4-17-2007

On Discrete Gauss-Hermite Functions and Eigenvectors of the Discrete Fourier Transform

Bal Santhanam

Thalanayar Santhanam

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

SCHOOL OF ENGINEERING
UNIVERSITY OF NEW MEXICO

On Discrete Gauss Hermite Functions and Eigenvectors of the Discrete Fourier Transform

Balu Santhanam
Department of Electrical and Computer Engineering
University of New Mexico, Albuquerque, NM: 87131
Tel: (505) 277 1611, Fax: (505) 277 1439
Email: bsanthan@ece.unm.edu

Thalanayar S. Santhanam
Department of Physics
Saint Louis University, Missouri, MO: 63103
Email: santhats@slu.edu

UNM Technical Report: EECE-TR-07-

Report Date: April 10, 2007

Abstract

The problem of furnishing an orthogonal basis of eigenvectors for the *discrete Fourier transform* (DFT) is fundamental to signal processing. Recent developments in the area of discrete fractional Fourier analysis also rely upon the ability to furnish a basis of eigenvectors for the DFT or its centralized version. However, none of the existing approaches are able to furnish a commuting matrix where both the eigenvalue spectrum and the eigenvectors are a close match to corresponding properties of the continuous differential Gauss-Hermite operator. Furthermore, any linear combination of commuting matrices produced by existing approaches also commutes with the DFT, thereby bringing up the question of uniqueness. In this paper, inspired by concepts from quantum mechanics in finite dimensions, we present an approach that furnishes a basis of orthogonal eigenvectors for both versions of the DFT. This approach also furnishes a commuting matrix whose eigenvalue spectrum is a very close approximation to that of the Gauss–Hermite differential operator and consequently a framework for a unique definition of the discrete Gauss–Hermite operator.

Keywords

Discrete Fourier transform, discrete Fractional Fourier transform, eigenvalues, commuting matrix, Gauss-Hermite differential operator, Gauss-Hermite functions, eigenvectors, centro-symmetric matrices, K-symmetric matrices, chirp signals.

1 Introduction

Conventional Fourier analysis treats frequency and time as orthogonal variables and consequently is only suitable for the analysis of signals with stationary frequency content. The *fractional Fourier transform* (FRFT), an angular generalization of the FRFT, enables the analysis of waveforms, such as linear-FM or chirps, that possess time-frequency coupling. The eigenfunctions of the FRFT are known to be *Gauss–Hermite* (G-H) functions

The continuous Fourier integral transform of a finite energy signal is defined via:

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt = \mathcal{F}(x(t)).$$

Gauss–Hermite (G-H) functions defined by:

$$H_n(t) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} h_n(t) \exp\left(-\frac{t^2}{2}\right),$$

where $h_n(t)$ is the n-th order Hermite polynomial are solutions to the second-order Hermite differential equation:

$$\frac{d^2x}{dt^2} - (t^2 + \lambda)x(t) = 0.$$

They are eigenfunctions of the G-H differential operator:

$$\mathcal{H}(x(t)) = (\mathcal{D}^2 - t^2 I)x(t) = -(2n+1)x(t).$$

with a corresponding eigenvalue of $\lambda_n = -(2n+1)$. They are also eigenfunctions of the Fourier integral operator:

$$\mathcal{F}(H_n(t)) = \exp\left(-jn\frac{\pi}{2}\right) H_n(t),$$

with a corresponding eigenvalue of $\lambda_n = \exp(-jn\pi/2)$. These G-H functions are also eigenfunctions of the *fractional Fourier transform* (FRFT) defined via:

$$\begin{aligned} X_\alpha(u) &= \int_{-\infty}^{\infty} x(t) K_\alpha(t,u) dt, \\ K_\alpha(t,u) &= \sum_{n=-\infty}^{\infty} \exp(-jn\alpha) H_n(t) H_n(u). \end{aligned}$$

Recent efforts to develop a discrete and computable version of the FRFT have focussed on the DFT and its centralized version and on generating an orthogonal basis of eigenvectors for the DFT by furnishing a commuting matrix that has a non-degenerate eigenvalue spectrum and shares a common basis of eigenvectors with the DFT. These approaches, however, do not yield a unique discretisation since the sum or product of matrices that commute with the DFT also commutes with the DFT. Our goal in this paper is of course to define the discrete equivalent of the G-H differential operator $\mathcal{H}$ that will furnish the basis for both the centered and off-centered versions of the DFT matrix. This framework will enable the definition of a discrete version of the FRFT and will also serve as the discrete equivalent of the G-H differential operator with eigenvalues and eigenvectors that very closely resemble those of the continuous counterpart.

Quantum mechanics as it pertains to the harmonic oscillator connects the canonical variables, position and momentum, through the Fourier integral operator $\mathcal{F}$ via [4, 3]:

$$\mathcal{F} = \exp\left(j\frac{\pi}{4}\left(\hat{p}^2 + \hat{q}^2 - 1\right)\right),$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators that are related through a similarity transformation:

$$\hat{p} = \mathcal{F}\hat{q}\mathcal{F}^\dagger,\ \hat{p} = -j\frac{d}{dq}$$

and p, q denote the eigenvalues of their corresponding operators. In the continuous case the expression inside the exponential is exactly the G-H differential operator:

$$q^2 + p^2 = -\frac{d^2}{dq^2} + q^2 = -\mathcal{H}(q).$$

Consequently G-H functions are also the eigenfunctions of the quantum harmonic oscillator. The position and momentum operators furthermore do not commute and their commutator corresponds to the identity:

$$[\hat{\mathbf{q}}, \hat{\mathbf{p}}] = \hat{\mathbf{q}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{q}} = j\mathbf{I}. \tag{1}$$

Existing approaches towards obtaining an orthogonal basis of eigenvectors for the DFT can be grouped into two basic categories. The first approach called the S matrix approach or the Harper matrix approach [7] is based on replacing the derivatives in the G-H differential equation with finite differences thereby converting the differential equation into a difference equation. The approach furnishes an almost tridiagonal (Harper) matrix that commutes with the DFT matrix and consequently furnishes a basis of orthogonal DFT eigenvectors when N is not a multiple of four. As shown in [7, 10], the Harper matrix does not converge to the G-H operator in the limit, but rather to the Mathieu differential operator. Furthermore the eigenvalue spectrum of the operator is different from the linear and uniform spacing needed to be considered a candidate for the discrete G-H operator.

The second approach pioneered by Grunbaum [9] and later refined in [8] is an algebraic approach that furnishes tridiagonal matrices that commute with both the centered and the off-centered versions of the DFT. It was shown in [9] that the commuting matrix in the limit converges to the G-H differential operator. However, the eigenvalues of the matrix do not exhibit the uniform integer spacing needed to be considered a viable candidate for the discrete G-H operator. Since the sum and product of the different commuting matrices also commutes with the DFT, numerous other commuting matrices can be furnished and the question of uniqueness of the commuting matrix approach arises.

Our goal in this paper is to present a framework for defining a unique discrete version of the G-H differential operator that: (a) furnishes orthogonal eigenvectors resembling G-H functions, (b) has an eigenvalue spectrum very close to that of the G-H operator, (c) converges to the G-H operator in the limit.

2 Discrete Gauss-Hermite Operator

2.1 Centered Case

Our goal in this paper is to present an approach that furnishes both a unique discrete version of the G-H differential operator and simultaneously a basis of orthogonal eigenvectors for the DFT or its centered version. Towards this purpose we borrow some ideas from quantum mechanics in finite dimensions [3, 4]. First we define a diagonal matrix $\mathbf{Q} \in \mathbf{R}^{N \times N}$ whose entries are given by:

$$Q_{rr} = q[r] = \sqrt{\frac{2\pi}{N}} r, \ |r| \leq \frac{(N-1)}{2}. \tag{2}$$

This operator constitutes the discrete equivalent of the position operator in infinite dimensional space. We then define the DFT matrix and its centered version as:

$$\begin{aligned} \{W_{oc}\}_{pq} &= \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N} pq\right) \ 0 \leq p, q \leq (N-1) \\ \{W_c\}_{pq} &= \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N} (p-a)(q-a)\right), \end{aligned}$$

with $a = (N-1)/2$. For purposes of discussion both these matrices will be denoted by $\mathbf{W}$. The version discussed will be clear from the appropriate context. The matrix $\mathbf{P}$ is defined by combining the matrices $\mathbf{Q}$ and $\mathbf{W}$ as:

$$\mathbf{P} = \mathbf{W}\mathbf{Q}\mathbf{W}^H$$

and this matrix constitutes the discrete equivalent of the momentum operator in infinite dimensions.

Following the approach in the continuous case, the matrix $\mathbf{T}_1$ that commutes with the CDFT is then defined as:

$$\mathbf{T}_1 = \mathbf{P}^H\mathbf{P} + \mathbf{Q}^H\mathbf{Q} = \mathbf{P}^2 + \mathbf{Q}^2. \tag{3}$$

2.2 Commutation Properties

We then proceed to show that this operator indeed commutes with the CDFT and consequently can be used to furnish that basis of CDFT eigenvectors that we seek. Substituting the expression for $\mathbf{P}$ into the definition of the commutor we have:

$$\mathbf{T} = \mathbf{W}\mathbf{Q}^2\mathbf{W}^H + \mathbf{Q}^2.$$

To demonstrate that this matrix indeed commutes with the matrix $\mathbf{W}$ we have:

$$\begin{aligned} \mathbf{TW} &= \mathbf{W}\mathbf{Q}^2 + \mathbf{Q}^2\mathbf{W} \\ \mathbf{WT} &= \mathbf{W}\mathbf{Q}^2 + \mathbf{W}^2\mathbf{Q}^2\mathbf{W}^H. \end{aligned}$$

Since $q^2[r] = q^2[-r], 0 \le r \le (N-1)$, the exchange matrix $\mathcal{J} = \mathbf{W}^2$ satisfies the relation,

$$\mathcal{J}\mathbf{Q}^2\mathcal{J} = \mathbf{Q}^2 \implies \mathbf{W}^2\mathbf{Q}^2\mathbf{W}^2 = \mathbf{Q}^2. \tag{4}$$

In other words the matrix $\mathbf{Q}^2$ needs to be a *centro symmetric* matrix for Eq. (4), a condition that is obviously met from the definition of the position operator $\mathbf{Q}$ in Eq. (2). Using Eq. (4) we can further write:

$$[\mathbf{T}, \mathbf{W}] = (\mathbf{TW} - \mathbf{WT}) = \mathbf{0} \tag{5}$$

This implies that the commutor defined in Eq. (3) can be used to furnish the basis of eigenvectors for the centered DFT. If we now define $\mathbf{C}_1 = [\mathbf{Q}, \mathbf{P}]$ and look at the commutator:

$$\begin{aligned} [\mathbf{W}, \mathbf{C}_1] &= [\mathbf{W}, \mathbf{QP} - \mathbf{PQ}] \\ &= \mathbf{W}(\mathbf{QP} - \mathbf{PQ}) - (\mathbf{QP} - \mathbf{PQ})\mathbf{W} \end{aligned}$$

Substituting the expression for $\mathbf{P}$ into this expression we have:

$$\begin{aligned} [\mathbf{W}, \mathbf{C}_1] &= \mathbf{WQWQW}^H - \mathbf{W}^2\mathbf{QW}^2\mathbf{WQ} \\ &- \mathbf{QWQ} + \mathbf{WQW}^H\mathbf{QW} \\ &= \mathbf{WQWQW}^H + \mathbf{WQW}^H\mathbf{QW} \\ &= \mathbf{WQWQW}^H + \mathbf{WQWW}^2\mathbf{QW}^2\mathbf{W}^H \\ &= \mathbf{0} \end{aligned} \tag{6}$$

where we have used the property that $q[-r] = -q[r]$, i.e.,

$$\mathbf{W}^2\mathbf{Q}\mathbf{W}^2 = -\mathbf{Q}. \tag{7}$$

In other words the matrix $\mathbf{Q}$ needs to *centro anti-symmetric* for Eq. (7) to be met. Note that relation in Eq. (7) is the stricter condition since it implies the relation in Eq. (4). This in turn implies that $\mathbf{C}_1$ and $\mathbf{W}$ share a common basis of eigenvectors. The commutor matrix in its general additive[1] form can therefore be written as:

$$\mathbf{T}_2 = c_1(\mathbf{P}^2 + \mathbf{Q}^2) + c_2\mathbf{C}_1^H\mathbf{C}_1 + c_3\mathbf{I}, \tag{8}$$

[1] We are not including product terms such as $(\mathbf{P^2} + \mathbf{Q^2})[\mathbf{Q}, \mathbf{P}]$ that also commute with $\mathbf{W}$

where the constants c_1, c_2, c_3 are chosen appropriately to obtain a matrix with a non-degenerate eigenvalue spectrum that is as close as possible to that of the G-H operator $\mathcal{H}$. Figure (2) depicts the eigenvalues of the two commuting matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ for $N = 128$ and for $c_1 = 1$, $c_2 = -c_3 = -\frac{\pi^2}{N^2}$. Note that the largest eigenvalue of the commuting matrix $\mathbf{T}_2$ is significantly smaller than that of the $\mathbf{T}_1$. The eigenvectors of $\mathbf{T}_2$ are the same as that of $\mathbf{T}_1$ except for a reordering of the eigenvectors.

Another key observation is the fact that the matrices $\mathbf{P}^2$, $\mathbf{QP}$, $\mathbf{PQ}$ are also $\mathbf{K} = \mathcal{J}$ symmetric since:

$$\begin{aligned}
\mathcal{J}\mathbf{P}^2\mathcal{J} &= \mathbf{W}^2\mathbf{W}\mathbf{Q}^2\mathbf{W}^H\mathbf{W}^2 \\
&= \mathbf{W}^3\mathbf{Q}^2\mathbf{W} = \mathbf{W}\mathbf{W}^2\mathbf{Q}^2\mathbf{W}^2\mathbf{W}^H = \mathbf{P}^2 \\
\mathcal{J}\mathbf{QP}\mathcal{J} &= \mathbf{W}^2\mathbf{QWQW}^H\mathbf{W}^2 \\
&= (\mathbf{W}^2\mathbf{Q}\mathbf{W}^2)\mathbf{W}(\mathbf{W}^2\mathbf{Q}\mathbf{W}^2)\mathbf{W}^H \\
&= \mathbf{QWQW}^H = \mathbf{QP} \\
\mathcal{J}\mathbf{PQ}\mathcal{J} &= \mathbf{W}^2\mathbf{WQW}^H\mathbf{QW}^2 \\
&= -\mathbf{WQWQW}^2 \\
&= \mathbf{WQW}^H\mathbf{Q} = \mathbf{PQ},
\end{aligned} \tag{9}$$

where we have used the fact that the position operator $\mathbf{Q}$ defined in Eq. (2) is $\mathcal{J}$-symmetric and $\mathbf{Q}^2$ is $\mathcal{J}$-symmetric. A direct consequence of this observation is the result that the commuting matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are both symmetric $\mathbf{K}$-symmetric matrices with $\mathbf{K} = \mathcal{J}$. In turn the framework from [6] on centrosymmetric matrices can be directly applied to furnish a basis of orthonormal symmetric and skew-symmetric eigenvectors.

2.3 Convergence to the G-H Operator

Here we show that the commutor matrix $\mathbf{T}_1$ defined here in the continuous limit converges to the Gauss-Hermite operator. This can be demonstrated by looking at:

$$\begin{aligned}
\mathbf{P}(\mathbf{x}) &= \sum_{s=0}^{N-1} P_{rs}x_s \\
&= \sum_{s=0}^{N-1}\sum_{m=0}^{N-1} \sqrt{\frac{2\pi}{N}}\frac{m}{N}\exp\left(j\frac{2\pi}{N}m(r-s)\right)x[s]
\end{aligned}$$

To enable the passage to the continuous limit, we can define the following quantities:

$$\begin{aligned}
q &= \sqrt{\frac{2\pi}{N}}r, \tilde{q} = \sqrt{\frac{2\pi}{N}}s,\ p = \sqrt{\frac{2\pi}{N}}m \\
&\Longrightarrow \quad dp = dq = d\tilde{q} = \sqrt{\frac{2\pi}{N}}.
\end{aligned}$$

Consequently we can write the sum as:

$$\begin{aligned}
\lim_{N\to\infty} \mathbf{Px} &= \int_{-\infty}^{\infty} d\tilde{q}\frac{1}{2\pi}\int_{-\infty}^{\infty} p\exp\left(jp(q-\tilde{q})\right)dp\,x(\tilde{q}) \\
&= -j\frac{d}{dq}x(q).
\end{aligned}$$

Therefore the commuting matrix $\mathbf{T}_1$ in the limit can be written as:

$$\begin{aligned}
\lim_{N\to\infty} \mathbf{Tx} &= \lim_{N\to\infty}(\mathbf{P}^2+\mathbf{Q}^2)x(q) \\
&= -\frac{d^2}{dq^2}x(q) + q^2x(q) = -\mathcal{H}(x(q))
\end{aligned}$$

If we now consider the commutator that constitutes the second part of the commuting matrix $\mathbf{T}_2$ we have:

$$\begin{aligned}\lim_{N\to\infty}[\mathbf{Q},\mathbf{P}]\mathbf{x} &= [\hat{\mathbf{q}},\hat{\mathbf{p}}]x(q) \\ &= (\hat{\mathbf{q}}\hat{\mathbf{p}}-\hat{\mathbf{p}}\hat{\mathbf{q}})x(q) \\ &= \hat{\mathbf{q}}\left(-j\frac{dx}{dq}\right)+j\frac{d}{dq}(qx(q)) \\ &= jx(q). \end{aligned} \tag{10}$$

This in turn implies that the commutator in the limit converges to a multiple of identity, a result that is consistent with Eq. (1). Specifically choosing the constants in the expression for the generalized commuting matrix $\mathbf{T}_2$ as $c_1 = 1, c_2 = -c_3 = -\frac{\pi^2}{N^2}$, we see that in the limit, the effect of the extra terms vanishes, and that this commuting matrix also converges to the operator $\mathcal{H}$. Thus the effect of this term is to truncate the spectrum when N is finite and it vanishes in the limit. As seen from the proof for convergence, the deviation of the eigenvalue spectrum at the end will also vanish asymptotically.

The matrix $\mathbf{T}_1$ therefore constitutes a unique discrete version of the G-H operator in the sense that: (a) it converges to the G-H operator in the limit, (b) its eigenvectors resemble sampled versions of G-H functions, (c) its eigenvalue spectrum is very close to that of the G-H operator, a feature not shared by any of the existing approaches.

2.4 The Off-Centered Case

Eq. (4) and Eq. (7) form the backbone of the approach for generating the eigenvectors of the centered DFT. However, these equations do not hold when we use the off-centered version of the DFT. Specifically in this case the cyclic flip matrix is given by:

$$\mathbf{W}^2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{J}_{N-1} \end{bmatrix} \neq \mathbf{J}_N$$

where $\mathbf{J}_N$ refers to the exchange matrix.

If we further explore the implications of Eq. (7) when $\mathbf{W}$ is the conventional DFT we obtain the relation:

$$\{Q\}_{((-m))_N,((-n))_N} = -\{Q\}_{mn}, \ \ 0 \le m,n \le (N-1), \tag{11}$$

where the notation $((.))_N$ denotes the modulo-N representation. This condition needs to be met for the earlier framework to apply. Obviously this relation is not met with the regular DFT and the centered position operator $\mathbf{Q}$ defined before in the case of the CDFT. If we further restrict ourselves to the case where $\mathbf{Q}$ is a diagonal matrix then Eq. (11) becomes

$$q[r] = -q[((-r))_N], \ \ 1 \le r \le (N-1), \ \ q[0] = 0.$$

The implication of Eq. (7) in the context of the DFT and the cyclic flip matrix is that the position operator $\mathbf{Q}$ needs to be *cyclo centro anti-symmetric*. The implication of Eq. (4) on the other hand is that the matrix $\mathbf{Q}^2$ needs to be *cyclo centro symmetric*. Our task there is therefore to define a modified position operator $\tilde{\mathbf{Q}}$ that satisfies Eq. (11). The only possible solution for a diagonal $\tilde{\mathbf{Q}}$ therefore needs the main diagonal to be anti-symmetric after exclusion of the first element and this would also require that in the case where N is even that $q[N/2+1] = 0$. Let us now define the modified position matrix $\mathbf{Q}$ as:

$$\tilde{\mathbf{Q}} = \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a}^T & \mathbf{Q}_{N-1} \end{pmatrix}, \tag{12}$$

where $\mathbf{Q}_{N-1}$ is the position matrix of order $(N-1)$ defined for the CDFT and $\mathbf{a}$ is any anti-symmetric vector. The discrete equivalent of the momentum operator is then given by:

$$\tilde{\mathbf{P}} = \mathbf{W}\tilde{\mathbf{Q}}\mathbf{W}^H,$$

where **W** now denotes the conventional DFT matrix. It is easily seen from Eq. (12) and the partitioned form of the cyclic flip matrix that both the conditions that are required of the ideal position operator are met because the sub-matrices $\mathbf{J}_{N-1}$ and $\mathbf{Q}_{N-1}$ satisfy the same relations seen in the case of the CDFT. The commutator of these matrices is then given by:

$$\tilde{\mathbf{C}} = [\tilde{\mathbf{Q}}, \tilde{\mathbf{P}}] = \tilde{\mathbf{Q}}\tilde{\mathbf{P}} - \tilde{\mathbf{P}}\tilde{\mathbf{Q}}$$

The uncorrected and corrected versions of the commuting matrix in the case of the off-centered DFT are then given by:

$$\begin{aligned} \mathbf{T}_1 &= \tilde{\mathbf{P}}^2 + \tilde{\mathbf{Q}}^2 \\ \mathbf{T}_2 &= \mathbf{T}_1 - \frac{\pi^2}{N^2}\tilde{\mathbf{C}}^H\tilde{\mathbf{C}}. \end{aligned} \tag{13}$$

Similar to what we saw in the case of the CDFT, both the commuting matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are K-centrosymmetric matrices with $\mathbf{K} = \mathbf{W}^2$ since Eq. (4) and Eq. (7) are satisfied. Consequently the eigenvectors that result from this approach will not have the symmetries present in the G-H functions or the eigenvectors from the CDFT approach because of the structure of the cyclic flip matrix . The eigenvectors in particular are conjugate symmetric and conjugate antisymmetric as seen in the appendix. In addition the eigenvalue spectrum of the commutor exhibits deviations from the eigenvalue spectrum seen in the case of the CDFT at two regions as seen in Fig. (2.4)(a) for the choice of $\mathbf{a} = \mathbf{0}$. Another choice for the vector

$$\mathbf{a} = \text{diag}(\mathbf{Q}_{N-1}) = [-(N-2)/2, \ldots, (N-2)/2]^T$$

removes the eigenvalue deviations in the linear region at the expense of increasing the deviation in the end as seen in Fig. (2.4)(b).

Suppose we relax the requirements so that the modified position operator just needs to satisfy Eq. (4) then we can remedy the eigenvalue situation by choosing a modified position operator of the form:

$$\begin{aligned} \tilde{\mathbf{Q}} &= \left(\sqrt{\frac{2\pi}{N}}\right)\mathbf{W}^2\text{diag}(0,1,2,\ldots,N-1) \\ &= \sqrt{\frac{2\pi}{N}}\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{J}_{N-1}\Lambda \end{pmatrix}, \end{aligned} \tag{14}$$

where $\Lambda = \text{diag}(1,2,3,\ldots N-1)$. It is easy to see that Eq. (4) is met with this form of the position operator since:

$$\mathbf{J}_{N-1}\Lambda\mathbf{J}_{N-1} = \Lambda\mathbf{J}_{N-1}\Lambda\mathbf{J}_{N-1}$$

and consequently the corresponding matrix $\mathbf{T}_1 = \mathbf{P}^2 + \mathbf{Q}^2$ commutes with the DFT matrix. The eigenvalues of the commutor are exactly the same as those seen for the CDFT as seen in Fig. (2.4)(c). The eigenvectors of the commuting matrix, however, will have the same problem of not having the symmetries seen in the G-H functions due again to the structure of the cyclic flip matrix, however, these eigenvectors are **K**-symmetric or **K**-antisymmetric with $\mathbf{K} = \mathbf{W}^2$ as described in the appendix. Note that the symmetric involution matrix **K** used in the definition and formulation of the **K**-symmetric matrix framework for computing the eigenvectors of the commuting matrix can in principle be any cyclic permutation matrix that commutes with the DFT and does not need to be the cyclic flip matrix $\mathbf{W}^2$ [11].

3 Appendix

The algorithm for generating the eigenvectors for both the DFT and its centralized version can be formulated as special cases of the eigenvalue problem for the broader class of generalized centrosymmetric matrices [5]. A matrix $\mathbf{M} \in \mathbf{R}^N$ is said to be *cyclo centrosymmetric* (CCS) or centrosymmetric with respect to $\mathbf{K} = \mathbf{W}^2$ if:

$$\mathbf{KMK} = \mathbf{M} \implies \mathbf{W}^2\mathbf{M}\mathbf{W}^2 = \mathbf{M}$$

More specifically in terms of its elements:

$$\{M\}_{((-i))_N,((-j))_N} = \{M\}_{ij}, \ \ 0 \leq i,j \leq (N-1).$$

If the matrix $\mathbf{M}$ is symmetric and CCS then we can express it in block matrix form as:

$$\mathbf{M} = \begin{pmatrix} k & \mathbf{c}^T & \mathbf{c}^T\mathbf{J}_M \\ \mathbf{c} & \mathbf{R}_M & \mathbf{S}_M \\ \mathbf{J}_M\mathbf{c} & \mathbf{J}_M\mathbf{S}_M\mathbf{J}_M & \mathbf{J}_M\mathbf{R}_M\mathbf{J}_M \end{pmatrix},$$

for $N = 2M+1$ and

$$\mathbf{M} = \begin{pmatrix} k_1 & \mathbf{c}_1^T & k_2 & \mathbf{c}_1^T\mathbf{J}_L \\ \mathbf{c}_1 & \mathbf{R}_L & \mathbf{c}_2 & \mathbf{S}_L \\ k_2 & \mathbf{c}_2^T & k_3 & \mathbf{c}_2^T\mathbf{J}_L \\ \mathbf{J}_L\mathbf{c}_1 & \mathbf{J}_L\mathbf{S}_L\mathbf{J}_L & \mathbf{J}_L\mathbf{c}_2 & \mathbf{J}_L\mathbf{R}_L\mathbf{J}_L \end{pmatrix}, \tag{15}$$

for $N = 2L+2$, where $\mathbf{R}$ and $\mathbf{S}$ are symmetric sub-matrices of the appropriate order. It is a simple exercise in matrix multiplication to show that the orthogonal similarity transformation $\mathbf{U}$ given by:

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{J}_M \\ \mathbf{0} & \mathbf{J}_M & -\mathbf{I}_M \end{pmatrix}, \ \text{for } N = 2M+1 \tag{16}$$

and

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_L & \mathbf{0} & \mathbf{J}_L \\ 0 & \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_L & \mathbf{0} & -\mathbf{I}_L \end{pmatrix}, \ \text{for } N = 2L+2 \tag{17}$$

block diagonalizes $\mathbf{M}$ into the form:

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & \mathbf{O} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix},$$

where the block matrices are given by:

$$\begin{aligned} \mathbf{D}_1 &= \begin{pmatrix} k & \sqrt{2}\mathbf{c}^T \\ \sqrt{2}\mathbf{c} & \mathbf{R}_M + \mathbf{S}_M\mathbf{J}_M \end{pmatrix} \\ \mathbf{D}_2 &= \mathbf{J}_M\mathbf{R}_M\mathbf{J}_M - \mathbf{J}_M\mathbf{S}_M, \ N = 2M+1 \end{aligned}$$

and

$$\begin{aligned} \mathbf{D}_1 &= \begin{pmatrix} k_1 & \sqrt{2}\mathbf{c}_1^T & k_2 \\ \sqrt{2}\mathbf{c}_1 & \mathbf{R}_L + \mathbf{S}_L\mathbf{J}_L & \sqrt{2}\mathbf{c}_2 \\ k_2 & \sqrt{2}\mathbf{c}_2^T & k_3 \end{pmatrix} \\ \mathbf{D}_2 &= \mathbf{J}_L\mathbf{R}_L\mathbf{J}_L - \mathbf{J}_L\mathbf{S}_L, \ N = 2L+2. \end{aligned}$$

Since $\mathbf{U}$ is a similarity transformation, the eigenvalues of the transformed matrix $\mathbf{D} = \mathbf{U}^{-1}\mathbf{M}\mathbf{U}$ are the same as the eigenvalues of $\mathbf{M}$. The eigenvectors of the transformed matrix are therefore related to the eigenvectors of $\mathbf{M}$ through the matrix $\mathbf{U}$. Further note that the transformation $\mathbf{U}$ is a symmetric involution, i.e., $\mathbf{U} = \mathbf{U}^{-1} = \mathbf{U}^T$. We can now furnish an orthogonal basis of eigenvectors for $\mathbf{D}$ by patching together the eigenvectors of $\mathbf{D}_1$ and $\mathbf{D}_2$. Specifically if we solve the smaller eigenvalue problems in the case where N is odd we obtain:

$$\begin{aligned} \begin{pmatrix} k & \sqrt{2}\mathbf{c}^T \\ \sqrt{2}\mathbf{c} & \mathbf{R}_M + \mathbf{S}_M\mathbf{J}_M \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \sqrt{2}w \end{pmatrix} &= \lambda \begin{pmatrix} \mathbf{u} \\ \sqrt{2}w \end{pmatrix}, \\ \mathbf{D}_2\mathbf{v} &= \rho\mathbf{v} \end{aligned}$$

We can construct $M+1$ K-symmetric and M K-antisymmetric unit-norm, orthogonal eigenvectors through concatenation via:

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2w \\ \mathbf{u} \\ \mathbf{Ju} \end{pmatrix}, \quad \mathbf{y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathbf{v} \\ -\mathbf{Jv} \end{pmatrix}.$$

In a similar fashion for the case where $N = 2L+2$, if we represent a general eigenvector of the matrix $\mathbf{D}_1$ as $[w_1 \; \mathbf{u} \; w_2]^T$ and a general eigenvector of the matrix $\mathbf{D}_2$ as $\mathbf{v}$, we can generate $L+2$ K-symmetric eigenvectors and L K-antisymmetric eigenvectors through concatenation as:

$$\mathbf{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2w_1 \\ \mathbf{u} \\ 2w_2 \\ \mathbf{Ju} \end{pmatrix}, \quad \mathbf{y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \mathbf{v} \\ 0 \\ -\mathbf{Jv} \end{pmatrix}.$$

As a special case if $\mathbf{K} = \mathbf{J}$, i.e., the exchange matrix, then the commuting matrix becomes centrosymmetric and consequently the eigenvectors from either eigen-subspace are either symmetric or antisymmetric as seen in case for the CDFT.

References

[1] J. H. McClellan and T. W. Parks, "Eigenvalues and Eigenvectors of the Discrete Fourier Transformation," *IEEE Trans. Audio and Electroacoustics*, Vol. 20, No. 1, March 1972.

[2] J. G. Vargas-Rubio and B. Santhanam, "On the Multiangle Centered Discrete Fractional Fourier Transform," *IEEE Signal Proc. letters*, Vol. 12, No. 4, pp. 273-276, Apr. 2005.

[3] R. Jagannathan, T. S. Santhanam, and R. Vasudevan, "Finite Dimensional Quantum Mechanics of a Particle," *Int. J. Theor. Phys.*, Vol. 20, pp. 755-783, 1981.

[4] T. S. Santhanam and A. R. Tekumalla, "Quantum Mechanics in Finite Dimensions," *Found. of Phys.*, Vol. 6, No. 5, pp. 583-587, 1976.

[5] D. Tao and M. Yasuda, "A Spectral Characterization of Generalized Real Symmetric Centrosymmetric and Generalized Real Symmetric Skew-Centrosymmetric Matrices," *SIAM Journal on Matrix Analysis and Applications*, Vol. 23, No. 3, pp. 885-895, 2002.

[6] A. Cantoni and P. Butler, "Eigenvalues and Eigenvectors of Symmetric Centrosymmetric Matrices," *Linear Algebra and Applications*, Vol. 13, pp. 275-288, 1976.

[7] B. W. Dickinson and K. Steiglitz, "Eigenvalues and Eigenvectors of the Discrete Fourier Transform," *IEEE Trans. ASSP*, Vol. 30, No. 1, pp. 25-31, Feb 1982.

[8] S. Clary and D. H. Mugler, "Shifted Fourier Matrices and Their Tridiagonal Commutors," *SIAM Jour. Matr. Anal.* & *Appl.*, Vol. 24, No. 3, pp. 809-821, 2003.

[9] F. A. Grunbaum, "The Eigenvectors of the Discrete Fourier Transform," *Jour. Math. Analy.* & *Appl.*, Vol. 88, No. 2, pp. 355-363, 1982.

[10] C. Candan, M. A. Kutay, H. M. Ozatkas, "The Discrete Fractional Fourier Transform," *IEEE Trans. Sig. Process.* , Vol. 48, No. 5, pp. 1329-1337, 2000.

[11] Paulo Jorge S. G. Ferreira, "A Group of Permutations that Commute with the Discrete Fourier Transform," *IEEE Trans. Sig. Process.*, Vol. 42, No. 2, 1994.

[12] B. Santhanam and T. S. Santhanam, "Discrete Gauss-Hermite Functions and Eigenvectors of the Centered Discrete Fourier Transform," *Proc. of ICASSP-07*, Hawai, April 2007.

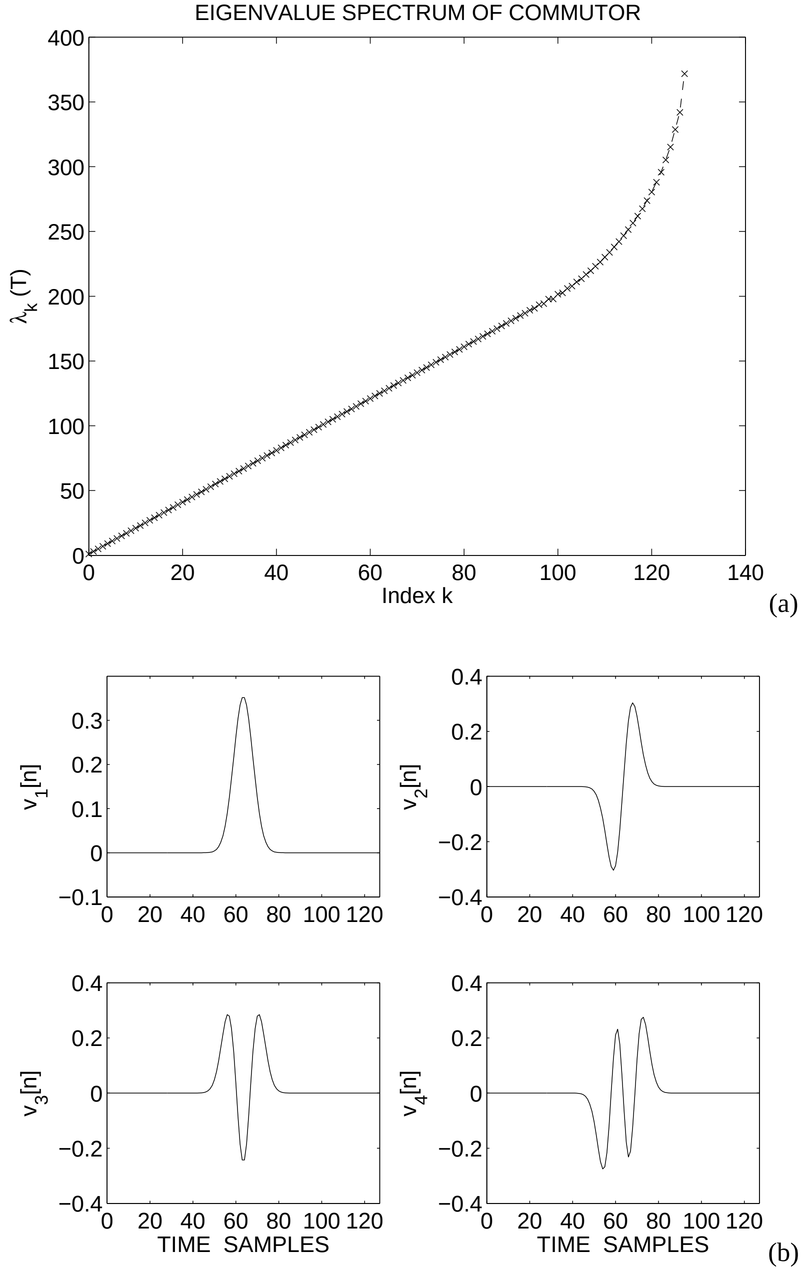

Figure 1: Decomposition of the commutor for the CDFT: (a) eigenvalues of the commutor $\mathbf{T}_1$, (b) eigenvectors of the commutor

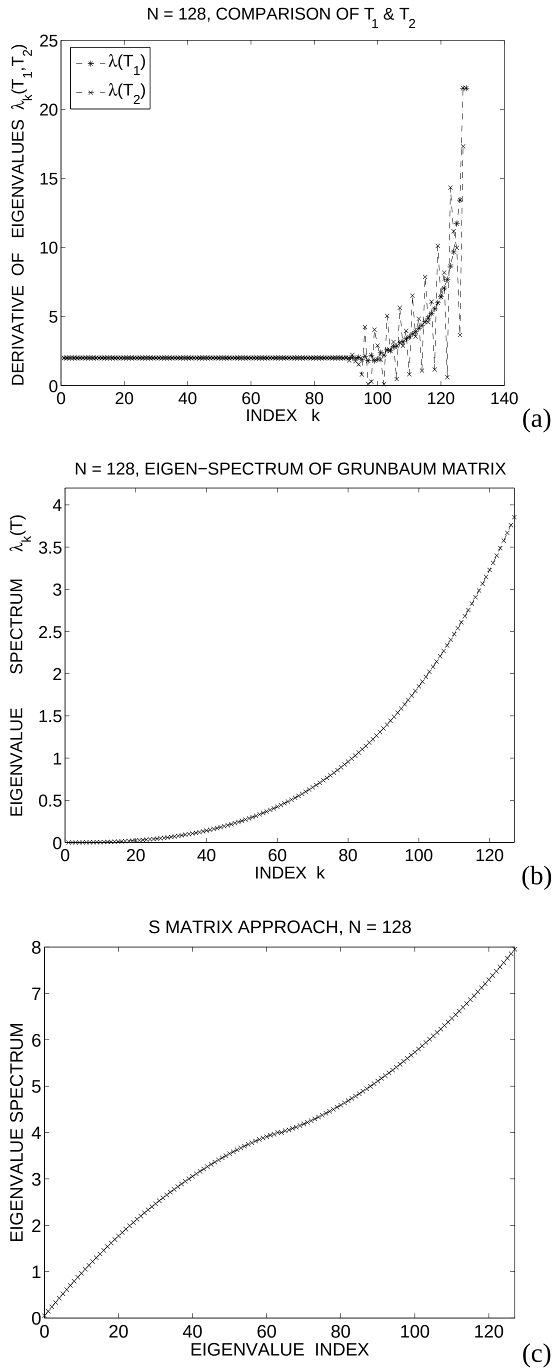

Figure 2: Comparison of the eigenvalue spectra: (a) eigenvalues of $\mathbf{T}_1$ and $\mathbf{T}_2$ for the CDFT, for $N = 128$, (b) eigenvalues of the Grnbaum commuting matrix approach, (c) eigenvalues of the Harper matrix approach.

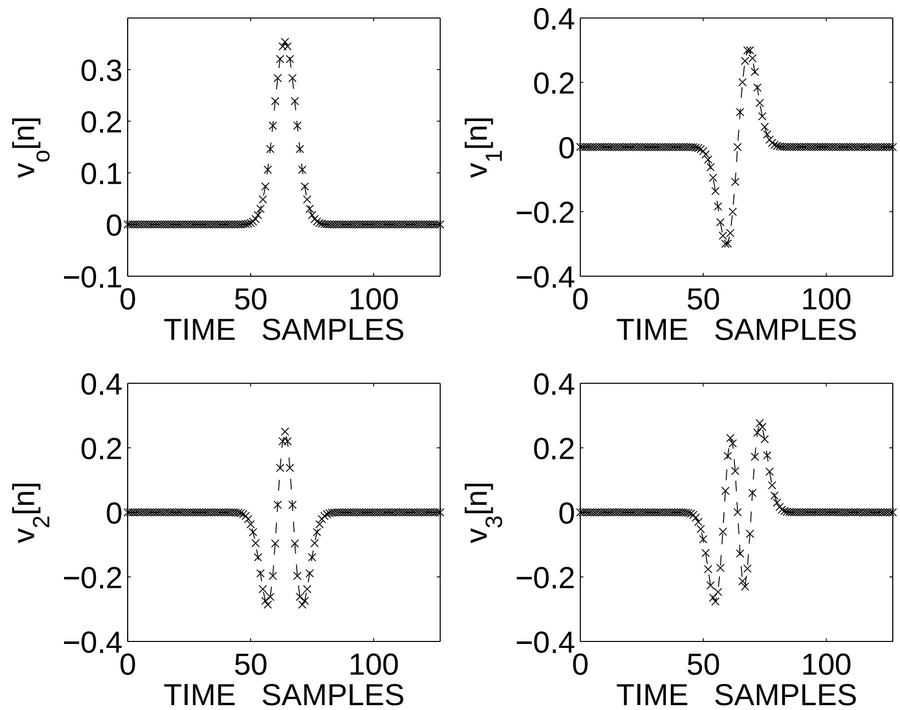

Figure 3: Eigenvectors of the commuting matrix $\mathbf{T}_1$ for the DFT with $N = 128$.

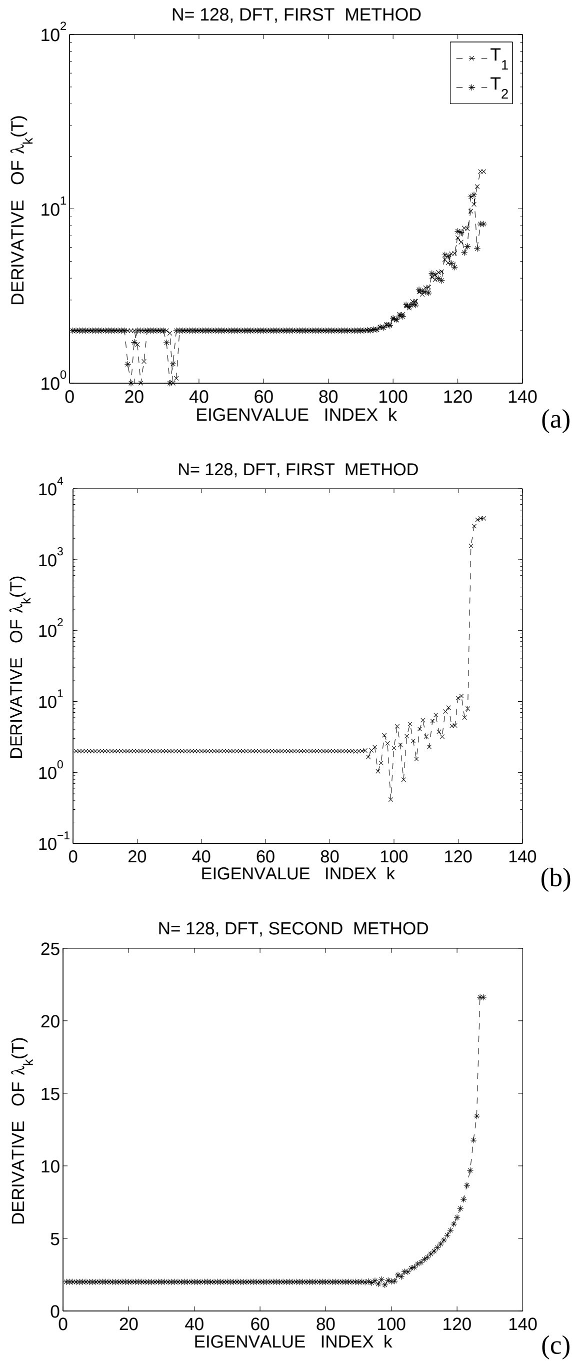

Figure 4: Eigenvalue spectrum of the commutor for the DFT: (a) when both Eq. (4) and Eq. (7) are satisfied with $\mathbf{a} = \mathbf{0}$, (b) with the second choice for the asymmetric vector $\mathbf{a}$, (c) when just Eq. (4) is satisfied.