}$ except at $\hat{t}$? B. F. Logan has shown us that such a network cannot exist; since if $h(\hat{t}) \neq 0$ and $\{h(\hat{t}+2\pi n) = 0\}_1^N$, then $H(s)$ (Laplace transform) has at least $N+1$ poles. The proof assumes the opposite, namely that $H(s)$ has N poles or less. Form $g(t) = h(t)+\sum_1^N a_j h(t+2\pi j)$. Now since $h(t)$ has the form of a finite linear combination of terms of the form $t^\alpha e^{\beta t}\Phi[0, \infty)$ we have that $g(t)$ is right continuous. If we can find non-trivial a_j so that $g(t)$ vanishes for $t > \tau$ then by right continuity $g(\hat{t}) = 0$. But $g(\hat{t}) = h(\hat{t})$, an impossibility. To get the choice of a_j form

$$F(s) = \prod_{j=1}^{k} 1-e^{2\pi(s+\lambda_j)},$$

where $\{-\lambda_j\}_1^k$ are the poles of $H(s)$. Viewed as a polynomial in $e^{+2\pi s}$ there are only real coefficients. Note $F(s)H(s)$ has no poles. So $F(s)H(s)e^{s(t-\hat{t})} \to 0$, uniformly in angle in the left half s-plane for $t > \hat{t}$. Taking the standard limit of Bromwich contours closed to the left, yields (via the Cauchy Integral Theorem and Jordan's Lemma) that the inverse transform vanishes for $t > \hat{t}$.

It is very easy to see that an ideal pulse can be approximated arbitrarily closely using lumped parameter networks (e.g., in Example 1, $\lim_{\theta\to 0} \rho^{\text{opt}} = \infty$).

An important case of data pulse design occurs if the output of a lumped parameter network in tandem with a tapped delay line is to be used as the data signalling pulse. The frequency transfer function of the output pulse can be written

$$H(\omega) = \sum_{j=1}^{N} \lambda_{j1}\, e^{-i\Delta_j\omega}\, G(\omega) = \langle \boldsymbol{\lambda}_1, \tilde{\mathbf{G}}(\omega, \boldsymbol{\Delta})\rangle$$

where $G(\omega)$ has M poles and $\boldsymbol{\Delta} = \{\Delta_j\}_{j=1}^N$ is a non-negative set of real numbers. It is straightforward to show that ISI can be completely eliminated in this case by setting $\boldsymbol{\lambda}_1^{\text{opt}}$ to be those coefficients of the delay polynomial obtained by the zero insertion technique used in the previous argument.† Since $H(\omega)$ is the transfer function of a network with parallel structure it is also possible to obtain the same $\boldsymbol{\lambda}_1^{\text{opt}}$ by using our procedure for maximizing ρ if we assume $g(t)$ is continuous. More importantly, when $N < M$ the zero insertion procedure is no longer applicable but the parallel structure procedure still yields the optimal data pulse design for this case.

In the past, filters have been constructed by choosing $\boldsymbol{\lambda}$ so that the resulting characteristic best approximates the ideal in some sense, say weighted least square or minimax. The question arises as to whether the problem we have posed can always be expressed, with p fixed ($1 \leqslant p \leqslant \infty$), as

$$\min_{\boldsymbol{\lambda}\in A,\, \tau\in[0,\, 2\pi]} \|P(\omega, \boldsymbol{\lambda}) - e^{i\omega\tau}\,\Phi_{(-\frac{1}{2}, +\frac{1}{2})}\|_p$$

over the interval $(-\infty, +\infty)$ and if $p < \infty$ the integral is permitted to employ any positive weight function.

† That is, ISI is completely eliminated if $N > M$.

Proposition. No such norm exists.

Proof. We give an example of a filter design problem that cannot be posed in such a norm minimization context.

Let
$$\Lambda = [-1, +1]\times[0, 2\pi]$$
and
$$P(\omega, \lambda) = (1-(2\omega)^2)(1+i\omega\lambda_1)\, e^{i\omega\lambda_2}\, \Phi_{(-\frac{1}{2},+\frac{1}{2})}.$$
Then
$$\|P(\omega, \lambda)-e^{i\omega\tau}\, \Phi_{(-\frac{1}{2},+\frac{1}{2})}\|_p = \|P(\omega, \lambda)\, e^{-i\omega\tau}-\Phi_{(-\frac{1}{2},+\frac{1}{2})}\|_p.$$
Setting $\theta = \lambda_2-\tau$ the problem becomes
$$\min\|[(1-(2\omega)^2)(1+i\omega\lambda_1)\, e^{i\omega\theta}-1]\Phi_{(-\frac{1}{2},+\frac{1}{2})}\|_p$$
where the constraint set is $[-1, +1]\times[0, 2\pi)$. If we can show that each ISI minimizing vector over $[-1, +1]\times\{0\}$ has $\lambda_1 \neq 0$ then each ISI minimizing vector over $[-1, +1]\times[0, 2\pi]$ cannot have both $\lambda_1 = 0$ and $\theta = 0$. To see that each ISI minimizing vector over $[-1, +1]\times\{0\}$ has $\lambda_1 \neq 0$ we note that $\rho(t, \lambda_1)$ associated with $(1-(2\omega)^2)(1+i\omega\lambda_1)$ does not peak at $t = 0$, $\lambda_1 = 0$. Rather, a simple computation shows $t = 0$, $\lambda_1 = 0$ is a critical point for $\rho(t, \lambda_1)$ and the Hessian matrix of the quadratic form expressing the local behaviour about (0, 0) has a negative determinant implying a saddle behaviour. Since $\rho(t, 0)$ peaks at $t = 0$ the ISI minimizing vector has $\lambda_1 \neq 0$. (The optimal pulses are asymmetric and there are at least two of them for if $\hat{\lambda}_1$ is a minimand so is $-\hat{\lambda}_1$.) From the above demonstration it follows that the Problem cannot be cast as such a norm minimization problem since an imaginary component of
$$[(1-(2\omega)^2)(1+\lambda_1 i\omega)\, e^{i\theta\omega}]\Phi_{(-\frac{1}{2},+\frac{1}{2})}$$
only serves to increase the error.

7. Extensions

We have seen in Section 4 that for parallel structures the Problem takes the form of determining $t^{\rm opt}$ and $\boldsymbol{\lambda}_1^{\rm opt}$ so that
$$\max_{t\in T,\, \lambda_1\in A} \frac{\langle \boldsymbol{\lambda}_1, \mathbf{h}(t)\rangle^2}{\langle \boldsymbol{\lambda}_1, \mathbf{B}(t)\boldsymbol{\lambda}_1\rangle} \quad \text{is achieved.}$$
In closing we show that inclusion of a frequency weighted power constraint on the spectral density function of $s(t)$, additive noise, crosstalk and the possibility of a multilevel formal does not alter the essential form of the functional that has occupied our attention in this paper.

(a) Elementary harmonic analysis shows that if a frequency weighted average power limitation on the waveform $s(t)$ is imposed, it amounts to the constraint on $\boldsymbol{\lambda}_1$
$$\langle \boldsymbol{\lambda}_1, \mathbf{A}_1\boldsymbol{\lambda}_1\rangle \leqslant P$$
where P is the allowable power and $\mathbf{A}_1$ is a constant positive definite operator. The resulting constraint set is closed and hence its intersection with $\boldsymbol{\Lambda}_1$ is compact. In finite dimensional problem a constraint set
$$0 \leqslant P_1 \leqslant \langle \boldsymbol{\lambda}_1, \mathbf{A}_1\boldsymbol{\lambda}_1\rangle \leqslant P_2$$
is compact and can be taken for $\boldsymbol{\Lambda}_1$.

(b) Additive second order stationary noise adds a positive constant σ^2 to $\langle \boldsymbol{\lambda}_1, \mathbf{B}_1(t)\boldsymbol{\lambda}_1 \rangle$. Parameterizing the argument of the resulting functional in this case by $\gamma\boldsymbol{\lambda}_1$ we notice that for each $\boldsymbol{\lambda}_1$ the functional increases with $|\gamma|$. The significance of this behaviour is that in applications σ^2 can be added to $\langle \boldsymbol{\lambda}_1, \mathbf{B}_1(t)\boldsymbol{\lambda}_1 \rangle$ in the form $\sigma^2 P^{-1}\langle \boldsymbol{\lambda}_1, \mathbf{A}_1\boldsymbol{\lambda}_1 \rangle$ since the optimal solution is on the surface of the ellipsoid $\langle \boldsymbol{\lambda}_1, \mathbf{A}_1\boldsymbol{\lambda}_1 \rangle = P$.

(c) Crosstalk considerations from neighbouring systems of the same type are accommodated by the (often time varying) quadratic functional $\langle \boldsymbol{\lambda}_1, \mathbf{A}_2(t)\boldsymbol{\lambda}_1 \rangle$.

(d) For $s(t)$ with a multi-level format it is evident that the effect is to merely introduce a constant multiplier into the quadratic forms $\langle \boldsymbol{\lambda}_1, \mathbf{h}(t) \rangle^2, \langle \boldsymbol{\lambda}_1, \mathbf{B}(t)\boldsymbol{\lambda}_1 \rangle$ $\langle \boldsymbol{\lambda}_1, \mathbf{A}_2(t)\boldsymbol{\lambda}_1 \rangle$ and $\langle \boldsymbol{\lambda}_1, \mathbf{A}_1\boldsymbol{\lambda}_1 \rangle$.

So the problem that results from incorporating the considerations (a), (b), (c), and (d) is no different in form than the problem we have treated which considers ISI alone.

We are grateful to F. X. Brophy for programming efforts associated with earlier stages of the procedure reported here. Discussions with him were invaluable in advancing the work to its present state.

References

Foschini, G. J. 1970 *Single transmitter design for binary transmission over a diversity of channels.* June 2, BTL internal memorandum.

Halmos, P. R. 1951 *Introduction to Hilbert space.* New York: Chelsea, p. 39.

Lee, Y. W. 1963 *Statistical theory of communication.* New York: Wiley, p. 487.

Rice, J. R. 1964 *Approximation of functions.* Reading, Mass.: Addison–Wesley, p. 124.

Treves, F. 1967 *Topological vector spaces, Distributions and Kernels.* New York: Academic Press, p. 148.