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Synthesis of Data Pulse Shaping Networks

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ABSTRACT

We treat the problem of identifying, from an ensemble of filters, that optimal filter design which maximizes a sampling signal to generalized noise power ratio. The noise power results from intersymbol interference and ambient noise. By focusing attention on networks of parallel structure a procedure for constructing a network solution is given. An example of a bandlimited network ensemble is presented.

1. INTRODUCTION

To reduce intersymbol interference (ISI) in synchronous data transmission systems, an objective has been to synthesize a pulse vanishing at all but one sampling time. Such a pulse gives infinite signal to ISI noise ratio (ρ). In practice, synthesizability limitations and timing inaccuracies result in pulse streams that are not ISI free. Because of the inevitability of ISI the problem becomes one of minimizing it subject to actual design constraints [1]-[7].

Now $t^{-1}\sin t$ requires the least bandwidth for transmitting data at a specified speed without ISI but its discontinuous spectrum must be approximated in practice. The pulse family obtained by convolving the flat spectrum of $t^{-1}\sin t$ with any other pulse spectrum offers many gradual rolloff spectra for which the corresponding time functions vanish at the required sampling times (since convolution in the frequency domain corresponds to multiplication in the time domain). Perhaps the best known of these resultant spectra is the raised cosine which has twice the bandwidth of $t^{-1}\sin t$. While smoothing the edges of the frequency response is an important step towards making the output pulse shape realizable, the solution to the actual pulse design problem is hardly complete.

There is another way to view this gap between the pulse designs suggested by theoretical ISI work and available network structures. Consider an ensemble of network transfer characteristics $\{P(i\omega,\lambda)\}_{\lambda\epsilon\Lambda}$ where λ is an adjustable vector parameter. A network from the family is to be selected and pulsed periodically by a fairly and independently signed sequence of delta functions. We then confront the problem of finding the settings of λ which would maximize ρ. But the maximization of ρ relative to λ can be seen to be a difficult task due to the nonlinear nature of the problem. Some nonlinear aspects of this problem can be removed if we analyze the special case when $P(i\omega,\lambda)$ depends linearly on some components of λ (the optimization problem remains nonlinear). For this case we shall outline a procedure which leads to the optimal settings for the components of λ. We shall demonstrate the procedure by exercising it for a particular network case.

2. PRELIMINARIES

The data signal $s(t)$ is composed of an equally spaced stream of independently signed copies of a single pulse $p(t)$ where both signs are equally probable and

$$s(t) = \sum_{n=-\infty}^{\infty} a_n p(t+\tau+2n\pi), \quad a_n = \pm 1.$$

For convenience we have normalized the system information rate to $1/2\pi$ bits/sec. Thus, the extraction of the information being conveyed by the polarity of each pulse is performed at the sample times $\{\tau+2n\pi\}_{n=-\infty}^{\infty}$. Hence, at time $t = 0$ the information-bearing portion of the signal is $a_0 p(\tau)$ and the ISI noise portion is given by

$$\sum_{n\neq 0} a_n p(\tau+2n\pi) .$$

The signal power is then $p^2(\tau)$ while the average ISI noise power is

$$\sum_{n\neq 0} p^2(\tau+2n\pi).$$

It then follows that the signal to ISI noise power ratio function $\rho_I(\tau)$ is given by

$$\rho_I(\tau) = \frac{p^2(\tau)}{\sum_{n\neq 0} p^2(\tau+2\pi n)}$$

$$= \left[\frac{\sum_{-\infty}^{+\infty} p^2(\tau+2\pi n)}{p^2(\tau)} - 1\right]^{-1}$$

(so long as $\rho(\tau) \neq 0$). Next we assume that $p(t)$, the data pulse shape, is parametrized by a vector $\lambda \in \Lambda$, that is $p(t) = p(t,\lambda)$. The parameter λ provides the flexibility needed to represent the impulse responses of a channel (or filter) ensemble. From the above expression for $\rho_I(\tau)$ we see that the problem of finding the network which yields minimum intersymbol interference can be portrayed as the search for λ^{opt} which maximizes the temporal maximum of $\alpha(\tau,\lambda)$ where

$$\alpha(\tau,\lambda) \triangleq \frac{p^2(\tau,\lambda)}{\sum_{-\infty}^{+\infty} p^2(\tau+2\pi n,\lambda)}$$

$$\text{since } \frac{d}{d\alpha}\left(\rho_I(\alpha) \triangleq \alpha(1-\alpha)^{-1}\right) > 0.$$

The sensitivity function $\rho_I(\tau,\lambda^{opt})$ is also of interest to us. A point t^{opt} of global maximum of $\rho_I(\tau,\lambda^{opt})$ is an optimal sampling time and $\rho_I(\tau,\lambda^{opt})$ describes the sensitivity of ρ to a uniform timing shift.

It is possible to consider two other types of noise in forming a generalized signal to noise ratio. Specifically, a frequency weighted average power limitation on the waveform $s(t)$ amounts to the constraint:

$$\int_{-\infty}^{\infty} |P(\omega,\lambda)|^2 W(\omega)d\omega = P.$$

Further, if ambient noise in the transmission medium is assumed to be white of variance σ^2, the generalized noise power increases by the addition of the constant σ^2. We will see how the inclusion of these two noise power terms is accounted for in a network synthesis procedure in the next section.

3. PARALLEL STRUCTURES

Henceforth we shall be concerned with certain network families which we term parallel structures. To introduce the structure we assume that the parameterization of the network family is split into two parts, i.e., $\Lambda = \Lambda_1 \times \Lambda_2$. Each $h(t,\lambda_1,\lambda_2)$ belongs to a Hilbert space H with induced norm $\|\cdot\|$ and satisfies

$$\sum \|h(t+2n\pi,\lambda_1,\lambda_2)\|^2 < \infty . \quad (1)$$

The dependence of $h(t,\lambda_1,\lambda_2)$ on λ_2 is linear and thus describes a parallel network formation:

$$h(t,\lambda_1,\lambda_2) = \langle\lambda_1,h(t,\lambda_2)\rangle \quad (2)$$

For example, if $H \equiv E^n$ then

$$h(t,\lambda_1,\lambda_2) = \sum_{K=1}^{n} \lambda_{1K} h(t,\lambda_2)$$

To minimize ISI the following problem is encountered

$$\max_{t,\lambda} \frac{\langle\lambda_1,h(t,\lambda_2)\rangle^2}{\sum \langle\lambda_1,h(t+2\pi n,\lambda_2)\rangle^2}$$

$$\text{where } \lambda = (\lambda_1,\lambda_2) \quad (3)$$

We notice that a generalized signal to noise ratio can be formed by including the average power limitation on $\langle\lambda_1,h(t,\lambda_2)\rangle$ and ambient noise of strength σ^2 in (3). Namely,

$$\max \frac{\langle\lambda_1,h(t,\lambda_2)\rangle^2}{\frac{\sigma^2}{P}\int_{-\infty}^{\infty} \langle\lambda_1,H(\omega,\lambda_2)\rangle^2 W(\omega)d\omega + \sum \langle\lambda_1,h(t+2\pi n,\lambda_2)\rangle^2} \quad (4)$$

Without sacrificing much generality we assume $h(t,\lambda_1,\lambda_2)$ is continuous on $T\times\Lambda$ where T is either $(-\infty,\infty)$ or $[0,\infty)$ and Λ_1 and Λ_2 are both closed subsets of E^n. To guarantee that a solution to the above problem exists* we require that the family $\{h(t_1\lambda_1,\lambda_2)\}$ satisfy two conditions:

A. a unimodal $L_2(T)$ function $h(t)$ exists for which $|h(t_1\lambda_1,\lambda_2)| \leq h(t)$ holds uniformly in λ.

B. a constant $A > 0$ exists such that $\sum h^2(t+2\pi n,\lambda_1,\lambda_2) \geq A$ holds uniformly in t and λ.

We give a procedure for attacking the problem in (3). (An argument similar to what follows can be used for solving the problem in (4). Both problems have the same form.) Two consistency conditions sufficient to allow this procedure to continue are listed along the way.

Now for each (t,λ_2) the Schwarz inequality implies

$$\phi(\xi,\eta) \overset{\Delta}{=} \sum <\xi,h(t+2\pi n,\lambda_2)><\eta,h(t+2\pi n,\lambda_2)>$$
$$\leq \|\xi\|\cdot\|\eta\| \sum \|h(t+2\pi n,\lambda_2)\|^2$$

for all ξ and η in H. Since $\phi(\xi,\eta)$ is a bounded bilinear functional there exists [8] a unique bounded linear operator $B(t,\lambda_2)$ on H such that $\phi(\xi,\eta) = <\xi,B(t,\lambda_2)\eta>$. By symmetry $B(t,\lambda_2)$ is self adjoint and since $\phi(\xi,\xi) \geq 0$ it is positive semidefinite. Let $B^{1/2}(t,\lambda_2)$ be its nonnegative definite self adjoint square root ([9], p. 488).

Condition I: B^{-1} is well defined on $\{h(t,\lambda_2)\}_{T\times\Lambda_2}$.

Again we have a self adjoint square root $B^{-1/2}(t,\lambda_2)$ and so the problem becomes

$$\max_{t,\lambda} \frac{<B^{+\frac{1}{2}}(t,\lambda_2)\lambda_1,B^{-\frac{1}{2}}(t,\lambda_2)h(t,\lambda_2)>^2}{<B^{\frac{1}{2}}(t,\lambda_2)\lambda_1,B^{\frac{1}{2}}(t,\lambda_2)\lambda_1>}.$$

By the Schwarz inequality

$$\sup_{t,\lambda_2} <B^{-\frac{1}{2}}(t,\lambda_2)h(t,\lambda_2),B^{-\frac{1}{2}}(t,\lambda_2)h(t,\lambda_2)>$$

bounds the quotient.

Condition II: Λ_1 contains a multiple of each vector of the form $\{B^{-1}(t,\lambda_2)h(t,\lambda_2)\}_{T\times\Lambda_2}$.

Then, for each (t,λ_2),

$$\max \frac{<\lambda_1,h(t,\lambda_2)>^2}{\sum <\lambda_1,h(t+2\pi n,\lambda_2)>^2}$$
$$= <h(t,\lambda_2),B^{-1}(t,\lambda_2)h(t,\lambda_2)> \qquad (5)$$

So if we can find a t^{opt}, λ_2^{opt}, maximizing the right-hand side of (5), then

$$<h\left(t^{opt},\lambda_2^{opt}\right),B^{-1}\left(t^{opt},\lambda_2^{opt}\right)h\left(t^{opt},\lambda_2^{opt}\right)> \qquad (6)$$

is the value α^{opt} that we have been seeking, and $\lambda_1^{opt} = B^{-1}\left(t^{opt},\lambda_2^{opt}\right)h\left(t^{opt},\lambda_2^{opt}\right)$ is an extremal λ_1. Letting

$$\alpha(\tau,\lambda^{opt}) = <\lambda_1^{opt},h\left(\tau,\lambda_2^{opt}\right)>^2 \cdot<\lambda_1^{opt},B\left(\tau,\lambda_2^{opt}\right)\lambda_1^{opt}>^{-1}$$

we point out that

$$\rho(\tau,\lambda^{opt}) = \alpha(\tau,\lambda^{opt})\left(1-\alpha(\tau,\lambda^{opt})\right)^{-1}$$

describes the corresponding sensitivity of ρ to a uniform timing shift when λ is optimally set. Note $\alpha^{opt}(1-\alpha^{opt})^{-1}$ is the

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* Detailed proofs of many assertions made here can be found in Reference 10.

maximum signal-to-noise ratio, the global maximum of $\rho(\tau,\lambda^{opt})$.

Recapping, we have shown (under quite general conditions) that the search for a t^{opt}, λ_1^{opt}, λ_2^{opt} can be accomplished by optimizing a function of t^{opt}, λ_2^{opt} alone and once this is done λ_1^{opt} is given by locating the peak of the sensitivity function of a single variable t given by (5). In lumped parameter network theory the function of t^{opt}, λ_2^{opt} may appear to be transcendental. In several R-C network examples we have examined we have found that it is a rational function optimization that is involved. In many applications Λ_2 is a single point and so the only true parameter is λ_1.

4. APPLICATIONS

In this section an illustrative example will be presented of a pulse filter configuration with parallel structure. In addition the procedure of the previous section will be exercised to obtain the parameter settings which optimize the signal to ISI noise ratio for this case.

4.1 BANDLIMITED NETWORKS

Consider an ensemble of filters with linearly independent transfer characteristics $\{P_j(\omega)\}_1^N$ that are differentiable everywhere, vanish off $|\omega| < 1/2$ and satisfy

$$\max_j \int_{-1/2}^{+1/2} |P_j'(\omega)|d\omega \triangleq M < \infty .$$

Let $\Lambda = \{(\lambda_1,\lambda_2,\ldots,\lambda_J)\}$ be a closed set in R^J excluding 0, with $a = \max_\Lambda \|\lambda\|$. Then

$$Q = \left\{\sum_1^J \lambda_j P_j(\omega)\right\}_\Lambda$$

is a parallel structure. For Condition A notice $\frac{aJM}{|t|}$ bounds any member Q (time response) for $t \geq 1$ since $|tp_j(t)| \leq M$. Now $p_*(t)$ exists since the time responses of Q are uniformly bounded in derivative since for all t

$$|p_j'(t)| \leq \int |\omega P_j(\omega)|d\omega \;,$$

$$\leq \frac{1}{4}\int |P_j(\omega)|^2 d\omega < \infty.$$

Now consider

$$\int_{-1/2}^{1/2} \left|\sum_{j=1}^N \lambda_j P_j(\omega)\right|^2 d\omega = \langle\lambda, B\lambda\rangle \geq \|\lambda\|^2 \underline{e}$$

where $\underline{e}$ = minimum eigenvalue of $B = [b_{ij}]$ and

$$[b_{ij}] = \left[\int_{-1/2}^{1/2} P_i(\omega)P_j(\omega)d\omega\right]$$ is a

constant nonsingular Gram matrix. The signal and ISI noise power which is invariant under a change in t satisfies

$$\sum_{n=-\infty}^{\infty}\left(\sum_{j=1}^N \lambda_j p_j(t+n2\pi)\right)^2 = \frac{1}{2\pi}\int_{-1/2}^{1/2} \cdot \left|\sum_{j=1}^N \lambda_j P_j(\omega)\right|^2 d\omega$$

$$\geq \frac{\|\lambda\|^2}{\pi}\underline{e} > 0$$

which is the fulfillment of the Condition B. We note that the optimization procedure is simplified in this case since B has no dependence on the time parameter t.

For a numerical illustration of the above consider the bandlimited family

$$P(\omega,\lambda_1) = \max\Big[0,(1-(2\omega)^2)^2 \cdot \left(\lambda_{11} + \lambda_{21}(2\omega)^2 + \lambda_{31}(2\omega)^4\right)\Big]$$

where

$$\Lambda_1 = \left\{\lambda_1 \neq 0, P(\omega,\lambda_1) \geq 0 \text{ for each } \omega\varepsilon\left(-\tfrac{1}{2}, +\tfrac{1}{2}\right)\right\}$$

Note $\Lambda\cup\{0\}$ is closed (and convex) since

$$\Lambda_1\cup\{0\} = \bigcap_{|\omega|<\frac{1}{2}}\left\{\lambda_1 | \langle\lambda_1,(1,\omega^2,\omega^4)\rangle \geq 0\right\}$$ an

intersection of half spaces.* For all $\gamma > 0$, $\gamma\Lambda = \Lambda$ so the spherical projection of Λ is identical to its intersection with the surface of the unit sphere and hence is closed. This condition sets up a well based extremal problem. From $\lambda_1 \neq 0$, $P(\omega,\lambda_1) \geq 0$, and the t-invariance of the S+N power, $t^{opt} = 0$. Following the procedure for parallel structures we evaluate the pertinent integrals and obtain

$$\beta(\lambda_1) = \frac{1}{2} \frac{\langle\lambda_1,(1,7^{-1},21^{-1})\rangle^2}{\langle\lambda_1,B\lambda_1\rangle}$$

where

$$B = 5\begin{pmatrix} 7^{-1} & (7\cdot11)^{-1} & 3(7\cdot11\cdot13)^{-1} \\ (7\cdot11)^{-1} & 3(7\cdot11\cdot13)^{-1} & (7\cdot11\cdot13)^{-1} \\ 3(7\cdot11\cdot13)^{-1} & (7\cdot11\cdot13)^{-1} & (11\cdot13\cdot17)^{-1} \end{pmatrix}.$$

We get $\alpha^{opt} = \alpha\left(0, \lambda_1^{opt}\right) = \frac{592}{630}$ which is achieved at any nonzero multiple of $\lambda_1^{opt} = (1, -2, 17)$. The positive multiples of λ_1^{opt} are in Λ_1 since $\left(1-(2\omega)^2\right)^2 (1-2^3\omega^2+2^4\cdot17\cdot\omega^4) = \left(1-(2\omega)^2\right)^2\left[\left(1-(2\omega)^2\right)^2 + 2^8\omega^4\right]$ which is positive on $|\omega| < 2^{-1}$. Note

$$\rho^{opt} = \frac{\alpha^{opt}}{1-\alpha^{opt}} = \frac{572}{58} \simeq 10 \text{ dB}.$$

The sensitivity of ρ to a sampling time shift Δ (for $|\Delta| < 2\pi$) can be analyzed from

$$\rho(\Delta) = p^2\left(\Delta,\lambda_1^{opt}\right)\left(2\langle\lambda_1^{opt},B\lambda_1^{opt}\rangle - p^2\left(\Delta,\lambda_1^{opt}\right)\right)^{-1}.$$

In the preceding illustration the parallel structures procedure was expedited by the fact that the value of t^{opt} was evident.

Let $\{n_k\}_0^\infty$ be a strictly increasing sequence of nonnegative integers with $n_0 = 0$. Generally for bandlimited families like

$$P_k(\omega,\lambda_1) = \left(1+(i2\omega)^2\right)^2\langle\lambda_1,\left((i\omega)^0,(i\omega)^{n_1},\ldots(i\omega)^{n_k}\right)\rangle$$

$$\left(\lambda_1 = (\lambda_{11},\ldots\lambda_{k1}) \neq 0\right)$$

the bound of $\alpha_k(\tau,\lambda_1)$, given by (iii), must be found first to locate t^{opt}. The authors have employed a computer program using Fast Fourier transform and quadrature routines to obtain $\langle h(t),B^{-1}h(t)\rangle$ for such problems. (It is not difficult to argue from Muntz's theorem [5] that if there is a subsequence e or even n_k's such that $\sum_e n_k^{-1} = \infty$ then $\rho_k^{opt} \uparrow \infty$.)

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* The envelope technique allows a simple explicit determination of Λ_1 but we do not need this here.