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***Abstract*–In PCM transmission systems, ensuring repeater timing and regenerative capabilities requires monitored enforcement of the ones density of the line waveform. One possible corrective monitor action is to pre-empt information bits and insert ones when necessary, and thus introduce errors into the pulse stream. The analysis of the resultant error rates for classes of digital and analog implementations of a PCM monitor is accomplished via theoretical consideration of certain stationary Markov processes.**

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I. Introduction

Practical considerations place certain requirements on the binary data sequence transmitted through PCM (pulse code modulation) systems. An important requirement stems from a voltage threshold associated with the normal operating characteristics of a PCM digital repeater. Generally, a segment of a waveform corresponding to a particular bit sequence must have a minimum number of ones to preserve timing and regenerative capability at each repeater. In practice, transmitter data streams can be monitored to adapt the source bit sequences to meet a given engineering requirement, but the monitoring action introduces uncorrectable errors into the bit stream. The rate at which errors are inserted depends on the particular monitor implementation.

To investigate two possible PCM monitor designs, we present a pair of mathematical models representing distinct implementations of the monitoring function and determine the error rate corresponding to each implementation. The analysis of the models has led to the formulation and solution of a class of problems in the theory of stationary Markov processes.

One monitor implementation is digital in form and consists of using a minimum ones density coder at the transmitter. The coder action guarantees that the transmitted sequence contains a minimum number of ones, M, in any L consecutive bits $(0 < M < L)$. It is evident that the pair (M, L) can be chosen *a priori* by consideration of the analog waveform that a conditioned bit stream would produce. That is, a corrected stream would have enough variation in the transmitted analog waveform for proper repeater operation. We can visualize the digital coder action as the sliding of an $L - 1$-bit window along the transmitted stream (left to right) in 1-bit increments where the bit following the rightmost window bit is set to one whenever the within window sum is $M - 1$. We will see that the discrete nature of this algorithm lends itself to analysis by identifying words at the coder output as states of a Markov chain. The inserted error rate then corresponds to the average percentage of time that the coder spends doing corrective action.

Next we shall examine the second monitor implementation form, which is on the one hand straightforward in terms of the motivation which underlies it, but on the other hand vague insofar as its implications on the allowable bit patterns. Namely, the specification is that a certain signal (to be explained in Section III) derived from the PCM line signal must not drive a repeater tank voltage below a critical threshold (floor). A coupler network can enforce this specification by an analog monitoring of the signal, and changing certain zeros to ones if the voltage drops below the critical level. This second implementation would be applicable to PCM systems where an analog implementation of the monitor is more economical than digital implementation. The particulars of the monitor design and the motivation for the type of design chosen are not within the scope of this paper. Rather, we shall present a mathematical model for the monitor voltage process which simulates the repeater tank voltage. Our domain is probabilistic analysis of this threshold crossing from which we deduce an upper bound for the error rate. The samples of the analog waveform processed by the monitor form a stationary Markov sequence of random geometric series.

II. A Minimum Ones Density Coder–Digital Monitor

We begin our error rate analysis of the first model of a PCM monitor by observing its operation in terms of L-bit word (or state) transitions. The Markovian nature of the digital PCM monitor will be apparent upon the labeling of states and identifying the transition matrix. We will set aside, for the moment, the particulars of the initialization of the monitor since an initial state distribution would have to be specified. As we shall see presently, since the Markov chain describing the monitor function is irreducible, the specification of an initial distribution is irrelevant.

A. Description of Operation

The coding of incoming data $a_j (a_j = 0, 1)$ into transmitted data b_j proceeds by choosing $b_j = a_j$ if $W\{b_k\}_{j-L+1}^{j-1} \geqslant M$ or $b_j = 1$ if $W\{b_k\}_{j-L+1}^{j-1} = M - 1$ where $W\{b_k\}_{j-L+1}^{j-1}$ is the number of ones (weight) in the previous $L - 1$ bits. We note that since all blocks of length L in the transmitted stream contain at least M ones, there are exactly $2^L - \sum_{r=0}^{M-1} \binom{L}{r}$ permissible words or states at the coder output. We can index each state by using the binary word it represents, e.g., $(0, 1, 1, \cdots, 1)$ corresponds to the 2^{L-1}-1th state. We note that each state belongs to one of two categories.

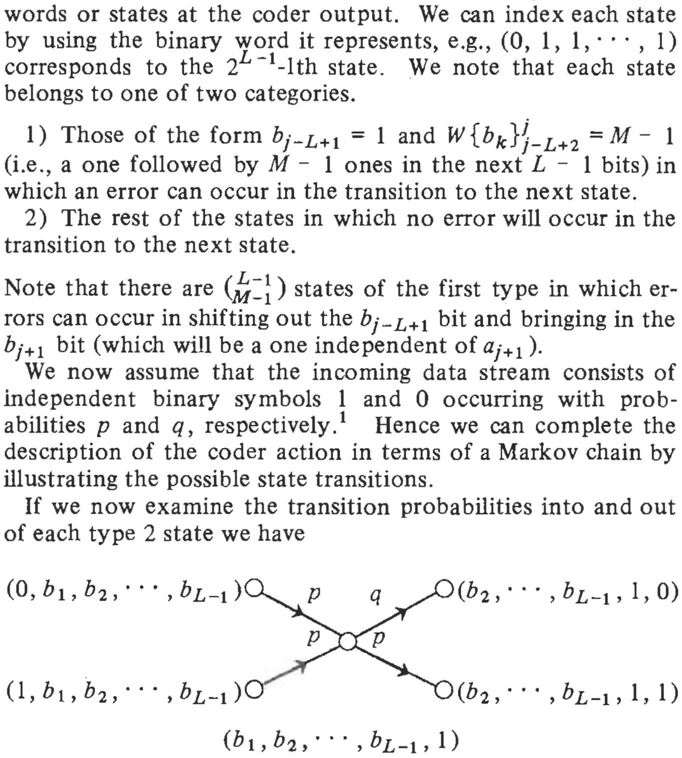
1) Those of the form $b_{j-L+1} = 1$ and $W\{b_k\}_{j-L+2}^{j} = M - 1$ (i.e., a one followed by $M - 1$ ones in the next $L - 1$ bits) in which an error can occur in the transition to the next state.

2) The rest of the states in which no error will occur in the transition to the next state.

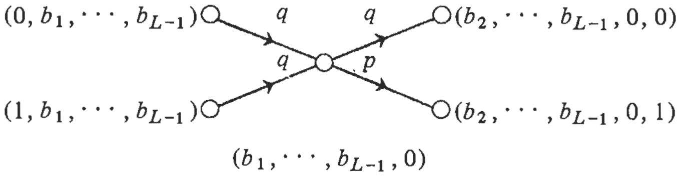
Note that there are $\binom{L-1}{M-1}$ states of the first type in which errors can occur in shifting out the b_{j-L+1} bit and bringing in the b_{j+1} bit (which will be a one independent of a_{j+1}).

We now assume that the incoming data stream consists of independent binary symbols 1 and 0 occurring with probabilities p and q, respectively.[1] Hence we can complete the description of the coder action in terms of a Markov chain by illustrating the possible state transitions.

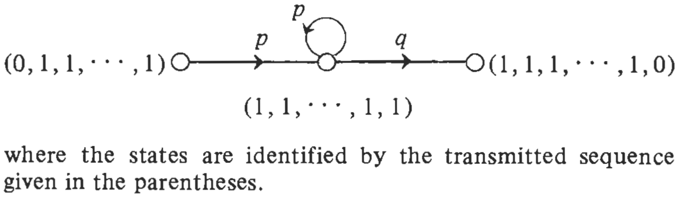
If we now examine the transition probabilities into and out of each type 2 state we have



or

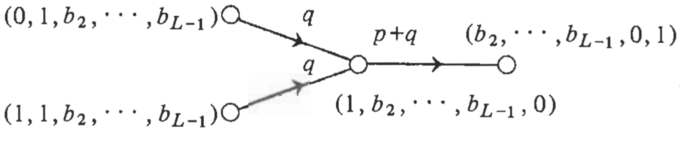


or

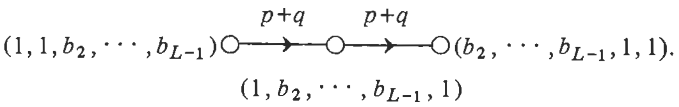


where the states are identified by the transmitted sequence given in the parentheses.

Similarly, for each type 1 state we have



or



It is evident that the above-illustrated transitions supply the entries of the stochastic matrix describing the state transitions of a Markov chain. It is easy to see that the chain is irreducible, that is, each state is eventually reachable from any other state because the only closed class is the set of all states.[2] By a theorem in finite Markov chains [1, pp. 335–356] an irreducible chain has a unique stationary limiting distribution which is independent of the initial distribution. We can now begin our search for this limiting distribution for two important cases.

B. Special Case; $p = q = 1/2$

Because of its simplicity, we treat a special case of great practical importance first. Letting $p = q = 1/2$, we get a probability equation for each possible state transition of the form

$$x_A = \tfrac{1}{2}x_B + \tfrac{1}{2}x_C$$

or

$$x_D = x_E$$

where x_A is the steady-state probability of being in state A. The obvious solution for these equations is for each x to be equal to the inverse of the number of states. Therefore

$$x_A = x_B = \cdots = \frac{1}{2^L - \sum_{r=0}^{M-1} \binom{L}{r}}.$$

Then the probability of error is the probability of being in a type 1 state times the probability that the next customer bit is a zero or

$$p_e = \frac{q\binom{L-1}{M-1}}{2^L - \sum_{r=0}^{M-1}\binom{L}{r}} = \frac{\frac{1}{2}\binom{L-1}{M-1}}{2^L - \sum_{r=0}^{M-1}\binom{L}{r}}. \quad (1)$$

C. General Case; p, q Arbitrary, $pq \neq 0$

Then the equations for the type 2 states are

$$x_{2k+1} = px_k + px_{k+2^{L-1}}$$

or

$$x_{2k} = qx_k + qx_{k+2^{L-1}}$$

for $k < 2^{L-1}$. Here the index denotes the binary representation of the monitor's contents. For the type 1 states we have

$$x_{2j} = qx_j + qx_{j+2^{L-1}}$$

or

$$x_{2i+1} = x_{i+2^{L-1}}$$

[1] The main case of interest is $p = q = 1/2$, but the analysis can be made more general easily enough, so we proceed with p, q unspecified, but $pq \neq 0$.

[2] The Markov chain terminology we use here is that found in [1].

for $i, j < 2^{L-1}$ and $W(2i+1) = W(2j) = M$ where $W(j)$ is the number of ones in the binary representation of j. Note that for $k < 2^{L-1}$

$$W(2k+1) = W(k + 2^{L-1}) = 1 + W(k) = 1 + W(2k). \quad (2)$$

Because of the symmetries involved in the state equations, we hypothesize that the solutions are of the form

$$x_k = y_{W(k)},$$

that is, all states with the same weight have the same probability of occurrence. Then, using (2), the state equations reduce to

$$y_{W(2k+1)}(1-p) = qy_{W(2k+1)} = py_{W(k)} \quad (3)$$

$$y_{W(2k)}(1-q) = py_{W(2k)} = qy_{W(2k+1)} \quad (4)$$

and since $W(k) = W(2k)$, it is apparent that the hypothesized form is indeed the solution.

There are exactly $\binom{L}{N}$ states with N ones in L bits (weight N), so we have

$$1 = \binom{L}{M} y_M + \binom{L}{M+1} y_{M+1} + \binom{L}{M+2} y_{M+2} + \cdots + \binom{L}{L} y_L \quad (5)$$

since there are no states of weight less than M. From (3) or (4) and (2)

$$y_{N+1} = \frac{p}{q} y_N$$

and (5) becomes

$$p^M q^{L-M} = \left[\binom{L}{M} q^{L-M} p^M + \binom{L}{M+1} q^{L-M-1} p^{M+1} + \cdots \binom{L}{L} p^L\right] y_M$$

or

$$y_M = \frac{p^M q^{L-M}}{1 - \sum_{r=0}^{M-1} \binom{L}{r} q^{L-r} p^r}. \quad (6)$$

The error probability is then

$$p_e = \binom{L-1}{M-1} q y_M = \frac{\binom{L-1}{M-1} p^M q^{L-M+1}}{1 - \sum_{r=0}^{M-1} \binom{L}{r} q^{L-r} p^r}. \quad (7)$$

We note briefly here that preliminary work, employing techniques associated with the strong law of large numbers, had realized fairly tight upper and lower bounds on the probability of error given by (7). The form of the bounds suggested the solution to the state equations for the Markov chain development reported here.

D. Some Calculations

It is obvious that (1) and (7) are identical for $p = q = 1/2$. For some cases of potential interest in PCM channels $p = q = 1/2$, $1 \gg \sum_{r=0}^{M-1} \binom{L}{r} q^{L-r} p^r$ and $p_e \approx \binom{L-1}{M-1} / 2^{L+1}$

$L = 16,\ M = 2$: $p_e \approx 15 \times 2^{-17} = 1.1 \times 10^{-4}$

$L = 32,\ M = 4$: $p_e \approx 4495 \times 2^{-33} = 5.2 \times 10^{-7}$

$L = 48,\ M = 6$: $p_e \approx 1.53 \times 10^6 \times 2^{-49} = 2.7 \times 10^{-9}$

$L = 64,\ M = 8$: $p_e \approx 5.53 \times 10^8 \times 2^{-65} = 1.5 \times 10^{-11}$.

The results are quite sensitive to q. For example, with $q = 0.6$ and $p = 0.4$,

$L = 16,\ M = 2$: $p_e \approx 1.13 \times 10^{-3}$

$L = 16,\ M = 8$: $p_e \approx 8.1 \times 10^{-8}$.

The error probability in the $L = 64, M = 8$ case is degraded by a factor of 5.4×10^3 if q is in fact 0.6 instead of 0.5.

Interest in error pairs comes from the fact that there are PCM line usages where an occurrence of a pair of consecutive errors is essentially no more degrading to performance than a singleton error. We note that the only states which can lead to consecutive errors are those of the form

$$(1, 1, M-2 \text{ ones in the next } L-2 \text{ bits}).$$

Thus the probability of two consecutive errors is

$$p_{2e} = \binom{L-2}{M-2} q^2 y_M = \frac{(M-1)q}{(L-1)} p_e,$$

which usually amounts to a negligible fraction of the total error rate p_e.

III. Analog Monitoring

We now give a mathematical development of the PCM line derived voltage process. The monitor uses this voltage to track what the repeater tank voltage would be if left unmonitored. Then we shall analyze this line derived voltage process to bound the percentage of time it spends below a floor level T expressed as a (small) percentage of its peak (all ones) level. The monitor pre-empts information bits with a dotting pattern[3] for the duration of the time for which consecutive voltage samples are below T. This pre-emption clearly results in the insertion of errors.

A. Error Rate Bound

Consider the random pulse train $v(t)$ obtained by setting the voltage level on each interval $(n, n+1)$ to 1 or 0, depending on whether an independent flip of a fair coin results in a head or a tail, respectively. (For simplicity we have scaled the time slots to unity.) Apply the signal $v(t)$ across a series resistor-capacitor connection and let $V_c(t)$ denote the voltage across the capacitor. Let τ denote the circuit time constant and set $r = e^{-\tau^{-1}}$. Sampling the capacitor voltage at the integers yields $\{V_c(n)\}_{-\infty}^{+\infty}$. Consider the problem of determining the percentage of samples for which the sampled output is below a threshold T. Letting $\chi_{r,T}(x) = \{^{0\ x \geq T}_{1\ x < T}$ we pose the problem as seeking

$$\lim_{N\to\infty} \frac{1}{2N+1} \left\{\sum_{-N}^{+N} \chi_{r,T}(V_c(n))\right\}. \quad (8)$$

To establish the existence of this limit notice

$$\{V_c(n) = (1-r)(a_n + a_{n-1} r + a_{n-2} r^2 + \cdots)\}_{-\infty}^{+\infty}$$

where $\{a_n\}_{-\infty}^{+\infty}$ are independent and identically distributed Bernoulli variates with 0 and 1 equiprobable. The stationarity of $\{V_c(n)\}_{-\infty}^{+\infty}$ is apparent. Now letting E stand for expectation, we have

$$E\{V_c(l)\, V_c(m)\} = \frac{(1-r)}{1+r} r^{|l-m|} \to 0 \ (|l-m| \to \infty).$$

[3] A dotting pattern is preferred to an all ones sequence because in practice the dotting pattern corresponds to a spectral power distribution which is superior from a crosstalk standpoint.

Thus we meet the hypothesis of a theorem in [2, p. 382] which is a variant of the Birkhoff-Khinchine ergodic theorem. We find (8) exists and equals $E(\chi_{r,T}(V_c(0))$ with probability one.

Now $E\{\chi_{r,T}(V_c(0))\}$ is merely the distribution function $F(r,T)$ of $V_c(0)$. Clearly, this distribution function vanishes for $T < 0$ and is 1 for $T > 1$. The determination of the distribution function $F(r, T)$ of such a random geometric series is a famous unsolved problem in probability theory [3]. However, $F(2^{-k^{-1}}, T)$ can be easily found for $T \leqslant (2^{k^{-1}} - 1)$ for any positive integer k. To accomplish this, rearrange the absolutely convergent series for $V_c(0)$ to get

$$\begin{aligned} V_0(0) = (1 - 2^{-k^{-1}}) \{&(a_0 + a_{-k} 2^{-1} + a_{-2k} 2^{-2} + \cdots) \\ &+ 2^{-k^{-1}} (a_1 + a_{-k-1} 2^{-1} + a_{-2k-2} 2^{-2} + \cdots) \\ &\vdots \\ &+ 2^{-(k-1)k^{-1}} (a_{-k+1} + a_{-2k+1} 2^{-2} + \cdots)\}. \end{aligned}$$

The random sums on each of the lines above are independent, and the ith sum is uniformly distributed on

$$[0, (1 - 2^{-k^{-1}}) \times 2^{1-ik^{-1}}] \quad [0 \leqslant i \leqslant k - 1).$$

So the distribution function of $V_c(0)$ is seen to be the resultant of a k-fold convolution of uniforms. Observe that on $[0, 2^{k^{-1}} - 1]$ the constancy of the uniforms gives that the repeated convolution reduces to the repeated integration

$$\left(\frac{1}{1 - 2^{-k^{-1}}}\right)^k \left(2 - \frac{1}{k} \int_0^x dx_{k-2}\right) \cdots \left(2^{2k^{-1}-1} \int_0^{x_2} dx_1\right) \cdot \left(2^{k^{-1}-1} \int_0^{x_1} dx_0\right) 2^{-1}.$$

Hence

$$F(2^{-k^{-1}}, T) = \frac{1}{2^{(k+1)/2} (1 - 2^{-k^{-1}})^k} \frac{T^k}{k!}. \tag{9}$$

Since (9) would not be changed if the return to zero point of each of the convolutants were increased by any positive number, we have that the right-hand side of (9) serves as an upper bound on $F(2^{-k^{-1}}, T)$ for all $T > 0$.

B. Example

Consider the following hypothetical situation. For a PCM system with a bit rate of 2 mb/s and with a nominal error rate of 10^{-6}, a monitor enforcing a 7 percent floor on the line-derived signal is to be examined. The time constant is nominally $100/\pi$; however, a ±23 percent variability is anticipated. The impact of the enforcement of this specification is by no means clear. While understanding the necessity for corrective monitoring of prohibited bit streams stemming from irregularities such as disconnects, concern has been expressed over the possibility of a deleterious effect on error rate during normal information transmission. However, it was felt that in the hypothetical situation outlined above, the threshold specification would not degrade the error rate significantly.

Employing (9) we can clarify the situation and defuse the issue concerning the corrective level monitoring by demonstrating that the monitor's contribution to the nominal line error rate is insignificant.

Transforming r to k and allowing for a 23 percent sensitivity analysis, the interval $17 < k < 27$ is of interest. By Stirling's approximation, the right-hand side of (9) increases as k decreases for $0 \leqslant T \leqslant 0.07$. So

$$\frac{(0.07)^{17}}{2^9 (1 - 2^{-1/17})^{17}\, 17!} < 10^{-13}$$

is an upper bound on the probability of error for even the smallest of time constants of the form $2^{-1/k}$ occurring in practice.

For completeness we next consider those values of r in the interval of interest which are not of the form $2^{-1/k}$. For convenience we unnormalize, seeking to upper-bound the 7 percent (of maximum) point of the distribution of $V_c(0)/(1 - r)$. Using primes to denote the unnormalized distribution functions, our objective is to upper-bound $F'(r, 0.07 \cdot (1/1 - k))$ for $r \in [2^{-1/17}, 2^{-1/27}]$. Notice that for each sequence of outcomes, $\{a_n\}_0^\infty$ $V_c(0)/1 - r$ is a monotone increasing function of r. So for each real x

$$F'(r^{**}, x) \leqslant F'(r^*, x) \quad (r^* < r^{**}).$$

Since F' is a distribution function

$$F'\left(r^*, 0.07 \frac{1}{1 - r^*}\right) < F'\left(r^*, 0.07 \frac{1}{1 - r^{**}}\right) \quad (r^* < r^{**}).$$

Combining the above two inequalities, we have that for

$$r \in [2^{-(1/k)}, 2^{-(1/k+1)}], \; F'\left(r, 0.07 \frac{1}{1 - r}\right) < F'\left(2^{-(1/k)}, 0.07 \frac{1}{1 - 2^{-(1/k+1)}}\right). \tag{10}$$

Now we can revert to normalized form and consider the 7 percent point of $V_c(0)$ directly (rather than $V_c(0)/1 - r$). Normalizing $V_c(0)/1 - 2^{-1/k}$, $[0, 1/1 - 2^{-1/k}] \to [0, 1]$; thus $0.7 \cdot (1/1 - 2^{-(1/k+1)}) \to 0.07(1 - 2^{-(1/k)}/1 - 2^{-(1/k+1)})$ and so corresponding to

$$F'\left(2^{-(1/k)}, 0.07 \frac{1}{1 - 2^{-(1/k+1)}}\right)$$

we have

$$F\left(2^{-(1/k)}, 0.07 \cdot \frac{1 - 2^{-(1/k)}}{1 - 2^{-(1/k+1)}}\right).$$

Substituting $T = 0.07 \cdot (1 - 2^{-(1/k)}/1 - 2^{-(1/k+1)})$ into (9) we get

$$F(r, 0.07) < \frac{(0.07)^k}{2^{(k+1/2)} (1 - 2^{-(1/k+1)})^k\, k!}, \quad 2^{-(1/k)} < r < 2^{-(1/k+1)}.$$

Numerical work shows $F(r, 0.07) < 10^{-12}$ for $2^{-(1/17)} < r < 2^{-(1/27)}$.

Indeed the corrective level monitoring of a maxentropic source contributes insignificantly to the nominal line error rate.

ACKNOWLEDGMENT

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theory of random geometric series to estimate the distribution of the response of a resonant circuit to a random pulse train.

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