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Commutators and Dyadic Paraproducts on weighted Lebesgue spaces

by

Dae-Won Chung

B.S., Chung-Ang University, Republic of Korea, 2002
M.S., Chung-Ang University, Republic of Korea, 2004

DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
Mathematics

The University of New Mexico

Albuquerque, New Mexico

July, 2010

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Dedication

To my parents, Seung-Guk and Soon-Rye, for their support, encouragement and love.

"And now these three remain: faith, hope and love. But the greatest of these is love" – Corinthians 13.13

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Abstract

We prove that the operator norm on weighted Lebesgue space $L^2(w)$ of the commutators of the Hilbert, Riesz and Beurling transforms with a BMO function b depends quadratically on the A_2-characteristic of the weight, as opposed to the linear dependence known to hold for the operators themselves. It is known that the operator norms of these commutators can be controlled by the norm of the commutator with appropriate Haar shift operators, and we prove the estimate for these commutators. For the shift operator corresponding to the Hilbert transform we use Bellman function methods, however there is now a general theorem for a class of Haar shift operators that can be used instead to deduce similar results. We invoke this general theorem to obtain the corresponding result for the Riesz transforms and the Beurling-Ahlfors operator. We can then extrapolate to $L^p(w)$, and the results are sharp for $1 < p < \infty$. We extend the linear bounds for the dyadic paraproduct on $L^2(w)$, [Be], into several variable setting using Bellman function arguments, that is,

we prove that the norm of the dyadic paraproduct on the weighted Lebesgue space $L^2_{\mathbb{R}^n}(w)$ is bounded with a bound that depends on $[w]_{A_2^d}$ and $\|b\|_{BMO^d}$ at most linearly. With this result, we can extrapolate to $L^p_{\mathbb{R}^n}(w)$ for $1 < p < \infty$. Furthermore, Bellman function arguments allow us to present the dimensionless linear bound in terms of the anisotropic weight characteristic.

Contents

Chapter 1

Introduction

In this dissertation we are primary interested in obtaining sharp weight inequalities for the commutators of the Hilbert, Riesz transforms and the Beurling-Ahlfors operators with multiplication by locally integrable function $b \in BMO$, and we are also concerned with the extension of the weighted norm estimates for the dyadic paraproduct into the several variable setting.

The study of singular integrals is one of the most important topics in harmonic analysis. The Hilbert transform is the prototypical example of a singular integral. A careful study of the Hilbert transform provided the understanding and the inspiration for the development of the general class of singular integrals. Almost simultaneously with the birth of singular integrals, a variety of questions related to weighted inequalities appeared. In 1960, Helson and Szegö first presented the boundedness of the Hilbert transform on $L^p(w)$ in [HS]. A better understanding of this subject was later obtained by Hunt, Muckenhoupt and Wheeden in 1973. They showed a new necessary and sufficient condition for the boundedness of the Hilbert transform on $L^p(w)$, the celebrated Muckenhoupt A_p condition in [HuMWh]. Precisely, we say the positive almost everywhere and locally integrable function w, a weight, satisfies the

Muckenhoupt A_p condition if:

$$[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{-1/(p-1)}\rangle_Q^{p-1} < \infty\,, \tag{1.1}$$

where we denote the average over the cube Q by $\langle\cdot\rangle_Q$ and the supremum is taken over all cubes. A year later, Coifman and Fefferman extended this result to a larger class of convolution type singular integrals with standard kernels (see [CoF]). Study of this subject has already reached a great level of perfection and has found applications in several branches of analysis, from complex function theory to partial differential equations.

There are still some unsolved problems in the area concerning the best constant in terms of the A_p-characteristic $[w]_{A_p}$. Precisely, one looks for a function $\phi(x)$, sharp in terms of its growth, such that:

$$\|Tf\|_{L^p(w)} \leq C\phi([w]_{A_p})\|f\|_{L^p(w)}\,. \tag{1.2}$$

These kinds of estimates for different singular integral operators are used often in the theory of partial differential equations. For instant, in [AsIS], the authors asked the question: Does the norm of the Beurling-Ahlfors operator on the weighted spaces $L^p(w)$ depend linearly on the A_p-characteristic of the weight $[w]_{A_p}$? Furthermore, they proved that the linear dependence of the weighted norm of the Beurling-Ahlfors operator on the A_p- characteristic provides the quasi-regularity of certain weak quasi-regular mapping. Although these type of problem (1.2) has attracted a lot of interest after [AsIS], we need to refer to the first result of this sort that was obtained by S. Buckley [Bu]. For the Hardy-Littlewood maximal function, he proved that $\phi(x) = x^{1/(p-1)}$ is the sharp rate of growth for all $1 < p < \infty$. He also showed in [Bu] that $\phi(x) = x^2$ works for the Hilbert transform in $L^2(w)$. S. Petermichl and S. Pott improved the result to $\phi(x) = x^{3/2}$, for the Hilbert transform in $L^2(w)$ in [PetPo]. More recently, S. Petermichl proved in [Pet2] the linear dependence, $\phi(x) = x$, for the Hilbert transform in $L^2(w)$

$$\|Hf\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}\,,$$

by estimating the operator norm of the dyadic shift. In fact, knowing a bound on $L^2(w)$ is crucial due to the extrapolation theorem. Rubio De Francia presented his famous, and mathematically convenient, extrapolation theorem in [Ru]. On the unweighted L^p-theory, one needs to find either the weak type or strong type estimate for an operator at two end points, then one can conclude the strong type estimate between endpoints by interpolation. However, on the weighted L^p-theory, it is enough to have a single strong type estimate for all weights $w \in A_{p_0}$ for an operator in $L^{p_0}(w)$, to obtain strong estimates in $L^p(w)$ for all $w \in A_p$, due to Rubio de Francia' extrapolation. Furthermore, a particular choice of the weight, $w \equiv 1$, yields the unweighted result. Recently, the sharp version of Rubio de Francia's extrapolation theorem was presented in [DGPerPet]. For example, when an operator T obeys linear bounds in $L^2(w)$, that is $\phi(x) = x$ in the inequality (1.2), the sharp extrapolation theorem will return the following bounds in $L^p(w)$, for $w \in A_p$:

$$\|Tf\|_{L^p(w)} \leq C(p)[w]_{A_p}^{\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(w)}\,. \tag{1.3}$$

It has been conjectured that the linear estimate holds for any Calderón-Zygmund operator T in $L^2(w)$. So far, it is known for only a small class of operators that the initial linear bound in $L^2(w)$ holds and is optimal, for instance the Beurling-Ahlfors operator [DV, PetV], the Hilbert transform [Pet2], Riesz transforms [Pet3], the martingale transform [Wi1], the square function [HukTV, Wi2], dyadic paraproduct [Be], well localized dyadic operators [L, LPetRe, CrMP, HyLReVa], and one-dimemsinal Calderón-Zygmund convolutions operators that are smooth averages of well localized operators [Va]. For some of them, not for all, the extrapolation bounds are optimal as well. For others (1.3) is optimal for $1 < p \leq 2$, but not for $p > 2$. For instance, the dyadic square function $\mathscr{S}_d$ obeys a linear bound when $p = 2$ (see [Wi2]) and this was extended to $1 < p < 2$ by extrapolation, and examples showed that the power $1/(p-1)$ is the best possible in [DGPerPet]. In [Le1], the author showed that for $p > 2$ the optimal power is at most $p/2(p-1)$. Recently, authors in [CrMP] showed for $p = 3$ the power is $1/2$ and by extrapolation they got

$$\|\mathscr{S}_d f\|_{L^p(w)} \leq C[w]_{A_p}^{\max\left\{\frac{1}{2},\frac{1}{p-1}\right\}} \|f\|_{L^p(w)} ,$$

and this bound is known to be optimal. Lerner [Le2] has very recently showed that this holds for Wilson's intrinsic square function, see [Wil2]. A modern introduction to weighted theory presenting related problems and much more can be found in the Lecture Notes by Carlos Perez [P4]. Most recently, in [PTV], the authors prove that the sharp bound of an arbitrary Calderón-Zygmund operator in $L^2(w)$ is $[w]_{A_2}\log(1+[w]_{A_2})$. Also, it is known that the initial linear bound in $L^2(w)$ holds for Calderón-Zygmund operators with sufficently smooth kernels [HyLReSaUrVa]. However, there is still an open conjecture between [PTV] and [HyLReSaUrVa] involving with smoothness levels of the kernel and dimensional constant (see [HyLReSaUrVa]).

In 2008, using Bellman function arguments, O. Beznosova proved that the dyadic paraproduct is bounded on $L^2(w)$ with the bound that depends on $[w]_{A_2^d}$ and $\|b\|_{BMO}$ at most linearly. The name *Paraproduct* was coined by Bony, in 1981 (see [Bo]), who used paraproducts to linearize the problem in the study of singularities of solutions of semilinear partial differential equations. After his work, the paraproducts have played an important role in harmonic analysis because they are examples of singular integral operators which are not translation-invariant. But they are not only examples; every singular integral operator which is bounded on L^2 decomposes into a paraproduct, an adjoint of a paraproduct, and an operator which behaves much like convolution operators. Moreover they arise as building blocks for more general operators such as multipliers.

In fact, the linear bound of the dyadic paraproducts in $\mathbb{R}^n$ are recovered in [HyLReVa, CrMP] using different methods in $L^2_{\mathbb{R}^n}(w)$. However, in this dissertation, to prove the linear bound of the dyadic paraproduct in $L^2_{\mathbb{R}^n}(w)$ we use the Bellman function arguments as well as [Be]. Furthermore, it turns out that the Bellman function proofs allow us to obtain dimensionless linear estimates in terms of anisotropic weight characteristic $[w]_{A_2^R}$.

Commutator operators are widely encountered and studied in many problems in PDEs, and Harmonic Analysis. One classical result of Coifman, Rochberg, and Weiss states in [CoRW] that, for the Calderón-Zygmund singular integral operator with a smooth kernel, $[b,T]f := bT(f) - T(bf)$ is a bounded operator on $L^p_{\mathbb{R}^n}$, $1 < p < \infty$, when b is a BMO function. Weighted estimates for the commutator have been studied in [ABKP], [P1], [P2], and [PPra]. Note that the commutator $[b,T]$ is more singular than the associated singular integral operator T, in particular, it does not satisfy the corresponding weak $(1,1)$ estimate. However one can find a weaker estimate in [P2]. In 1997, C. Pérez obtained the following result concerning commutators of singular integrals in [P2], for $1 < p < \infty$,

$$\|[b,T]f\|_{L^p(w)} \le C\|b\|_{BMO}[w]^2_{A_\infty}\|M^2 f\|_{L^p(w)}\,,$$

where $M^2 = M \circ M$ denotes the Hardy-Littlewood maximal function iterated twice. With this result and Buckley's sharp estimate for the maximal function [Bu] one can immediately conclude that

$$\|[b,T]\|_{L^p(w)\to L^p(w)} \le C[w]^2_{A_\infty}[w]^{\frac{2}{p-1}}_{A_p}\|b\|_{BMO}\,.$$

In this dissertation we show that for T the Hilbert, Riesz, Beurling transform, for $1 < p \le 2$ one can drop the $[w]_{A_\infty}$ term, in the above estimate, and this is sharp (Theorem 1.0.3). However for $p > 2$, the $L^p(w)$-norm of $[b,T]$ is bounded above by $\|b\|_{BMO^d}[w]^2_{A_p}$. For $T = H$ the Hilbert transform we prove, using Bellman function techniques similar to those used in [Be], [Pet2], the following Theorem.

Theorem 1.0.1. *There exists a constant $C > 0$, such that*

$$\|[b,H]\|_{L^p(w)\to L^p(w)} \le C\|b\|_{BMO}[w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}\|f\|_{L^p(w)}\,,$$

and this is sharp for $1 < p < \infty$.

Most of the work goes into showing the quadratic estimate for $p = 2$, sharp extrapolation [DGPerPet] then provides the right rate of growth for $p \neq 2$. Our method involves the use of the dyadic paraproduct π_b and its adjoint π_b^*, both of which obey linear estimates in $L^2(w)$, see [Be], like the Hilbert transform. It also uses Petermichl description of the Hilbert transform as an average of dyadic shift operators S, [Pet1], and reduces the estimate to obtaining corresponding estimates for the commutator $[b, S]$. After we decompose this commutator in three parts:

$$[b, S]f = [\pi_b, S]f + [\pi_b^*, S]f + [\lambda_b, S]f \,,$$

we estimate each commutator separately. This decomposition has been used before to analyze the commutator, [Pet1], [L], [LPetPiWic]. For precise definitions and detail derivations, see Section 2.2.1. The first two commutators immediately give the desired quadratic estimates in $L^2(w)$ from the known linear bounds of the operators commuted. For the third commutator we can prove a better than quadratic bound, in fact a linear bound. The following Theorem will be the crucial part of the proof.

Theorem 1.0.2. *There exists a constant* $C > 0$, *such that*

$$\|[\lambda_b, S]\|_{L^2(w)\to L^2(w)} \leq C[w]_{A_2^d}\|b\|_{BMO^d} \,, \tag{1.4}$$

for all $b \in BMO^d$ *and* $w \in A_2^d$.

This theorem is an immediate consequence of results in [HyLReVa], [LPetRe] and [CrMP], since the operator $[\lambda_b, S]$ belongs to the class of Haar shift operators for which they can prove linear bounds. We present a different proof of this result and others, using Bellman function techniques and bilinear Carleson embedding theorems, very much in the spirit of [Pet1] and [Be]. These arguments were found independently by the author, and we think they can be of interest.

We then observe that for any Haar shift operator T as defined in [LPetRe] the commutator $[\lambda_b, T]$ is again a Haar shift operator, and therefore it obeys linear bounds in the A_2-characteristic of the weight as in Theorem 1.0.2. As a consequence, we obtain quadratic bounds for all commutators of Haar shift operators and BMO function b. In particular, this holds true for Haar shift operators in $\mathbb{R}^n$ whose averages recover the Riesz transforms [Pet3] and for martingale transforms in $\mathbb{R}^2$ whose averages recover the Beurling-Ahlfors operator [PetV], [DV]. Extrapolation will provide $L^p(w)$ bounds which turn out to be sharp for the Riesz transforms and Beurling-Ahlfors operators as well. The following Theorem holds

Theorem 1.0.3. *Let T_τ be the first class of Haar shift operators of index τ. Its convex hull include the Hilbert transform, Riesz transforms, the Beurling-Ahlfors operator and so on. Then, there exists a constant $C(\tau, n, p)$ which only depend on τ, n and p such that*

$$\|[b, T_\tau]\|_{L^p(w)\to L^p(w)} \leq C(\tau, n, p)[w]_{A_p}^{2\max\{1, \frac{1}{p-1}\}}\|b\|_{BMO}$$

We are now ready to explain the organization of this dissertation. In Chapter 2 we will introduce notations and discuss some useful results about weighted Haar systems. We also present our main results for the commutator and the dyadic paraproduct in Chapter 2. In Chapter 3 we will introduce a number of Lemmas and Theorems that will be used. In Chapter 4 we will prove our results about the commutator of the Hilbert transform. In Chapter 5 we present the $L^p(w)$ estimate of the commutator with a Haar shift operator, Theorem 1.0.3. In Chapter 5 we also provide the sharpness for the commutators of Hilbert, Riesz transforms and Beurling-Ahlfors operators. Finally, In Chapter 6 we will prove our results about the multivariable dyadic paraproduct.

Chapter 2

Preliminaries

Let us now introduce the notation which will be used frequently through this dissertation. Even though the A_p conditions have already been introduced in (1.1), we will state the special case of this condition when $p = 2$, namely A_2^d since we will refer repeatedly to this. A weight w, which is positive almost everywhere and locally integrable function defined on $\mathbb{R}$, belongs to A_2^d class if

$$[w]_{A_2^d} := \sup_{I \in \mathcal{D}} \langle w \rangle_I \langle w^{-1} \rangle_I < \infty \,. \tag{2.1}$$

Here we take the supremum over all dyadic interval in $\mathbb{R}$. Note that if $w \in A_2$ then $w \in A_2^d$ and $[w]_{A_2^d} \leq [w]_{A_2}$. Intervals of the form $[k2^{-j}, (k+1)2^{-j})$ for integers j, k are called dyadic intervals. Let us denote $\mathcal{D}$ the collection of all dyadic intervals, and let us denote $\mathcal{D}(J)$ the collection of all dyadic subintervals of J. We use the symbol $\langle \cdot, \cdot \rangle$ for the standard inner product, that is

$$\langle f, g \rangle = \int fg \,.$$

Given a weight w in $\mathbb{R}$ a measurable function f belongs to $L^p_{\mathbb{R}}(w)$, $1 < p < \infty$, if

$$\|f\|_{L^p(w)} := \left(\int |f(x)|^p w(x)\, dx \right)^{1/p} < \infty \,.$$

Through out this dissertation, we denote a constant by c or C which may change line by line and we keep indicating its dependence on various parameters using a parenthesis, e.g. $C(n,p)$ will mean a constant depending on the dimension n and on the parameter p.

2.1 Haar systems in $\mathbb{R}$

For any interval I, there is a Haar function defined by

$$h_I(x) = \frac{1}{|I|^{1/2}}(\chi_{I_+}(x) - \chi_{I_-}(x)),$$

where χ_I denotes the characteristic function of the interval I, $\chi_I(x) = 1$ if $x \in I$, $\chi_I(x) = 0$ otherwise, and $I_\pm$ are the right and left halves of I. It is a well known fact that the Haar system $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal system in $L^2_{\mathbb{R}}$. In fact, the Haar system was introduced by Alfréd Haar, in 1909, to see the existence of an orthonormal system for $L^2[0,1]$, so that convergence will be uniform for continuous functions. It is also now known as the first wavelet.

We also consider the different grids of dyadic intervals parametrized by α, r, defined by

$$\mathcal{D}^{\alpha,r} = \{\alpha + rI : I \in \mathcal{D}\},$$

for $\alpha \in \mathbb{R}$ and positive r. For each grid $\mathcal{D}^{\alpha,r}$ of dyadic intervals, there are corresponding Haar functions h_I, $I \in \mathcal{D}^{\alpha,r}$ that are an orthonormal system in $L^2_{\mathbb{R}}$. Let us introduce a proper orthonormal system for $L^2_{\mathbb{R}}(w)$. The weighted or disbalanced Haar function associated to an interval I and a weight w is

$$h_I^w := \frac{1}{w(I)^{1/2}}\left[\frac{w(I_-)^{1/2}}{w(I_+)^{1/2}}\chi_{I_+} - \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}}\chi_{I_-}\right],$$

where $w(I) = \int_I w$. We define the weighted inner product by $\langle f,g\rangle_w = \int fgw$. Then, every function $f \in L^2(w)$ can be written as

$$f = \sum_{I \in \mathcal{D}} \langle f, h_I^w \rangle_w h_I^w \,,$$

where the sum converges a.e. in $L^2(w)$. Moreover,

$$\|f\|_{L^2(w)}^2 = \sum_{I \in \mathcal{D}} |\langle f, h_I^w \rangle_w|^2 \,. \tag{2.2}$$

Again $\mathcal{D}$ can be replaced by $\mathcal{D}^{\alpha,r}$ and the corresponding weighted Haar functions are an orthonormal system in $L^2(w)$. For convenience we will observe basic properties of the disbalanced Haar system. First observe that $\langle h_K, h_I^w \rangle_w$ could be non-zero only if $I \supseteq K$, moreover, for any $I \supseteq K$,

$$|\langle h_K, h_I^w \rangle_w| \le \langle w \rangle_K^{1/2} \,. \tag{2.3}$$

Here is the the calculation that provides (2.3),

$$\begin{aligned} |\langle h_K, h_I^w \rangle_w| &= \left| \int \frac{\chi_{K_+}(x) - \chi_{K_-}(x)}{|K|^{1/2} w(I)^{1/2}} \left[\frac{w(I_-)^{1/2}}{w(I_+)^{1/2}} \chi_{I_+}(x) - \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}} \chi_{I_-}(x) \right] w(x) dx \right| \\ &\le \frac{1}{|K|^{1/2} w(I)^{1/2}} \underbrace{\int_K \left| \frac{w(I_-)^{1/2}}{w(I_+)^{1/2}} \chi_{I_+}(x) + \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}} \chi_{I_-}(x) \right| w(x) dx}_{A} \,. \end{aligned}$$

If $K \subset I_+$, then $A \le w(I_-)^{1/2} w(I_+)^{-1/2} w(K)$. Thus

$$|\langle h_K, h_I^w \rangle_w| \le \langle w \rangle_K^{1/2} \sqrt{\frac{w(K) w(I_-)}{w(I_+) w(I)}} \le \langle w \rangle_K^{1/2} \,.$$

Similarly, if $K \subset I_-$. If $K = I$, then $A \le 2\, w(K_-)^{1/2} w(K_+)^{1/2}$. Thus

$$|\langle h_K, h_I^w \rangle_w| = |\langle h_K, h_K^w \rangle_w| \le \langle w \rangle_K^{1/2} \frac{2\sqrt{w(K_-) w(K_+)}}{w(K)} \le \langle w \rangle_K^{1/2} \,.$$

Estimate (2.3) implies that $|\langle h_{\hat{J}}, h_{\hat{J}}^{w^{-1}} \rangle_{w^{-1}} \langle h_J, h_J^w \rangle_w| \le \sqrt{2} [w]_{A_2^d}^{1/2}$, where $\hat{J}$ is the parent of J,

$$\begin{aligned} |\langle h_{\hat{J}}, h_{\hat{J}}^{w^{-1}} \rangle_{w^{-1}} \langle h_J, h_J^w \rangle_w| &\le \langle w^{-1} \rangle_{\hat{J}}^{1/2} \langle w \rangle_J^{1/2} = \langle w^{-1} \rangle_{\hat{J}}^{1/2} \left(\frac{1}{|J|} \int_J w \right)^{1/2} \\ &= \langle w^{-1} \rangle_{\hat{J}}^{1/2} \left(\frac{2}{|\hat{J}|} \int_J w \right)^{1/2} \le \sqrt{2} \langle w^{-1} \rangle_{\hat{J}}^{1/2} \langle w \rangle_{\hat{J}}^{1/2} \\ &\le \sqrt{2}\, [w]_{A_2^d}^{1/2} \,. \end{aligned} \tag{2.4}$$

Also, one can deduce similarly the following estimate

$$|\langle h_J, h_J^{w^{-1}}\rangle_{w^{-1}}\langle h_{J_-}, h_J^w\rangle_w| \leq \sqrt{2}[w]_{A_2^d}^{1/2}. \tag{2.5}$$

For $I \supsetneq J$, h_I^w is constant on J. We will denote this constant by $h_I^w(J)$. Then $h_{\hat{J}}^w(J)$ is the constant value of $h_{\hat{J}}^w$ on J and $|h_{\hat{J}}^w(J)| \leq w(J)^{-1/2}$, as can be seen by the following estimate,

$$|h_{\hat{J}}^w(J)| = \begin{cases} w(\hat{J}_-)^{1/2}w(\hat{J})^{-1/2}w(\hat{J}_+)^{-1/2} \leq w(J)^{-1/2} & \text{if} \quad J = \hat{J}_+ \\ w(\hat{J}_+)^{1/2}w(\hat{J})^{-1/2}w(\hat{J}_-)^{-1/2} \leq w(J)^{-1/2} & \text{if} \quad J = \hat{J}_-. \end{cases} \tag{2.6}$$

Let us define the weighted averages, $\langle g\rangle_{J,w} := w(J)^{-1}\int_J g(x)w(x)dx$. As with the standard Haar system, we can write the weighted averages

$$\langle g\rangle_{J,w} = \sum_{I\in\mathcal{D}:I\supsetneq J}\langle g, h_I^w\rangle_w h_I^w(J). \tag{2.7}$$

In fact, here is the derivation of (2.7).

$$\begin{aligned}\langle g\rangle_{J,w} &= \frac{1}{w(J)}\int_J\sum_{I\in\mathcal{D}}\langle g, h_I^w\rangle_w h_I^w(x)w(x)dx = \frac{1}{w(J)}\int_J\sum_{I\in\mathcal{D}:I\supsetneq J}\langle g, h_I^w\rangle_w h_I^w(J)w(x)dx \\ &= \frac{1}{w(J)}\int_J w(x)dx\sum_{I\in\mathcal{D}:I\supsetneq J}\langle g, h_I^w\rangle_w h_I^w(J) = \sum_{I\in\mathcal{D}:I\supsetneq J}\langle g, h_I^w\rangle_w h_I^w(J).\end{aligned}$$

Also, we will be using system of functions $\{H_I^w\}_{I\in\mathcal{D}}$ defined by

$$H_I^w = h_I\sqrt{|I|} - A_I^w\chi_I \quad \text{where} \quad A_I^w = \frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{2\langle w\rangle_I}. \tag{2.8}$$

Then, $\{w^{1/2}H_I^w\}$ is orthogonal in L^2 with norms satisfying the inequality

$$\|w^{1/2}H_I^w\|_{L^2} \leq \sqrt{|I|\langle w\rangle_I},$$

refer to [Be] or Section 2.2, where the calulation is performed for corresponding system in $\mathbb{R}^n$ defined in (2.14). Moreover, by Bessel's inequality we have, for all $g \in L^2$,

$$\sum_{I\in\mathcal{D}}\frac{1}{|I|\langle w\rangle_I}\langle g, w^{1/2}H_I^w\rangle^2 \leq \|g\|_{L^2}^2. \tag{2.9}$$

Since unconditional bases provide an efficient way to represent an arbitrary function in terms of known functions, it is very useful concept in functional analysis. It is a well known fact that the Haar system generates an unconditional basis in L^p for $1 < p < \infty$. We refer to [Wo] for more detailed statements. The authors in [CoJS] showed that the weighted Haar system forms an unconditional basis on L^2, and it was shown in [TV] that if the weight w satisfies the A_2-condition then the unweighted Haar system generates an unconditional basis on $L^2(w)$. Through out the dissertation, there are many manipulations with infinite sums associated with Haar functions. However, those manipulations are legitimate because Haar systems used in this dissertation are unconditional bases on corresponding function spaces.

2.2 Haar systems in $\mathbb{R}^n$

In this dissertation, the result of [Be] will generalize to the setting of $\mathbb{R}^n$ (Theorem 2.4.4 and Theorem 2.4.5). Thus, we need to introduce the appropriate n-dimensional Haar systems. We will denote the family of dyadic cubes in $\mathbb{R}^n$ by $\mathcal{D}^n$. For any $Q \in \mathcal{D}^n$, we set $\mathcal{D}_1^n(Q) \equiv \{Q' \in \mathcal{D}^n : Q' \subset Q, \ell(Q') = \ell(Q)/2\}$, the class of 2^n dyadic sub-cubes of Q, where we denote the side length of cubes by $\ell(Q)$. We will also denote the class of all dyadic sub-cubes of Q by $\mathcal{D}^n(Q)$. Then we can write $\mathcal{D}^n(Q) = \bigcup_{j=0}^{\infty} \mathcal{D}_j^n(Q)$. We refer to [Wil1] for the following lemma.

Lemma 2.2.1. *Let $Q \in \mathcal{D}^n$. Then, there are $2^n - 1$ pairs of sets $\{(E_{j,Q}^1, E_{j,Q}^2)\}_{j=1,\ldots,2^n-1}$ such that:*

(1) for each j, $|E_{j,Q}^1| = |E_{j,Q}^2|$.

(2) for each j, $E_{j,Q}^1$ and $E_{j,Q}^2$ are non-empty unions of cubes from $\mathcal{D}_1^n(Q)$;

(3) for each j, $E_{j,Q}^1 \cap E_{j,Q}^2 = \emptyset$;

(4) *for every* $j \neq k$, *one of the following must hold:*

(a) $E^1_{j,Q} \cup E^2_{j,Q}$ *is entirely contained in either* $E^1_{k,Q}$ *or* $E^2_{k,Q}$*;*

(b) $E^1_{k,Q} \cup E^2_{k,Q}$ *is entirely contained in either* $E^1_{j,Q}$ *or* $E^2_{j,Q}$*;*

(c) $\big(E^1_{j,Q} \cup E^2_{j,Q}\big) \cap \big(E^1_{k,Q} \cup E^2_{k,Q}\big) = \emptyset$.

We can construct such a set by induction on n. It is clear when $n = 1$. We assume that Lemma 2.2.1 is true for $n-1$ and let $\widetilde{Q}$ be the $(n-1)$-dimensional cube and $\big\{(E^1_{j,\widetilde{Q}}, E^2_{j,\widetilde{Q}})\big\}_{j=1,\dots,2^{n-1}-1}$ be the corresponding pairs of sets for $\widetilde{Q}$. We can get the first pair of sets by $(E^1_{1,Q}, E^2_{1,Q}) := (\widetilde{Q} \times I_-, \widetilde{Q} \times I_+)$ where I is a dyadic interval so that $|I| = \ell(\widetilde{Q})$, and $\widetilde{Q} \times I = Q$. We also have the last $2^n - 2$ pairs of sets as follows.

$$\begin{aligned}
&\{(E^1_{2j,Q},E^2_{2j,Q}), (E^1_{2j+1,Q}, E^2_{2j+1,Q})\}_{j=1,\dots,2^{n-1}-1} \\
&\qquad := \big\{(E^1_{j,\widetilde{Q}} \times I_-, E^2_{j,\widetilde{Q}} \times I_-),\, (E^1_{j,\widetilde{Q}} \times I_+, E^2_{j,\widetilde{Q}} \times I_+)\big\}_{j=1,\dots,2^{n-1}-1}.
\end{aligned}$$

To save space, we denote $E^1_{j,Q} \cup E^2_{j,Q}$ by $E_{j,Q}$ and, by (1) in Lemma 2.2.1, we have $|E_{j,Q}| = 2|E^i_{j,Q}|$ for $i = 1, 2$. Note that the sets $E_{j,Q}$ are rectangles. Also note that we assign $E_{1,Q} = Q$, $E_{2,Q} = E^1_{1,Q}$ and $E_{3,Q} = E^2_{1,Q}$ and so on. With such a choice, we have

$$\begin{aligned}
Q = E_{1,Q} &= E^1_{1,Q} \cup E^2_{1,Q} \\
&= E_{2,Q} \cup E_{3,Q} \\
&= (E_{4,Q} \cup E_{6,Q}) \cup (E_{5,Q} \cup E_{7,Q}) \\
&\quad \vdots \\
&= E_{2^{n-1},Q} \cup E_{2^{n-1}+1,Q} \cup \cdots \cup E_{2^n-1,Q} \\
&= E^1_{2^{n-1},Q} \cup E^2_{2^{n-1},Q} \cup E^1_{2^{n-1}+1,Q} \cup E^2_{2^{n-1}+1,Q} \cup \cdots \cup E^1_{2^n-1,Q} \cup E^2_{2^n-1,Q},
\end{aligned}$$

in fact,

$$Q = \bigcup_{j=2^k}^{2^k-1} E_{j,Q} = \bigcup_{j=2^k}^{2^k-1} (E^1_{j,Q} \cup E^2_{j,Q}),$$

the sets $E_{j,Q}$ in that range of j's are disjoint, and

$$\mathcal{D}_1^n(Q) = \{E^1_{2^{n-1},Q}\,, E^2_{2^{n-1},Q}\,, E^1_{2^{n-1}+1,Q}\,, E^2_{2^{n-1}+1,Q}\,, \dots, E^1_{2^n-1,Q}\,, E^2_{2^n-1,Q}\}\,.$$

As a consequence of Lemma 2.2.1, we can introduce the proper weighted Haar system for $L^2_{\mathbb{R}^n}(w)$, $\{h^w_{j,Q}\}_{1\le j\le 2^n-1, Q\in\mathcal{D}^n}$, where

$$h^w_{j,Q} := \frac{1}{\sqrt{w(E_{j,Q})}}\left[\sqrt{\frac{w(E^1_{j,Q})}{w(E^2_{j,Q})}}\chi_{E^2_{j,Q}} - \sqrt{\frac{w(E^2_{j,Q})}{w(E^1_{j,Q})}}\chi_{E^1_{j,Q}}\right].$$

Note that when $n=1$, this reduces to the one dimensional disbalanced Haar system. Due to its construction, $h^w_{j,Q}$'s satisfy that

$$\int h^w_{j,Q}(x)w(x)dx = 0\,, \quad \text{for all} \quad j\,, \tag{2.10}$$

and

$$\int h^w_{j,Q}(x)\, h^w_{i,Q}(x)w(x)dx = \delta_{ij}\,, \quad \text{for all } i \text{ and } j\,. \tag{2.11}$$

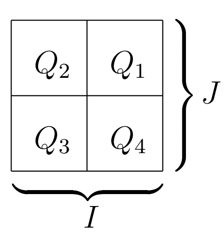
Then, every function $f \in L^2_{\mathbb{R}^n}(w)$ can be written as

$$f = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle f, h^w_{j,Q}\rangle_w h^w_{j,Q}\,.$$

Moreover,

$$\|f\|^2_{L^2_{\mathbb{R}^n}(w)} = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}|\langle f, h^w_{j,Q}\rangle_w|^2\,.$$

For better understanding, we now observe the example of 2-dimensional case. Let us consider the dyadic box $Q = I\times J \in \mathcal{D}^2$ as the figure below, in this case $\widetilde{Q} = J$, the $2-1=1$ dimensional cube in the construction.



Then, there are three pairs of sets

$$\left\{(Q_3 \cup Q_4, Q_1 \cup Q_2),\ (Q_3, Q_4),\ (Q_2, Q_1)\right\},$$

and three weighted Haar functions associated with Q

$$\begin{aligned}
h^w_{1,Q} &= \frac{1}{\sqrt{w(Q)}}\left[\sqrt{\frac{w(Q_3 \cup Q_4)}{w(Q_1 \cup Q_2)}}\chi_{Q_1 \cup Q_2} - \sqrt{\frac{w(Q_1 \cup Q_2)}{w(Q_3 \cup Q_4)}}\chi_{Q_3 \cup Q_4}\right] \\
h^w_{2,Q} &= \frac{1}{\sqrt{w(Q_3 \cup Q_4)}}\left[\sqrt{\frac{w(Q_3)}{w(Q_4)}}\chi_{Q_4} - \sqrt{\frac{w(Q_4)}{w(Q_3)}}\chi_{Q_3}\right] \\
h^w_{3,Q} &= \frac{1}{\sqrt{w(Q_1 \cup Q_2)}}\left[\sqrt{\frac{w(Q_2)}{w(Q_1)}}\chi_{Q_1} - \sqrt{\frac{w(Q_2)}{w(Q_1)}}\chi_{Q_2}\right].
\end{aligned}$$

One can easily see that the functions $h^w_{j,Q},\ j = 1, 2, 3$, satisfy (2.10).

$$\begin{aligned}
&\int h^w_{1,Q}\, h^w_{2,Q} w dx \\
&= \frac{1}{\sqrt{w(Q)}\sqrt{w(Q_3 \cup Q_4)}}\sqrt{\frac{w(Q_1 \cup Q_2)}{w(Q_3 \cup Q_4)}}\left(\int_{Q_3}\sqrt{\frac{w(Q_4)}{w(Q_3)}}wdx - \int_{Q_4}\sqrt{\frac{w(Q_3)}{w(Q_4)}}wdx\right) \\
&= \frac{1}{\sqrt{w(Q)}\sqrt{w(Q_3 \cup Q_4)}}\sqrt{\frac{w(Q_1 \cup Q_2)}{w(R_3 \cup Q_4)}}\left(\sqrt{w(Q_4)w(Q_3)} - \sqrt{w(Q_3)w(Q_4)}\right) \\
&= 0\,.
\end{aligned}$$

Similarly, one can check the other cases of (2.11).

With the particular choice of $w \equiv 1$, set $h_{j,Q} = h^1_{j,Q}$, the family $\{h_{j,Q} : Q \in \mathcal{D}^n, 1 \le j \le 2^n - 1\}$ is an orthonormal system for $L^2_{\mathbb{R}^n}$ and complete. Again, for all $f \in L^2_{\mathbb{R}^n}$, we have

$$\|f\|^2_{L^2_{\mathbb{R}^n}} = \sum_{Q \in \mathcal{D}^n} \sum_{j=1}^{2^n-1} |\langle f, h_{j,Q}\rangle|^2\,.$$

For all $Q' \in \mathcal{D}^n_1(Q)$, the $h_{j,Q}$'s and $h^w_{j,Q}$'s are constant on Q', we will also denote this constant by $h_{j,Q}(Q')$ and $h^w_{j,Q}(Q')$ respectively. As in the one dimensional weighted

Haar system (2.7), we can obtain the weighted average of f over $E_{j,Q}$ for some $1 \leq j \leq 2^n - 1$,

$$\begin{aligned}
\langle f \rangle_{E_{j,Q},w} &= \frac{1}{w(E_{j,Q})} \int_{E_{j,Q}} \sum_{Q' \in \mathcal{D}^n} \sum_{i=1}^{2^n-1} \langle f, h^w_{i,Q'} \rangle h^w_{i,Q'}(x) w(x) dx \\
&= \frac{1}{w(E_{j,Q})} \Bigg(\int_{E_{j,Q}} \sum_{Q' \in \mathcal{D}^n : Q' \supsetneq Q} \sum_{i : E_{i,Q} \supsetneq E_{i,Q}} \langle g, h^w_{i,Q'} \rangle h^w_{i,Q'}(E_{j,Q}) w(x) dx \\
&\qquad\qquad + \int_{E_{j,Q}} \sum_{i : E_{i,Q} \supsetneq E_{j,Q}} \langle g, h^w_{i,Q'} \rangle h^w_{i,Q'}(E_{j,Q}) w(x) dx \Bigg) \\
&= \sum_{Q' \in \mathcal{D}^n : Q' \supsetneq Q} \sum_{i=1}^{2^n-1} \langle g, h^w_{i,Q'} \rangle h^w_{i,Q'}(E_{j,Q}) + \sum_{i : E_{i,Q} \supsetneq E_{j,Q}} \langle g, h^w_{i,Q'} \rangle h^w_{i,Q'}(E_{j,Q}) \\
&= \sum_{Q' \in \mathcal{D}^n : Q' \supseteq Q} \sum_{i : E_{i,Q'} \supsetneq E_{j,Q}} \langle g, h_{i,Q'} \rangle h_{i,Q'}(E_{j,Q}) \,. \qquad (2.12)
\end{aligned}$$

Furthermore, for $j = 1$, $E_{1,Q} = Q$, we have

$$\langle f \rangle_{E_{1,Q},w} = \langle f \rangle_{Q,w} = \sum_{Q' \in \mathcal{D}^n : Q' \supsetneq Q} \sum_{j=1}^{2^n-1} \langle f, h^w_{j,Q'} \rangle_w h^w_{j,Q'}(Q) \,. \qquad (2.13)$$

Because it is occasionally more convenient to deal with simpler functions, it might be good to have an orthogonal system in $L^2_{\mathbb{R}^n}(w)$, similar to the one dimensional case defined in (2.8). Let us define

$$H^w_{j,Q} := h_{j,Q} \sqrt{|E_{j,Q}|} - A^w_{j,Q} \, \chi_{E_{j,Q}} \,, \qquad (2.14)$$

where

$$A^w_{j,Q} := \frac{\langle w \rangle_{E^2_{j,Q}} - \langle w \rangle_{E^1_{j,Q}}}{2 \langle w \rangle_{E_{j,Q}}} \,.$$

Then, the family of functions $\{w^{1/2} H^w_{j,Q}\}_{j,Q}$ is an orthogonal system for $L^2_{\mathbb{R}^n}$ with norms satisfying the inequality $\|w^{1/2} H^w_{j,Q}\|_{L^2_{\mathbb{R}^n}} \leq \sqrt{|E_{j,Q}| \langle w \rangle_{E_{j,Q}}}$. In order to see the orthogonality, it is enough to check that each function has zero mean with respect to the measure induced by the weight w, i.e. wdx,

$$\begin{aligned}\int H^w_{j,Q}(x)w(x)dx &= \sqrt{|E_{j,Q}|}\int h_{j,Q}(x)w(x)dx - \int_{E_{j,Q}} A^w_{j,Q}w(x)dx \\ &= \int_{E^2_{j,Q}} w(x)dx - \int_{E^1_{j,Q}} w(x)dx - \frac{|E_{j,Q}|}{2}\big(\langle w\rangle_{E^2_{j,Q}} - \langle w\rangle_{E^1_{j,Q}}\big) \\ &= \int_{E^2_{j,Q}} w(x)dx - \int_{E^1_{j,Q}} w(x)dx - \int_{E^2_{j,Q}} w(x)dx + \int_{E^1_{j,Q}} w(x)dx = 0\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\|w^{1/2}H^w_{j,Q}\|^2_{L^2_{\mathbb{R}^n}} &= \|H^w_{j,Q}\|^2_{L^2_{\mathbb{R}^n}(w)} = \int \big(H^w_{j,Q}(x)\big)^2 w(x)dx \\ &= |E_{j,Q}|\int \big(h_{j,Q}(x)\big)^2 w(x)dx + \int_{E_{j,Q}} \big(A^w_{j,Q}\big)^2 w(x)dx \\ &\qquad - 2\sqrt{|E_{j,Q}|}\int_{E_{j,Q}} h_{j,Q}(x)A^w_{j,Q}\,w(x)dx \\ &= \int_{E_{j,Q}} w(x)dx + |E_{j,Q}|\frac{\big(\langle w\rangle_{E^2_{j,Q}} - \langle w\rangle_{E^1_{j,Q}}\big)^2}{4\langle w\rangle_{E_{j,Q}}} - |E_{j,Q}|\frac{\big(\langle w\rangle_{E^2_{j,Q}} - \langle w\rangle_{E^1_{j,Q}}\big)^2}{2\langle w\rangle_{E_{j,Q}}} \\ &= |E_{j,Q}|\left(\langle w\rangle_{E_{j,Q}} - \frac{\big(\langle w\rangle_{E^2_{j,Q}} - \langle w\rangle_{E^1_{j,Q}}\big)^2}{4\langle w\rangle_{E_{j,Q}}}\right) \\ &\le |E_{j,Q}|\langle w\rangle_{E_{j,Q}}\,.\end{aligned}$$

By Bessel's inequality in $L^2_{\mathbb{R}^n}$ one gets, for all $g \in L^2_{\mathbb{R}^n}$,

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\frac{\langle gw^{1/2}, H^w_{j,Q}\rangle^2}{|E_{j,Q}|\,\langle w\rangle_{E_{j,Q}}} \le \|g\|^2_{L^2_{\mathbb{R}^n}}\,. \tag{2.15}$$

Then by setting $g = fw^{1/2}$ in (2.15) one gets, for all $f \in L^2_{\mathbb{R}^n}(w)$,

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\frac{\langle f, H^w_{j,Q}\rangle^2_w}{|E_{j,Q}|\langle w\rangle_{E_{j,Q}}} \le \|f\|^2_{L^2_{\mathbb{R}^n}}(w)\,. \tag{2.16}$$

2.3 The commutator: Hilbert transform case

A basic example which lies at the source of the theory of singular integrals is given by the Hilbert transform. The Hilbert transform of a function $f(y)$ is given formally by the principal value integral

$$H(f) := p.v. \frac{1}{\pi} \int \frac{f(y)}{x-y}\, dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy\,.$$

The bilinear operator

$$fH(g) + H(f)g$$

maps $L^2 \times L^2$ into $\mathcal{H}^1$, here H is the Hilbert transform and $\mathcal{H}^1$ is the real Hardy space defined by

$$\mathcal{H}^1(\mathbb{R}) := \{f \in L^1(\mathbb{R}) :\ Hf \in L^1(\mathbb{R})\}$$

with norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_{L^1} + \|Hf\|_{L^1}\,.$$

The dual of $\mathcal{H}^1$ is BMO. This is the celebrated Fefferman-Stein duality Theorem [FS], we define BMO after Definition 2.3.1. Thus we will pair with a BMO function b. Using that $H^* = -H$, we obtain that

$$\langle\, fH(g) + H(f)g,\, b\,\rangle = \langle\, f,\, H(g)b - H(gb)\,\rangle\,.$$

Hence the operator $g \mapsto H(g)b - H(gb)$ should be L^2 bounded. For more detail we refer [G]. This expression $H(g)b - H(gb)$ is called the commutator of H with the BMO function b. More generally, we define as follows.

Definition 2.3.1. The commutator of the Hilbert transform H with a function b is defined as

$$[b, H](f) = bH(f) - H(bf)\,.$$

Our main concern in this dissertation is to prove that the commutator $[b, H]$, for $b \in BMO$, as an operator from $L^2_{\mathbb{R}}(w)$ into $L^2_{\mathbb{R}}(w)$ is bounded by the square of the A_2-characteristic, $[w]_{A_2}$, of the weight times the BMO norm, $\|b\|_{BMO}$, of b where

$$\|b\|_{BMO} := \sup_I \frac{1}{|I|} \int_I | b(x) - \langle b \rangle_I | dx \,.$$

The supremum is taken over all intervals in $\mathbb{R}$. Note that when we restrict the supremum to dyadic intervals this will define BMO^d and we denote this dyadic BMO norm by $\| \cdot \|_{BMO^d}$. We now state our main results.

Theorem 2.3.2. *There exists C such that for all $w \in A_2$,*

$$\|[b, H]\|_{L^2(w) \to L^2(w)} \leq C[w]^2_{A_2} \|b\|_{BMO} \,,$$

for all $b \in BMO$.

Once we have boundedness and sharpness for the crucial case $p = 2$, we can carry out the power of the A_p-characteristic, for any $1 < p < \infty$, using the sharp extrapolation theorem [DGPerPet] to obtain Theorem 1.0.1. Furthermore, an example of C. Pérez [P3] shows this quadratic power is sharp. In [Pet1], S. Petermichl showed that the norm of the commutator of the Hilbert transform is bounded by the supremum of the norms of the commutator of certain shift operators. This result follows after writing the kernel of the Hilbert transform as a well chosen average of certain dyadic shift operators discovered by Petermichl. More precisely, S. Petermichl showed there is a non zero constant C such that

$$\|[b, H]\| \leq C \sup_{\alpha, r} \|[b, S^{\alpha, r}]\| \,, \tag{2.17}$$

where the dyadic shift operator $S^{\alpha, r}$ is defined by

$$S^{\alpha, r} f = \sum_{I \in \mathcal{D}^{\alpha, r}} \langle f, h_I \rangle (h_{I_-} - h_{I_+}) \,.$$

Petermichl [Pet2] showed that shift operator obeys a linear bound in $L^2(w)$

$$\|Sf\|_{L^2(w)} \leq C[w]_{A_2^d}\|f\|_{L^2(w)}\,, \tag{2.18}$$

similar result holds for $S^{\alpha,r}$, where C is independent of the dyadic intervals used.

Let us consider a compactly supported $b \in BMO^d$ and $f \in L^2$. Expanding b and f in the Haar system associated to the dyadic intervals $\mathcal{D}$,

$$b(x) = \sum_{I\in\mathcal{D}}\langle b, h_I\rangle h_I(x), \quad f(x) = \sum_{J\in\mathcal{D}}\langle f, h_J\rangle h_J(x)\,;$$

formally, we get the multiplication of b and f to be broken into three terms,

$$bf = \pi_b^*(f) + \pi_b(f) + \lambda_b(f)\,, \tag{2.19}$$

where π_b is the dyadic paraproduct, π_b^* is its adjoint and $\lambda_b(\cdot) = \pi_{(\cdot)}b$, defined as follows

$$\pi_b^*(f)(x) := \sum_{I\in\mathcal{D}}\langle b, h_I\rangle\langle f, h_I\rangle h_I^2(x)\,,$$

$$\pi_b(f)(x) := \sum_{I\in\mathcal{D}}\langle b, h_I\rangle\langle f\rangle_I h_I(x)\,,$$

$$\lambda_b(f)(x) := \sum_{I\in\mathcal{D}}\langle b\rangle_I\langle f, h_I\rangle h_I(x)\,.$$

It is an exercise to verify that the sum of these three terms returns formally the product bf. You can see the detailed proof of the n-dimensional analogue of this decomposition in p. 25. Thus, we have

$$[b, S] = [\pi_b^*, S] + [\pi_b, S] + [\lambda_b, S]\,, \tag{2.20}$$

where

$$S(f) = \sum_{I\in\mathcal{D}}\langle f, h_I\rangle(h_{I_-} - h_{I_+})$$

and we can estimate each term separately. Notice that both π_b and π_b^* are bounded operators in $L^p(w)$ for $b \in BMO$ [Be], despite the fact that multiplication by b is a bounded operator in $L^p(w)$ only when b is bounded (L^∞). Therefore, λ_b can not be a bounded operator in $L^p(w)$. However $[\lambda_b, S]$ will be bounded on $L^p(w)$ and will be better behaved than $[b, S]$. Decomposition (2.20) was used to analyze the commutator with the shift operator first by Petermichl in [Pet1], but also Lacey in [L] and authors in [LPetPiWic] to analyze the iterated commutators. Since all estimates are independent on the dyadic grid, through out this dissertation we only deal with the dyadic shift operator S associated to the standard dyadic grid $\mathcal{D}$. For a single shift operator the hypothesis required on b and w are that they belong to dyadic BMO^d and A_2^d with respect to the underlying dyadic grid defining the operator. However since ultimately we want to average over all grids, we will need b and w belonging to BMO^d and A_2^d for all shifted and scaled dyadic grids, that we will have if $b \in BMO$ and $w \in A_2$, non-dyadic BMO and A_2. Beznosova has proved linear bounds for π_b and π_b^*, [Be],

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w)}$$

together with Petermichl's (2.18) linear bounds for S, [Pet2], this immediately provides the quadratic bounds for $[\pi_b, S]$ and $[\pi_b^*, S]$. Theorem 2.3.2 will be proved once we show the quadratic estimate holds for $[\lambda_b, S]$. We can actually obtain a better linear estimate as in Theorem 1.0.2. Some terms in (2.20) do also obey linear bounds.

Theorem 2.3.3. *There exists C such that*

$$\|\pi_b^* S\|_{L^2(w)} + \|S\pi_b\|_{L^2(w)} \leq C[w]_{A_2^d}\|b\|_{BMO^d}.$$

for all $b \in BMO^d$.

Note the three operators $[\lambda_b, S]$, $\pi_b^* S$ and $S\pi_b$ are generalized Haar shift operators for which there are now two different proofs of linear bounds on $L^2(w)$ with respect to $[w]_{A_2^d}$, [LPetRe] and [CrMP], and in this dissertation we present a third proof.

2.4 The dyadic paraproduct

For the locally integrable functions b and f, the dyadic paraproduct is defined by

$$\pi(b,f) := \sum_{I\in\mathcal{D}} \langle b, h_I\rangle \langle f\rangle_I h_I\,,$$

on the real line. Thus the dyadic paraproduct is a bilinear operation. It is now well known fact that the dyadic paraproduct is bounded on L^p if $b \in BMO^d$ (see [Per1]). Thus, after we fix b in BMO^d, we consider $\pi(b,f)$ as a linear operator acting on f and we write $\pi_b(f)$ for this linear operator. The question (1.2) for the dyadic paraproduct was solved in [Be] by Beznosova and it is now known that the linear estimate holds for the dyadic paraproduct in $L^2(w)$.

Theorem 2.4.1 (O. Beznosova)**.** *The norm of dyadic paraproduct on the weighted Lebesgue space $L^2_{\mathbb{R}}(w)$ is bounded from above by a constant multiple of the product of the A^d_2-characteristic of the weight w and the BMO^d norm of b.*

One of main contribution of this dissertation is to extend Theorem 2.4.1 to the multivariable setting in the spirit of [Be], that is using Bellman function arguments. This allows to establish the dimension free estimates in terms of anisotropic weight characteristic. Thus we need to consider the class of anisotropic A_2-weights and the class of anisotropic BMO functions which are defined as follows.

Definition 2.4.2. A locally integrable and positive almost everywhere function w on the space $\mathbb{R}^n$ belongs to class of A^R_p weights, $1 < p < \infty$ if

$$[w]_{A^R_p} := \sup_R \langle w\rangle_R \langle w^{-1/(p-1)}\rangle_R^{p-1} < \infty\,,$$

where the supremum is taken over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Definition 2.4.3. A locally integrable function on $\mathbb{R}^n$ belongs to BMO^R if

$$\|b\|_{BMO^R} := \sup_R \frac{1}{|R|}\int_R |\, b(x) - \langle b\rangle_R|\, dx < \infty\,,$$

where the supremum runs over all rectangles $R \subset \mathbb{R}^n$ with sides parallel to the coordinate axes.

Since a cube is a particular case of a rectangle, it is easy to observe that $\|b\|_{BMO} \leq \|b\|_{BMO^R}$. Thus, $BMO \supset BMO^R$ when $n \geq 2$. In [K], the example

$$b(x) = \sum_{k=1}^{\infty} \chi_{[0,2^{-k+1}]\times[0,1/k]}(x), \qquad x = (x_1, x_2) \in \mathbb{R}^2$$

was presented which is in $BMO(\mathbb{R}^2)$ but not in $BMO^R(\mathbb{R}^2)$. For more details, see [K] which include the John-Nirenberg inequality, the Muckenhoupt embedding for the anisotropic weights and more.

As well as in the one dimensional case, one can define

$$\|b\|_{BMO^d_{\mathbb{R}^n}} := \sup_{Q\in\mathcal{D}^n} \frac{1}{|Q|} \int_Q |b(x) - \langle b\rangle_Q| dx\,, \tag{2.21}$$

for a locally integrable function on $\mathbb{R}^n$. The function b is said to have dyadic bounded mean oscillation if $\|b\|_{BMO^d_{\mathbb{R}^n}} < \infty$, and we denote the class of all locally integrable functions b on $\mathbb{R}^n$ with dyadic bounded mean oscillation by $BMO^d_{\mathbb{R}^n}$. Notably one can replace (2.21) by

$$\|b\|^2_{BMO^d_{\mathbb{R}^n}} = \sup_{Q\in\mathcal{D}^n} \frac{1}{|Q|} \sum_{Q\in\mathcal{D}^n(Q)} \sum_{j=1}^{2^n-1} |\langle b, h_{j,Q}\rangle|^2\,. \tag{2.22}$$

In the anisotropic case, it is known that the John-Nirenberg inequality holds for all $b \in BMO^R$ and any rectangle $R \subset \mathbb{R}^n$,

$$|\{x \in R \,|\, |b(x) - \langle b\rangle_R| > \lambda\}| \leq e^{1+2/e}|R| \exp\left(-\frac{2/e}{\|b\|_{BMO^R}}\lambda\right), \qquad \lambda > 0\,. \tag{2.23}$$

Note that the John-Nirenberg inequality is dimensionless in the anisotropic case. As an easy consequence of (2.23), we have a self improving property for the anisotropic

BMO class. For any rectangle $R \in \mathbb{R}^n$, there exists a constant $C(p)$ independent of the dimension n such that

$$\left(\frac{1}{|R|}\int_R |\,b(x) - \langle b\rangle_R|^p dx\right)^{1/p} \leq C(p)\,\|b\|_{BMO^R}\,. \tag{2.24}$$

One can easily check that

$$\int_{E_{i,Q'}} |\,b(x) - \langle b\rangle_{E_{i,Q'}}|^2 dx = \int_{E_{i,Q'}} b^2(x)dx - |E_{i,Q'}|\langle b\rangle^2_{E_{i,Q'}}\,.$$

Then

$$\begin{aligned}
&\int_{E_{i,Q'}} |\,b(x) - \langle b\rangle_{E_{i,Q'}}|^2 dx - |\langle b, h_{i,Q'}\rangle|^2 \\
&\quad = \int_{E_{i,Q'}} b^2(x)dx - |E_{i,Q'}|\langle b\rangle^2_{E_{i,Q'}} - \frac{|E_{i,Q'}|}{4}\left(\langle b\rangle_{E^2_{i,Q'}} - \langle b\rangle_{E^1_{i,Q'}}\right)^2 \\
&\quad = \int_{E_{i,Q'}} b^2(x)dx - \frac{|E_{i,Q'}|}{4}\left(\langle b\rangle_{E^1_{i,Q'}} + \langle b\rangle_{E^2_{i,Q'}}\right)^2 - \frac{|E_{i,Q'}|}{4}\left(\langle b\rangle_{E^2_{i,Q'}} - \langle b\rangle_{E^1_{i,Q'}}\right)^2 \\
&\quad = \int_{E_{i,Q'}} b^2(x)dx - \frac{|E_{i,Q'}|}{2}\left(\langle b\rangle^2_{E^1_{i,Q'}} + \langle b\rangle^2_{E^2_{i,Q'}}\right) \\
&\quad = \int_{E^1_{i,Q'}} b^2(x)dx - |E^1_{i,Q'}|\langle b\rangle^2_{E^1_{i,Q'}} + \int_{E^2_{i,Q'}} b^2(x)dx - |E^2_{i,Q'}|\langle b\rangle^2_{E^2_{i,Q'}} \\
&\quad = \int_{E^1_{i,Q'}} |\,b(x) - \langle b\rangle_{E^1_{i,Q'}}|^2 dx + \int_{E^2_{i,Q'}} |\,b(x) - \langle b\rangle_{E^2_{i,Q'}}|^2 dx\,.
\end{aligned}$$

By iterating this process, one gets

$$\int_{E_{i,Q'}} |\,b(x) - \langle b\rangle_{E_{i,Q'}}|^2 dx = \sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subsetneq E_{j,Q'}} |\langle b, h_{j,Q}\rangle|^2\,.$$

Note that the sets $E_{i,Q'}$ are rectangles. Using the self improving property (2.24), we have, for $i = 1, ..., 2^n - 1$, $Q' \in \mathcal{D}^n$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \langle b, h_{j,Q}\rangle^2 \leq C\|b\|^2_{BMO^R}\,. \tag{2.25}$$

We now define the multivariable dyadic paraproduct. As we have seen in (2.19) the product of two square integrable functions can be written as the sum of two dyadic paraproducts and a diagonal term in a single variable case. Moreover, the diagonal term is the adjoint of one dyadic paraproduct i.e. for all $f, g \in L^2_{\mathbb{R}}$,

$$fg = \pi_g^*(f) + \pi_g(f) + \lambda_g(f)\,. \tag{2.26}$$

Thus, we expect to have analogous decomposition. Let us assume that $f, g \in L^2_{\mathbb{R}^n}$. Expanding f and g in the Haar system,

$$f = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle f, h_{j,Q}\rangle h_{j,Q}\,, \quad g = \sum_{Q'\in\mathcal{D}^n}\sum_{i=1}^{2^n-1}\langle g, h_{i,Q'}\rangle h_{i,Q'}$$

and multiplying these sums formally we can get

$$f(x)g(x) = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\sum_{Q'\in\mathcal{D}^n}\sum_{i=1}^{2^n-1}\langle f, h_{j,Q}\rangle\langle g, h_{i,Q'}\rangle h_{j,Q}(x)h_{i,Q'}(x) = (I) + (II) + (III)\,.$$

Here, (I) is the diagonal term $Q' = Q,\ j = i$;

$$(I) = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle f, h_{j,Q}\rangle\langle g, h_{j,Q}\rangle h^2_{j,Q}(x) = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle f, h_{j,Q}\rangle\langle g, h_{j,Q}\rangle\frac{\chi_{E_{j,Q}}(x)}{|E_{j,Q}|}\,. \tag{2.27}$$

The second term (II) is the upper triangle term corresponding to those $Q' \supsetneq Q$, all i, j and $Q' = Q$ so that $E_{i,Q'} \supsetneq E_{j,Q}$.

$$\begin{aligned}(II) &= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\Bigg(\sum_{Q'\in\mathcal{D}^n:Q'\supsetneq Q}\sum_{i=1}^{2^n-1}\langle f, h_{j,Q}\rangle\langle g, h_{i,Q'}\rangle h_{i,Q'}(x)h_{j,Q}(x) \\ &\qquad\qquad + \sum_{Q'=Q}\sum_{i:E_{i,Q'}\supsetneq E_{j,Q}}\langle f, h_{j,Q}\rangle\langle g, h_{i,Q'}\rangle h_{i,Q'}(x)h_{j,Q}(x)\Bigg) \\ &= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\sum_{Q'\in\mathcal{D}^n:Q'\supseteq Q}\sum_{i:E_{i,Q'}\supsetneq E_{j,Q}}\langle f, h_{j,Q}\rangle\langle g, h_{i,Q'}\rangle h_{i,Q'}(E_{j,Q})h_{j,Q}(x) \\ &= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle f, h_{j,Q}\rangle\langle g\rangle_{E_{j,Q}}h_{j,Q}(x)\,, \end{aligned} \tag{2.28}$$

where $h_{i,Q'}(E_{j,Q}) = h_{i,Q'}(x)$ for $x \in E_{j,Q}$. In the second equality we used formula (2.12) for the average of g on $E_{j,Q}$. Similarly, the third term is the lower triangle corresponding to those $Q' \subsetneq Q$, all i,j and $Q' = Q$ so that $E_{i,Q} \subsetneq E_{j,Q}$.

$$
\begin{aligned}
(III) = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \Bigg(& \sum_{Q'\in\mathcal{D}^n : Q'\subsetneq Q} \sum_{i=1}^{2^n-1} \langle f, h_{j,Q}\rangle \langle g, h_{i,Q'}\rangle h_{i,Q'}(x) h_{j,Q}(x) \\
& + \sum_{Q'=Q} \sum_{i: E_{i,Q'}\subsetneq E_{j,Q}} \langle f, h_{j,Q}\rangle \langle g, h_{i,Q'}\rangle h_{i,Q'}(x) h_{j,Q}(x) \Bigg) \\
= \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} & \langle g, h_{j,Q}\rangle \langle f\rangle_{E_{j,Q}} h_{j,Q}(x)\,.
\end{aligned} \tag{2.29}
$$

If we consider the sum (2.27) as an operator acting on f, then we can easily check that (III) is its adjoint operator, in fact, here is the derivation.

$$
\begin{aligned}
&\left\langle \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle g, h_{j,Q}\rangle \langle f, h_{j,Q}\rangle \frac{\chi_{E_{j,Q}}}{|E_{j,Q}|}, \sum_{R\in\mathcal{D}^n} \sum_{i=1}^{2^n-1} \langle q, h_{i,R}\rangle h_{i,R} \right\rangle \\
&\qquad = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \sum_{R\in\mathcal{D}^n} \sum_{i=1}^{2^n-1} \langle f, h_{j,Q}\rangle \langle g, h_{j,Q}\rangle \langle q, h_{i,R}\rangle \frac{\langle \chi_{E_{j,Q}}, h_{i,R}\rangle}{|E_{j,Q}|} \\
&\qquad = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f, h_{j,Q}\rangle \langle g, h_{j,Q}\rangle \sum_{R\in\mathcal{D}^n : R\supseteq Q} \sum_{i: E_{i,R}\supsetneq E_{j,Q}} \langle q, h_{i,R}\rangle h_{i,R}(E_{j,Q}) \\
&\qquad = \left\langle f, \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle g, h_{j,Q}\rangle \langle q\rangle_{E_{j,Q}} h_{j,Q} \right\rangle .
\end{aligned}
$$

We now can define the multivariable dyadic paraproduct by pairing the dyadic BMO function. In $\mathbb{R}^n$, the dyadic paraproduct is an operator π_b , given by

$$
\pi_b f(x) = \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle h_{j,Q}(x)\,. \tag{2.30}
$$

Note that the construction of the Haar systems are not unique. One can actually construct different Haar systems [DPetV]. Furthermore, the dyadic paraproduct depends on the choice of the Haar functions. Thus, one can establish the different dyadic paraproducts associated with different Haar functions. But the decomposition (2.26) holds for all of them. To close this chapter, we state our main results for the dyadic paraproduct.

Theorem 2.4.4. *For $1 < p < \infty$ there exists a constant $C(n,p)$ only depending on p and dimension n such that for all weights $w \in A_p^d$ and $b \in BMO_{\mathbb{R}^n}^d$*

$$\|\pi_b\|_{L^p_{\mathbb{R}^n}(w) \to L^p_{\mathbb{R}^n}(w)} \leq C(n,p)[w]_{A_p^d}^{\max\{1,\frac{1}{p-1}\}} \|b\|_{BMO^d_{\mathbb{R}^n}} .$$

Theorem 2.4.5. *There exists a constant C which doesn't depend on the dimensional constant such that for all weight $w \in A_2^R$ and $b \in BMO_{\mathbb{R}^n}^R$*

$$\|\pi_b\|_{L^2_{\mathbb{R}^n}(w) \to L^2_{\mathbb{R}^n}(w)} \leq C\,[w]_{A_2^R} \|b\|_{BMO^R_{\mathbb{R}^n}} .$$

The proofs for both Theorems are presented in Chapter 7, and they only depend on the results presented in Charter 3.

We will finish this Chapter by including a comparison to the standard tensor product Haar basis in $\mathbb{R}^n$, $\{h^s_{\sigma,Q}\}$, with the Haar basis introduced in Section 2.2 and associated paraproducts. Let us denote the Haar function associated with a dyadic interval $I \in \mathcal{D}$ by $h^0_I = |\,I|^{-1/2}(\chi_{I_+} - \chi_{I_-})$ and normalized characteristic functions $h^1_I = |\,I|^{-1/2}\chi_I$. Here 0 stands for mean value zero and 1 for the indicator. Also we consider a set of signatures $\Sigma = \{0,1\}^{\{1,\dots,n\}} \setminus \{(1,...,1)\}$ which contains $2^n - 1$ signatures. These are all n-tuples with entries 0 and 1, but excluding n-tuple whose entries are all 1. Then, for each dyadic cube $Q = I_1 \times \cdots \times I_n$, one can get the standard tensor product Haar basis in $\mathbb{R}^n$ by

$$h^s_{\sigma,Q}(x_1,...,x_n) = h^{\sigma_1}_{I_1}(x_1) \times \cdots \times h^{\sigma_n}_{I_n}(x_n)\,,$$

where $\sigma = (\sigma_1,...,\sigma_n) \in \Sigma$. Notice that all $h^s_{\sigma,Q}$ are supported on Q. In this case, we have the paraproduct associated to the standard tensor product Haar basis:

$$\pi^s_b f(x) = \sum_{Q \in \mathcal{D}^n} \langle f \rangle_Q \sum_{\sigma \in \Sigma} \langle b, h^s_{\sigma,Q}\rangle h^s_{\sigma,Q}(x)\,. \tag{2.31}$$

Observe that, for each dyadic cube $Q \in \mathcal{D}^n$,

$$\text{span}\{h^s_{\sigma,Q}\}_{\sigma\in\Sigma} = \text{span}\{h_{j,Q}\}_{j=1,\dots,2^n-1} = \text{span}\{\chi_{\widetilde{Q}}\}_{\widetilde{Q}\in\mathcal{D}^n_1(Q)}\,. \tag{2.32}$$

Changing the basis, we can see that two multivariable paraproducts, (2.30) and (2.31), are different, that is

$$\pi_b^s f(x) = \sum_{Q\in\mathcal{D}^n} \langle f\rangle_Q \sum_{\sigma\in\Sigma} \langle b, h^s_{\sigma,Q}\rangle h^s_{\sigma,Q} \neq \sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle f\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle h_{j,Q} = \pi_b f(x)\,.$$

Chapter 3

Main tools

In this chapter, we are going to introduce several theorems and lemmas which will be used to prove our main results. In Section 3.1 we recall some embedding theorems and weighted inequalities in one dimensional case. Some of them will be stated in multivariable setting and be proven via Bellman function arguments in Section 3.2.

3.1 Embedding theorems and weighted inequalities in $\mathbb{R}$

To prove Theorem 2.3.2 we need several theorems and lemmas. Some of them will be given in this dissertation with detailed arguments. If not, you can find the proof in the indicated references. First we recall that the dyadic square function is defined by $f \mapsto \mathscr{S}_d f$ where

$$\mathscr{S}_d f(x) := \left(\sum_{I \in \mathcal{D}} |\langle f, h_I \rangle|^2 \frac{\chi_I(x)}{|I|} \right)^{1/2}.$$

It is well known in [HukTV] that if $w \in A_2$, the norm of the dyadic square function is bounded in $L^2(w)$, with a bound that depends linearly on the A_2-characteristic of the weight.

Theorem 3.1.1. *There is a constant c so that for all $w \in A_2$ the square function $\mathscr{S}_d : L^2(w) \to L^2(w)$ has operator norm $\|\mathscr{S}_d\|_{L^2(w)\to L^2(w)} \le c[w]_{A_2}$.*

Another main result in [Pet2] is a two-weighted bilinear embedding theorem. The author in [Pet2] proved this Theorem by a Bellman function argument.

Theorem 3.1.2 (Petermichl's Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w\rangle_I\langle v\rangle_I \le Q$ for all intervals I and let $\{\alpha_I\}$ be a non-negative sequence so that the three estimates below hold for all J*

$$\sum_{I\in\mathcal{D}(J)} \frac{\alpha_I}{\langle w\rangle_I} \le Q\, v(J) \tag{3.1}$$

$$\sum_{I\in\mathcal{D}(J)} \frac{\alpha_I}{\langle v\rangle_I} \le Q\, w(J) \tag{3.2}$$

$$\sum_{I\in\mathcal{D}(J)} \alpha_I \le Q\,|J|\,. \tag{3.3}$$

Then there is c such that for all $f \in L^2(w)$ and $g \in L^2(v)$

$$\sum_{I\in\mathcal{D}} \alpha_I \langle f\rangle_{I,w}\langle g\rangle_{I,v} \le cQ\|f\|_{L^2(w)}\|g\|_{L^2(v)}\,.$$

Replacing α_I, f, and g by $\alpha_I\langle w\rangle_I\langle v\rangle_I|\,I|$, $fw^{-1/2}$ and $gv^{-1/2}$ respectively yields the following Corollary.

Corollary 3.1.3 (Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w\rangle_I\langle v\rangle_I \le Q$ for all intervals I . Let $\{\alpha_I\}$ be a sequence of nonnegative numbers such that for all dyadic intervals $J \in \mathcal{D}$ the following three inequalities hold with some constant $Q > 0$,*

$$\sum_{I\in\mathcal{D}(J)} \alpha_I\langle v\rangle_I\,|\,I| \le Q\, v(J) \tag{3.4}$$

$$\sum_{I\in\mathcal{D}(J)} \alpha_I \langle w\rangle_I \, |I| \le Q\, w(J) \tag{3.5}$$

$$\sum_{I\in\mathcal{D}(J)} \alpha_I \langle w\rangle_I \langle v\rangle_I |\, I| \le Q\, |J|\,. \tag{3.6}$$

Then for any two nonnegative function f, $g \in L^2$

$$\sum_{I\in\mathcal{D}} \alpha_I \langle f w^{1/2}\rangle_I \langle g v^{1/2}\rangle_I \, |I| \le CQ\|f\|_{L^2}\|g\|_{L^2}$$

holds with some constant $C > 0$.

Both Bilinear Embedding Theorems are key tools in our estimate. One version of such a theorem appeared in [NTV]. The original version of the next lemma also appeared in [Pet2].

We use the notation $\Delta_I f := \frac{1}{2}\Big(\langle f\rangle_{I_+} - \langle f\rangle_{I_-}\Big)$. Let us introduce the operator defined by

$$S_b(f) := \sum_{I\in\mathcal{D}} \Delta_I b \,\langle f, h_I\rangle_I h_{I_-}\,,$$

which will be used for the estimate of the commutator $[\lambda_b, S]$ in Section 4.1. The operator $S^I_{b,w^{-1}}$ in the following Lemma is the truncated operator of the composition of S_b with multiplication by w^{-1}, that is

$$S^I_{b,w^{-1}}(f) := \sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1} f, h_L\rangle h_L\,.$$

Lemma 3.1.4. *There is a constant c such that*

$$\|S^I_{b,w^{-1}}\chi_I\|_{L^2(w)} \le c\|b\|_{BMO^d}[w]_{A_2^d} w^{-1}(I)^{1/2}$$

for all intervals I and weights $w \in A_2^d$.

Proof. We will prove this Lemma by duality. It is sufficient to prove the inequality

$$|\langle S^I_{b,w^{-1}}\chi_I, f\rangle_w| \le c\|b\|_{BMO^d}[w]_{A_2^d} w^{-1}(I)^{1/2}\|f\|_{L^2(w)}\,, \tag{3.7}$$

for positive test functions f.

$$
\begin{aligned}
|\langle S^I_{b,w^{-1}}\chi_I, f\rangle_w| &= \left|\left\langle \sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}, h_L\rangle h_{L_-}, \sum_{J\in\mathcal{D}}\langle f, h^w_J\rangle_w h^w_J\right\rangle_w\right| \\
&= \left|\sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}, h_L\rangle \sum_{J\in\mathcal{D}}\langle f, h^w_J\rangle_w\langle h_{L_-}, h^w_J\rangle_w\right| \\
&\leq \left|\sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}, h_L\rangle \sum_{J\in\mathcal{D}:J\supsetneq L}\langle f, h^w_J\rangle_w\langle h_{L_-}, h^w_J\rangle_w\right| \qquad (3.8)\\
&\quad + \sum_{L\in\mathcal{D}(I)} |\Delta_L b\,\langle w^{-1}, h_L\rangle\langle f, h^w_{L_-}\rangle_w\langle h_{L_-}, h^w_{L_-}\rangle_w| \qquad (3.9)\\
&\quad\quad + \sum_{L\in\mathcal{D}(I)} |\Delta_L b\,\langle w^{-1}, h_L\rangle\langle f, h^w_{L}\rangle_w\langle h_{L_-}, h^w_{L}\rangle_w|\,. \qquad (3.10)
\end{aligned}
$$

Using (2.2), (2.3), Hölder's inequality, the fact $\langle w\rangle_{L_-} \leq 2\langle w\rangle_L$, and

$$
\begin{aligned}
|\Delta_I b| &= \frac{1}{2}|\,\langle b\rangle_{I_+} - \langle b\rangle_{I_-}| = \frac{1}{2}|\,\langle b\rangle_{I_+} - \langle b\rangle_I + \langle b\rangle_I - \langle b\rangle_{I_-}| \\
&\leq \frac{1}{2}\left(\frac{1}{|I_+|}\int_{I_+}|b-\langle b\rangle_I| + \frac{1}{|I_-|}\int_{I_-}|\,b-\langle b\rangle_I|\right) \\
&\leq \frac{1}{2}\left(\frac{2}{|I|}\int_I|\,b-\langle b\rangle_I| + \frac{2}{|I|}\int_I|\,b-\langle b\rangle_I|\right) \\
&\leq 2\|b\|_{BMO^d}\,, \qquad (3.11)
\end{aligned}
$$

we can estimate (3.9) and (3.10),

$$
\begin{aligned}
&\sum_{L\in\mathcal{D}(I)} |\Delta_L b\,\langle w^{-1}, h_L\rangle\langle f, h^w_{L_-}\rangle_w\langle h_{L_-}, h^w_{L_-}\rangle_w| + \sum_{L\in\mathcal{D}(I)} |\Delta_L b\,\langle w^{-1}, h_L\rangle\langle f, h^w_{L}\rangle_w\langle h_{L_-}, h^w_{L}\rangle_w| \\
&\leq 2\|b\|_{BMO^d}\|f\|_{L^2(w)}\left[\left(\sum_{L\in\mathcal{D}(I)}\langle w^{-1}, h_L\rangle^2\langle h_{L_-}, h^w_{L_-}\rangle^2_w\right)^{1/2}\right. \\
&\qquad\qquad\left. + \left(\sum_{L\in\mathcal{D}(I)}\langle w^{-1}, h_L\rangle^2\langle h_{L_-}, h^w_{L}\rangle^2_w\right)^{1/2}\right] \\
&\leq 2\|b\|_{BMO^d}\|f\|_{L^2(w)}\left[\left(\sum_{L\in\mathcal{D}(I)}\langle w^{-1}, h_L\rangle^2\langle w\rangle_{L_-}\right)^{1/2} + \left(\sum_{L\in\mathcal{D}(I)}\langle w^{-1}, h_L\rangle^2\langle w\rangle_{L_-}\right)^{1/2}\right] \\
&\leq 4\sqrt{2}\|b\|_{BMO^d}\|f\|_{L^2(w)}\left(\sum_{L\in\mathcal{D}(I)}\langle w^{-1}, h_L\rangle^2\langle w\rangle_L\right)^{1/2}. \qquad (3.12)
\end{aligned}
$$

Applying Theorem 3.1.1. with $f = w^{-1}\chi_I$,

$$\begin{aligned}\sum_{L\in\mathcal{D}(I)} \langle w^{-1}, h_L\rangle^2 \langle w\rangle_L &\le \sum_{L\in\mathcal{D}} \langle w^{-1}\chi_I, h_L\rangle^2 \langle w\rangle_L \\ &= \sum_{L\in\mathcal{D}} \langle w^{-1}\chi_I, h_L\rangle^2 \frac{1}{|L|}\int w(x)\chi_L(x)dx \\ &= \int \sum_{L\in\mathcal{D}} \langle w^{-1}\chi_I, h_L\rangle^2 \frac{\chi_L(x)}{|L|} w(x)dx \\ &= \|\mathscr{S}(w^{-1}\chi_I)\|^2_{L^2(w)} \le c[w]^2_{A_2^d}\|w^{-1}\chi_I\|^2_{L^2(w)} = c[w]^2_{A_2^d} w^{-1}(I)\,.\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) gives us

$$(3.9) + (3.10) \le c\|b\|_{BMO^d}[w]_{A_2^d} w^{-1}(I)^{1/2}\|f\|_{L^2(w)}\,. \tag{3.14}$$

We can estimate (3.8) using (2.7) and (3.11) as follows.

$$\left|\sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}, h_L\rangle \sum_{J\in\mathcal{D}:J\supsetneq L} \langle f, h_J^w\rangle_w \langle h_{L_-}, h_J^w\rangle_w\right| \tag{3.15}$$

$$\begin{aligned}&\le \sum_{L\in\mathcal{D}(I)} |\Delta_L b|\,|\langle w^{-1}, h_L\rangle| \left|\left(\sum_{J\in\mathcal{D}:J\supsetneq L} \langle f, h_J^w\rangle_w h_J^w(L)\right)\langle h_{L_-}, w\rangle\right| \\ &\le 2\|b\|_{BMO^d} \sum_{L\in\mathcal{D}(I)} |\langle w^{-1}, h_L\rangle\langle w, h_{L_-}\rangle| \langle f\rangle_{L,w} \\ &\le 2\|b\|_{BMO^d} \sum_{L\in\mathcal{D}} |\langle w^{-1}, h_L\rangle\langle w, h_{L_-}\rangle| \langle f\rangle_{L,w}\langle \chi_I\rangle_{L,w^{-1}}\,.\end{aligned} \tag{3.16}$$

In the last inequality, we can check

$$\langle \chi_I\rangle_{L,w^{-1}} = \frac{1}{w^{-1}(L)}\int_{L\cap I} w^{-1} = \begin{cases} 1, & \text{if } \; L\subseteq I \\ w^{-1}(I)/w^{-1}(L) < 1, & \text{if } \; I \subseteq L \\ 0, & \text{otherwise.}\end{cases}$$

If we show

$$\sum_{L\in\mathcal{D}} |\langle w^{-1}, h_L\rangle\langle w, h_{L_-}\rangle| \langle f\rangle_{L,w}\langle \chi_I\rangle_{L,w^{-1}} \le [w]_{A_2^d} w^{-1}(I)^{1/2}\|f\|_{L^2(w)}\,, \tag{3.17}$$

then we can finish proving the lemma. To see (3.17), let us assume that for w, $v = w^{-1}$, $\alpha_I = |\langle w^{-1}, h_L\rangle\langle w, h_{L_-}\rangle|$ and $Q = c\,[w]_{A_2^d}$ satisfy the embedding condition (3.1), (3.2), and (3.3). Then, by Theorem 3.1.2. and $\|\chi_I\|_{L^2(w^{-1})} = w^{-1}(I)^{1/2}$, we can see (3.17). The only remaining thing to do is to check the embedding conditions. We refer to [Pet2] for the proof of the embedding conditions. □

We will also need the Weighted Carleson Embedding Theorem from [NTV], and some other inequalities for weights.

Theorem 3.1.5 (Weighted Carleson Embedding Theorem)**.** *Let $\{\alpha_J\}$ be a non-negative sequence such that for all dyadic intervals I*

$$\sum_{J\in\mathcal{D}(I)} \alpha_J \le Q w^{-1}(I)\,.$$

Then for all $f \in L^2(w^{-1})$

$$\sum_{J\in\mathcal{D}} \alpha_J \langle f\rangle^2_{J,w^{-1}} \le 4Q\|f\|^2_{L^2(w^{-1})}\,.$$

Theorem 3.1.6 (Wittwer's sharp version of Buckley's inequality)**.** *There exist a positive constant C such that for any weight $w \in A_2^d$ and dyadic interval $I \in \mathcal{D}$,*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \frac{\big(\langle w\rangle_{I_+} - \langle w\rangle_{I_-}\big)^2}{\langle w\rangle_I}\,|\,I| \le C[w]_{A_2^d}\langle w\rangle_J\,.$$

We refer to [Wi1] for the proof. You can find extensions of Theorem 3.1.5 and 3.1.6 to the doubling positive measure σ in [Per2]. One can find the Bellman function proof of the following three Lemmas in [Be].

Lemma 3.1.7. *For all dyadic interval J and all weights w.*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} |I||\Delta_I w|^2 \frac{1}{\langle w\rangle_I^3} \le \langle w^{-1}\rangle_J\,.$$

Lemma 3.1.8. *Let w be a weight and $\{\alpha_I\}$ be a Carleson sequence of nonnegative numbers. If there exist a constant $Q > 0$ such that*

$$\forall\, J \in \mathcal{D}\,, \quad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \alpha_I \leq Q\,,$$

then

$$\forall\, J \in \mathcal{D}\,, \quad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \frac{\alpha_I}{\langle w^{-1}\rangle_I} \leq 4Q\langle w\rangle_J$$

and therefore if $w \in A_2^d$ the for any $J \in \mathcal{D}$ we have

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \langle w\rangle_I \alpha_I \leq 4Q[w]_{A_2^d}\langle w\rangle_J\,.$$

Lemma 3.1.9. *If $w \in A_2^d$ then there exists a constant $C > 0$ such that*

$$\forall J \in \mathcal{D}\,, \quad \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \left(\frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2 |\, I|\, \langle w\rangle_I \langle w^{-1}\rangle_I \leq C[w]_{A_2^d}\,.$$

3.2 Embedding theorems and weighted inequalities in $\mathbb{R}^n$

We now state several multi-variable versions of Embedding Theorems and weighted inequalities which appeared in previous section. In general, once we have a Bellman function proof for a certain property in $\mathbb{R}$ then we can extend a property into $\mathbb{R}^n$ with the same Bellman function. This process is essentially trivial when we use the Haar system in $\mathbb{R}^n$ introduced in Section 2.2, and it allows to do the "induction in scales argument" at once, instead of once per each $j = 1, ..., 2^n - 1$, which then introduces a dimensional constant of order 2^n in the estimates. We will present several lemmas and associated Embedding theorems and weight properties. One can find the proof of these lemmas and one dimensional analogues of the propositions in [Be] or indicated references.

Lemma 3.2.1. *The following function*

$$B(F,f,u,M) = 4A\left(F - \frac{f^2}{u+M}\right)$$

is defined on domain $\mathfrak{D}$ *which is given by*

$$\mathfrak{D} = \left\{(F,f,u,M) \in \mathbb{R}^4 \middle|\, F,f,u,M > 0 \text{ and } f^2 \leq Fu,\ \ M \leq u\right\},$$

and B *satisfies the following size and convexity property in* $\mathfrak{D}$*:*

$$0 \leq B(F,f,u,M) \leq 4A\,F\,, \tag{3.18}$$

and for all (F,f,u,M), (F_1,f_1,u_1,M_1) *and* $(F_2,f_2,u_2,M_2) \in \mathfrak{D}$,

$$B(F,f,u,M) - \frac{B(F_1,f_1,u_1,M_1)+B(F_2,f_2,u_2,M_2)}{2} \geq \frac{f^2}{u^2}m\,, \tag{3.19}$$

where

$$(F,f,u,M) = \left(\frac{F_1+F_2}{2}, \frac{f_1+f_2}{2}, \frac{u_1+u_2}{2}, m + \frac{M_1+M_2}{2}\right) \text{ and } m \geq 0\,.$$

One can find the proof of Lemma 3.2.1 in [NTV].

Theorem 3.2.2 (Multivariable Version of Weighted Carleson Embedding Theorem)**.** *Let* w *be a weight and* $\left\{\alpha_{j,Q}\right\}_{Q,j}$, $Q \in \mathcal{D}^n$, $j = 1,...,2^n-1$, *be a sequence of non-negative numbers such that for all dyadic cubes* $Q' \in \mathcal{D}^n$ *and a positive constant* $A > 0$,

$$\frac{1}{\left|E_{i,Q'}\right|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \alpha_{j,Q}\langle w\rangle^2_{E_{j,Q}} \leq A\langle w\rangle_{E_{i,Q'}}\,. \tag{3.20}$$

Then for all positive $f \in L^2_{\mathbb{R}^n}$

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\alpha_{j,Q}\langle fw^{1/2}\rangle_{E_{j,Q}} \leq CA\|f\|^2_{L^2_{\mathbb{R}^n}} \tag{3.21}$$

holds with some constant $C > 0$.

Proof. For any subset of dyadic cube $Q' \in \mathcal{D}^n$ and fixed $i = 1, ..., 2^n - 1$, let

$$F_{i,Q'} = \langle f^2 \rangle_{E_{i,Q'}}, \quad f_{i,Q'} = \langle f w^{1/2} \rangle_{E_{i,Q'}}, \quad u_{i,Q'} = \langle w \rangle_{E_{i,Q'}},$$

$$F^k_{i,Q'} = \langle f^2 \rangle_{E^k_{i,Q'}}, \quad f^k_{i,Q'} = \langle f w^{1/2} \rangle_{E^k_{i,Q'}}, \quad u^k_{i,Q'} = \langle w \rangle_{E^k_{i,Q'}}, \text{ for } k = 1, 2,$$

and

$$M_{i,Q'} = \frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q}\, u^2_{j,Q} = \frac{M^1_{i,Q'} + M^2_{i,Q'}}{2} + \frac{1}{|E_{i,Q'}|} \alpha_{i,Q'}\, u^2_{i,Q'},$$

where $M^1_{i,Q'} = \frac{1}{|E^1_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E^1_{i,Q'}} \alpha_{j,Q}\, u^2_{j,Q}$ and similarly for $M^2_{i,Q'}$. Note that $F = F_{i,Q'}$, $f = f_{i,Q'}$, $u = u_{i,Q'}$, $M = M_{i,Q}$, $F_k = F^k_{i,Q'}$, $f_k = f^k_{i,Q'}$, $u_k = u^k_{i,Q'}$, and $M_k = M^k_{i,Q}$, for $k = 1, 2$, belong to the domain $\mathfrak{D}$ of the function B defined in Lemma 3.2.1. Furthermore,

$$(F, f, u, M) = \left(\frac{F_1 + F_2}{2}, \frac{f_1 + f_2}{2}, \frac{u_1 + u_2}{2}, m + \frac{M_1 + M_2}{2} \right),$$

so we can use the both size condition (3.18) of B and convexity property (3.19). For fixed dyadic cube Q', using $E_{1,Q'} = Q'$, $|E^1_{i,Q'}| = |E^2_{i,Q'}| = |E_{i,Q}|/2$ as well as $\{E_{j,Q'}\}_{j=2^m}^{2^{m+1}-1} = \{E^1_{j,Q'}, E^2_{j,Q'}\}_{j=2^m}^{2^{m+1}-1}$ and the property (3.19), we have

$$\begin{aligned}
4A|Q'|\langle f^2 \rangle_{Q'} &= 4A|E_{1,Q'}|\langle f^2 \rangle_{E_{1,Q'}} = 4A|E_{1,Q'}|F_{1,Q'} \\
&\geq |E_{1,Q'}| B(F_{1,Q'},\, f_{1,Q'},\, u_{1,Q'},\, M_{1,Q'}) \\
&\geq \sum_{k=1}^{2} |E^k_{1,Q'}| B(F^k_{1,Q'},\, f^k_{1,Q'},\, u^k_{1,Q'},\, M^k_{1,Q'}) + \alpha_{1,Q'}\, f^2_{1,Q'} && (3.22) \\
&= \sum_{j=2}^{3} |E_{j,Q'}| B(F_{j,Q'},\, f_{j,Q'},\, u_{j,Q'},\, M_{j,Q'}) + \alpha_{1,Q'}\, f^2_{1,Q'} && (3.23) \\
&\geq \sum_{j=2}^{3} \sum_{k=1}^{2} |E^k_{j,Q'}| B(F^k_{j,Q'},\, f^k_{j,Q'},\, u^k_{j,Q'},\, M^k_{j,Q'}) + \sum_{j=1}^{3} \alpha_{j,Q'}\, f^2_{j,Q'} && (3.24) \\
&= \sum_{j=4}^{7} |E_{j,Q'}| B(F_{j,Q'},\, f_{j,Q'},\, u_{j,Q'},\, M_{j,Q'}) + \sum_{j=1}^{3} \alpha_{j,Q'}\, f^2_{j,Q'}\,. && (3.25)
\end{aligned}$$

If we iterate this process $n-2$ times more, we get:

$$4A|Q'|\langle f^2\rangle_{Q'} \geq \sum_{j=2^{n-1}}^{2^n-1}\sum_{k=1}^{2}\left|E^k_{j,Q'}\right|B(F^k_{j,Q'},\, f^k_{j,Q'},\, u^k_{j,Q'},\, M^k_{j,Q'}) + \sum_{j=1}^{2^n-1}\alpha_{j,Q'}\, f^2_{j,Q'}\,.$$

Due to our construction of Haar system, for all $j = 2^{n-1}, 2^{n-1}+1, ..., 2^n-1$, and $k = 1, 2$, $E^k_{j,Q'}$'s are mutually disjointed and $\left|E^k_{j,Q'}\right| = |Q'|/2^n$ i.e. $\{E^1_{j,Q'}, E^2_{j,Q'}\}_{j=2^{n-1},...,2^n-1}$ is the set $\mathcal{D}^n_1(Q')$ of dyadic sub-cubes of Q' . Thus,

$$\begin{aligned} 4A|Q'|\langle f^2\rangle_{Q'} &\geq |Q'|B(F_{Q'}, f_{Q'}, u_{Q'}, M_{Q'}) \\ &\geq \sum_{k=1}^{2^n}|Q'_k|B(F_{Q'_k}, f_{Q'_k}, u_{Q'_k}, M_{Q'_k}) + \sum_{j=1}^{2^n-1}\alpha_{j,Q'}\, f^2_{j,Q'}\,. \end{aligned} \tag{3.26}$$

In the inequality (3.26), Q'_k's are enumerations of 2^n dyadic sub-cubes of Q' . Iterating this procedure and using the fact $B \geq 0$ yields that

$$\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j=1}^{2^n-1}\alpha_{j,Q}\langle fw^{1/2}\rangle_{j,Q} \leq CA|Q'|\langle f^2\rangle_{Q'}\,,$$

which completes the proof. □

The proof of the following Lemma appeared in [Pet2].

Lemma 3.2.3. *The following function*

$$B(F,f,w,G,g,v,M,N,K) = B_1(F,f,w,M)+B_2(G,g,v,N)+B_3(F,f,w,G,g,v,K)\,,$$

where

$$\begin{aligned} &B_1(F,f,w,M) = F - \frac{f^2}{w+\frac{M}{A^2}}, \quad B_2(G,g,v,N) = G - \frac{g^2}{v+\frac{N}{A^2}}\,, \\ &B_3(F,f,w,G,g,v,K) = \inf_{a>0}\left(F+G-\frac{f^2}{w+\frac{aM}{A^2}}-\frac{g^2}{v+\frac{N}{aA^2}}\right). \end{aligned}$$

is defined on domain $\mathfrak{D}$ *which is given by*

$$\begin{aligned} \mathfrak{D} = \{(F, f,&w, G, g, v, M, N, K) \in \mathbb{R}^9_+ \mid \\ &0 < wv < A,\ f^2 \leq Fw,\ g^2 \leq Gv, M \leq A^2 w,\ N \leq A^2 v,\ K \leq A\}\,, \end{aligned}$$

and B satisfies the following size and convexity property in $\mathfrak{D}$:

$$0 \leq B(F,f,w,G,g,v,M,N,K) \leq 2(F+G)\,, \tag{3.27}$$

and for all (F,f,w,G,g,v,M,N,K), $(F_i,f_i,w_i,G_i,g_i,v_i,M_i,N_i,K_i) \in \mathfrak{D}$, *where* $i = 1,2$, *and for some constant* C,

$$\begin{aligned} &B(F,f,w,G,g,v,M,N,K) \\ &\quad \geq \frac{B(F_1,f_1,w_1,G_1,g_1,v_1,M_1,N_1,K_1)+B(F_2,f_2,w_2,G_2,g_2,v_2,M_2,N_2,K_2)}{2} + \frac{Cfg\kappa}{Awv}\,, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} &(F,f,w,G,g,v,M,N,K) \\ &\quad = \Big(\frac{F_1+F_2}{2},\frac{f_1+f_2}{2},\frac{w_1+w_2}{2},\frac{G_1+G_2}{2},\frac{g_1+g_2}{2},\frac{v_1+v_2}{2}, \\ &\qquad\qquad \frac{K}{v}\kappa+\frac{M_1+M_2}{2},\frac{K}{w}\kappa+\frac{N_1+N_2}{2},\kappa+\frac{K_1+K_2}{2}\Big)\,. \end{aligned} \tag{3.29}$$

Theorem 3.2.4 (Multivariable Version of Petermichl's the Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w\rangle_{Q'}\langle w\rangle_{Q'} < A$ and $\{\alpha_{j,Q}\}_{Q,j}$ be a sequence of nonnegative numbers such that, for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1,...,2^n-1$, the three inequalities below holds with some constant $A > 0$,*

$$\begin{aligned} \frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle w\rangle_{E_{j,Q}}} &\leq A\langle v\rangle_{E_{i,Q'}} \\ \frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle v\rangle_{E_{j,Q}}} &\leq A\langle w\rangle_{E_{i,Q'}} \\ \frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \alpha_{j,Q} &\leq A\,. \end{aligned}$$

Then for all $f \in L^2_{\mathbb{R}^n}(w)$ and $g \in L^2_{\mathbb{R}^n}(v)$

$$\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j=1}^{2^n-1} \alpha_{j,Q}\langle f\rangle_{E_{j,Q},w}\langle g\rangle_{E_{j,Q},v} \leq CA\|f\|_{L^2_{\mathbb{R}^n}(w)}\|g\|_{L^2_{\mathbb{R}^n}(v)}$$

holds with some constant $C > 0$.

Proof. For any dyadic cube $Q' \in \mathcal{D}^n$, and fixed $i = 1, ..., 2^n - 1$, let

$$\begin{aligned}
F_{i,Q'} &= \langle f^2 w\rangle_{E_{i,Q'}}, \quad f_{i,Q'} = \langle fw\rangle_{E_{i,Q'}}, \quad w_{i,Q'} = \langle w\rangle_{E_{i,Q'}},\\
G_{i,Q'} &= \langle g^2 v\rangle_{E_{i,Q'}}, \quad g_{i,Q'} = \langle gv\rangle_{E_{i,Q'}}, \quad v_{i,Q'} = \langle v\rangle_{E_{i,Q'}},\\
F^k_{i,Q'} &= \langle f^2 w\rangle_{E^k_{i,Q'}}, \quad f^k_{i,Q'} = \langle fw\rangle_{E^k_{i,Q'}}, \quad w^k_{i,Q'} = \langle w\rangle_{E^k_{i,Q'}},\\
G^k_{i,Q'} &= \langle g^2 v\rangle_{E^k_{i,Q'}}, \quad g^k_{i,Q'} = \langle gv\rangle_{E^k_{i,Q'}}, \quad v^k_{i,Q'} = \langle v\rangle_{E^k_{i,Q'}},
\end{aligned}$$

for $k = 1, 2$. Also we define, for all dyadic cube $Q' \in \mathcal{D}^n$ and fixed $i = 1, ..., 2^n - 1$,

$$\begin{aligned}
M_{i,Q'} &= \frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle v\rangle_{E_{j,Q}}} K_{E_{j,Q}},\\
N_{i,Q'} &= \frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle w\rangle_{E_{j,Q}}} K_{E_{j,Q}},\\
K_{i,Q'} &= \frac{1}{|E_{i,Q'}|} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \alpha_{j,Q},
\end{aligned}$$

$\kappa_{j,Q'} = \frac{2\alpha_{j,Q'}}{|E_{j,Q'}|}$, and similarly for $M^k_{i,Q'}$, $N^k_{i,Q'}$ and $K^k_{i,Q'}$. Then all variables

$$(F, f, w, G, g, v, M, N, K) = (F_{i,Q'}, f_{i,Q'}, w_{i,Q'}, G_{i,Q'}, g_{i,Q'}, v_{i,Q'}, M_{i,Q'}, N_{i,Q'}, K_{i,Q'}),$$

$$(F_k, f_k, w_k, G_k, g_k, v_k, M_k, N_k, K_k) = (F^k_{i,Q'}, f^k_{i,Q'}, w^k_{i,Q'}, G^k_{i,Q'}, g^k_{i,Q'}, v^k_{i,Q'}, M^k_{i,Q'}, N^k_{i,Q'}, K^k_{i,Q'}),$$

for $k = 1, 2$, belong to the domain $\mathfrak{D}$ of the function B defined in Lemma 3.2.3 and satisfy (3.29) with $\kappa = \kappa_{E_{j,Q'}}$. Then, by the properties (3.27) and (3.28), we have

$$\begin{aligned}
2|Q'|(\langle f^2 w\rangle_{Q'} + \langle g^2 v\rangle_{Q'}) &= 2|E_{1,Q'}|(\langle f^2 w\rangle_{E_{1,Q'}} + \langle g^2 v\rangle_{E_{1,Q'}})\\
&\geq |E_{1,Q'}| B(F_{1,Q'}, f_{1,Q'}, w_{1,Q'}, G_{1,Q'}, g_{1,Q'}, v_{1,Q'}, M_{1,Q'}, N_{1,Q'}, K_{1,Q'})\\
&\geq \sum_{k=1}^{2} |E^k_{1,Q'}| B(F^k_{1,Q'}, f^k_{1,Q'}, w^k_{1,Q'}, G^k_{1,Q'}, g^k_{1,Q'}, v^k_{1,Q'}, M^k_{1,Q'}, N^k_{1,Q'}, K^k_{1,Q'})\\
&\qquad + \frac{C}{A} \frac{\alpha_{E_{1,Q'}}}{\langle w\rangle_{E_{1,Q'}} \langle v\rangle_{E_{1,Q'}}} \langle fw\rangle_{E_{1,Q'}} \langle gv\rangle_{E_{1,Q'}}.
\end{aligned}$$

Iterating this procedure similar with the proof of Theorem 3.2.2 yields that

$$\frac{C}{A} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j=1}^{2^n-1} \frac{\alpha_{E_{i,Q'}}}{\langle w\rangle_{E_{i,Q'}} \langle v\rangle_{E_{i,Q'}}} \langle fw\rangle_{E_{i,Q'}} \langle gv\rangle_{E_{i,Q'}} \leq 2|Q'|(\langle f^2 w\rangle_{Q'} + \langle g^2 v\rangle_{Q'}).$$

Then, we can conclude that

$$\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j=1}^{2^n-1}\frac{\alpha_{E_{i,Q'}}}{\langle w\rangle_{E_{i,Q'}}\langle v\rangle_{E_{i,Q'}}}\langle fw\rangle_{E_{i,Q'}}\langle gv\rangle_{E_{i,Q'}}\leq cA(\|f\|^2_{L^2_{\mathbb{R}^n}(w)}+\|g\|^2_{L^2_{\mathbb{R}^n}(v)}) \tag{3.30}$$

Finally, letting

$$f=\sqrt{\frac{\|g\|_{L^2_{\mathbb{R}^n}(v)}}{\|f\|_{L^2_{\mathbb{R}^n}(w)}}}f \text{ and } g=\sqrt{\frac{\|f\|_{L^2_{\mathbb{R}^n}(w)}}{\|g\|_{L^2_{\mathbb{R}^n}(v)}}}g$$

yields the desired result. □

Changing $\alpha_{j,Q}$, f and g by $\alpha_{j,Q}\langle v\rangle_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}|E_{j,Q}|$, $fw^{-1/2}$ and $gv^{-1/2}$ respectively in Theorem 3.2.4, we can get the following Corollary.

Corollary 3.2.5 (Multivariable Version of the Bilinear Embedding Theorem)**.** *Let w and v be weights so that $\langle w\rangle_{Q'}\langle w\rangle_{Q'}<A$ and $\{\alpha_{j,Q}\}_{Q,j}$ be a sequence of nonnegative numbers such that, for all dyadic cubes $Q'\in\mathcal{D}^n$ and $i=1,...,2^n-1$, the three inequalities below holds with some constant $A>0$,*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\alpha_{j,Q}\langle v\rangle_{E_{j,Q}}|E_{j,Q}|\leq A\langle v\rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\alpha_{j,Q}\langle w\rangle_{E_{j,Q}}|E_{j,Q}|\leq A\langle w\rangle_{E_{i,Q'}}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\alpha_{j,Q}\langle w\rangle_{E_{j,Q}}\langle v\rangle_{E_{j,Q}}|E_{j,Q}|\leq A\,.$$

Then for all $f,g\in L^2_{\mathbb{R}^n}$

$$\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j=1}^{2^n-1}\alpha_{j,Q}\langle fw^{1/2}\rangle_{E_{j,Q}}\langle gv^{1/2}\rangle_{E_{j,Q}}|E_{j,Q}|\leq CA\|f\|_{L^2_{\mathbb{R}^n}}\|g\|_{L^2_{\mathbb{R}^n}}$$

holds with some constant $C>0$.

Lemma 3.2.6. *Let $\mathfrak{D}$ be given by those $(u,v,l)\in\mathbb{R}^3$ such that $u,v>0$, $uv\geq 1$ and $0\leq l\leq 1$. Then the function*

$$B(u,v,l):=u-\frac{1}{v(1+l)}$$

is defined on $\mathfrak{D}$, $0 \le B(x) \le u$ *for all* $x = (u, v, l) \in \mathfrak{D}$ *and satisfies the following differential inequalities on* $\mathfrak{D}$*:*

$$\frac{\partial B}{\partial l}(u, v, l) \ge \frac{1}{4v} \tag{3.31}$$

and

$$-(du, dv, dl) d^2 B(u, v, l)(du, dv, dl)^t \ge 0\,, \tag{3.32}$$

where $d^2 B(u, v, l)$ *denotes the Hessian matrix of the function* B *evaluated at* (u, v, l). *Moreover, condition (3.31) and (3.32) imply the following convexity condition. For all* x, x_j*'s* $\in \mathfrak{D}$, $j = 1, 2$, *such that* $x - \frac{1}{2}(x_1 + x_2) = (0, 0, \beta)$,

$$B(x) - \frac{1}{2}\sum_{j=1}^{2} B(x_j) \ge \frac{1}{4v}\beta\,. \tag{3.33}$$

One can find the proof of Lemma 3.2.6 in [Be].

Proposition 3.2.7. *Let* w *be a weight, so that* w^{-1} *is also a weight. Let* $\alpha_{j,Q}$ *be a Carleson sequence of nonnegative numbers i.e., there is a constant* $A > 0$ *such that, for all* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$,

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \le A\,. \tag{3.34}$$

Then, for all $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$,

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle w^{-1} \rangle_{E_{j,Q}}} \le 4A \langle w \rangle_{E_{i,Q'}}\,, \tag{3.35}$$

and if $w \in A_2^d$ *then for any* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$, *we have*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \langle w \rangle_{E_{j,Q}} \alpha_{j,Q} \le 4 \cdot 2^{2(n-1)} A [w]_{A_2^d} \langle w \rangle_{E_{i,Q'}}\,. \tag{3.36}$$

Furthermore, if $w \in A_2^R$ *then for any* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^{n-1}$, *we have*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \langle w \rangle_{E_{j,Q}} \alpha_{j,Q} \le 4A [w]_{A_2^R} \langle w \rangle_{E_{i,Q'}}\,. \tag{3.37}$$

Proof. Fix a dyadic cube Q' and i, Set $u_{i,Q'} = \langle w \rangle_{E_{i,Q'}}$, $v_{i,Q'} = \langle w^{-1} \rangle_{E_{i,Q'}}$ and

$$l_{i,Q'} = \frac{1}{A|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \alpha_{j,Q} \,.$$

For each dyadic cube Q', k and $k = 1, 2$, let $u^k_{i,Q'} = \langle w \rangle_{E^k_{i,Q'}}$, $v^k_{i,Q'} = \langle w^{-1} \rangle_{E^k_{i,Q'}}$ and

$$l^k_{i,Q'} = \frac{1}{A|E^k_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E^k_{i,Q'}} \alpha_{j,Q} \,.$$

Then, it is easy to see $\big(u_{i,Q'}, v_{i,Q'}, l_{i,Q'}\big)$, $\big(u^k_{i,Q'}, v^k_{i,Q'}, l^k_{i,Q'}\big) \in \mathfrak{D}$ by Hölder's inequality and (3.34). Moreover,

$$l_{i,Q'} = \frac{l^1_{i,Q'} + l^2_{i,Q'}}{2} + \frac{\alpha_{i,Q'}}{A} \,.$$

Then, by using the size condition of $B(x)$ in Lemma 3.2.6, we have

$$\begin{aligned} \big|E_{j_0,Q'}\big| \langle w \rangle_{E_{j_0,Q'}} &\geq \big|E_{j_0,Q'}\big| \, B(u_{j_0,Q'}, v_{j_0,Q'}, l_{j_0,Q'}) \\ &\geq \sum_{k=1}^{2} \big|E^k_{j_0,Q'}\big| B(u^k_{j_0,Q'}, v^k_{j_0,Q'}, l^k_{j_0,Q'}) + \frac{\alpha_{j_0,Q'}}{4\langle w^{-1} \rangle_{E_{j_0,Q'}} A} \qquad (3.38) \\ &\geq \sum_{k=1}^{2^n} |Q'_k| B(u_{Q'_k}, v_{Q'_k}, l_{Q'_k}) + \frac{1}{4A} \sum_{j=j_0}^{2^n-1} \frac{\alpha_{j_0,Q'}}{\langle w^{-1} \rangle_{E_{j_0,Q'}}} \,. \qquad (3.39) \end{aligned}$$

We use the convexity condition (3.32) for the inequality (3.38). We can get the inequality (3.39) by repeating the process (3.38) several times. Iterating this process and using the fact that $B \geq 0$ we have

$$\langle w \rangle_{E_{i,Q'}} \geq \frac{1}{4A|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \frac{\alpha_{j,Q}}{\langle w^{-1} \rangle_{E_{j,Q}}} \,,$$

which completes the proof of (3.35).Observe that in the case $w \in A_2^R$ then

$$\frac{1}{\langle w^{-1} \rangle_{E_{j,Q}}} \geq \frac{\langle w \rangle_{E_{j,Q}}}{[w]_{A_2^R}} \,.$$

Now (3.37) follows from (3.35). Observe if $w \in A_2^d$ then

$$[w]_{A_2} \geq \langle w \rangle_Q \langle w^{-1} \rangle_Q \geq \left(\frac{|E_{j,Q}|}{|Q|} \right)^2 \langle w \rangle_{E_{j,Q}} \langle w^{-1} \rangle_{E_{j,Q}} = 2^{-2(n-1)} \langle w \rangle_{E_{j,Q}} \langle w^{-1} \rangle_{E_{j,Q}} \,.$$

Thus, we can have (3.36) from (3.37). □

We refer to [Be] for the following lemma.

Lemma 3.2.8. *The following function*

$$B(u,v) = v - \frac{1}{u}$$

is defined on domain $\mathfrak{D}$ *which is given by*

$$\mathfrak{D} = \{\, (u,v) \in \mathbb{R}^2 |\ u, v > 0 \ \textit{and}\ uv \geq 1\} \,,$$

and B *satisfies the following size and differential condition in* $\mathfrak{D}$ *:*

$$0 \leq B(u,v) \leq v \,, \tag{3.40}$$

and

$$-(du,dv)d^2B(u,v)(du,dv)^t = \frac{2}{u^3}|du|^2 \,. \tag{3.41}$$

Furthermore, (3.41) implies the following convexity property. For all $(u,v), (u_1,v_1)$ *and* $(u_2,v_2) \in \mathfrak{D}$ *, where* $(u,v) = \big((u_1,u_2)/2, (v_1,v_2)/2\big)$*:*

$$B(u,v) - \frac{B(u_1,v_1) + B(u_2,v_2)}{2} \geq C\frac{1}{u^3}(u_1 - u_2)^2 \,. \tag{3.42}$$

The following proposition includes the multidimensional analogues to corresponding one-dimensional results in [Pet2] for both regular and anisotropic cases.

Proposition 3.2.9. *There exist a positive constant* C *so that for all weight* w *and* w^{-1} *and for all dyadic cubes* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$*:*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \frac{\big(\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}\big)^2}{\langle w\rangle^3_{E_{j,Q}}}\big|E_{j,Q}\big| \leq C\langle w^{-1}\rangle_{E_{i,Q'}} \tag{3.43}$$

and, if $w \in A^d_2$*, the following inequality holds for all dyadic cubes* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$ *:*

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\ \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(\frac{\langle w\rangle_{E^1_{j,Q}} - \langle w\rangle_{E^2_{j,Q}}}{\langle w\rangle_{E_{j,Q}}}\right)^2 \big|E_{j,Q}\big|\langle w^{-1}\rangle_{E_{j,Q}}$$

$$\leq C2^{2(n-1)}[w]_{A^d_2}\langle w^{-1}\rangle_{E_{i,Q'}} \,. \tag{3.44}$$

Moreover, if $w \in A_2^R$, *the following inequality holds for all dyadic cubes* $Q' \in \mathcal{D}^n$ *and* $i = 1, ..., 2^n - 1$:

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w^{-1} \rangle_{E_{j,Q}} \le C[w]_{A_2^R} \langle w^{-1} \rangle_{E_{i,Q'}} . \tag{3.45}$$

Proof. Due to the construction of $E_{i,Q'}$, it is enough to show when $i = 1$. For any subset of dyadic cube $Q' \in \mathcal{D}^n$ and fixed j, $E_{j,Q'}$, set

$$u_{j,Q'} = \langle w \rangle_{E_{j,Q'}}, v_{j,Q'} = \langle w^{-1} \rangle_{E_{j,Q'}}, u^i_{j,Q'} = \langle w \rangle_{E^i_{j,Q'}} \text{ and } v^i_{j,Q'} = \langle w^{-1} \rangle_{E^i_{j,Q'}} \tag{3.46}$$

for $i = 1, 2$. By Hölder's inequality $(v_{j,Q'}, u_{j,Q'})$ and $(v^i_{j,Q'}, u^i_{j,Q'})$ belong to $\mathfrak{D}$ which is defined in Lemma (3.2.8), for all i, j and Q'. Then, by the construction of our Haar functions,

$$|E_{1,Q'}| \langle w^{-1} \rangle_{E_{1,Q'}} \ge |E_{1,Q'}| B(u_{1,Q'}, v_{1,Q'}) \tag{3.47}$$

$$\ge \sum_{i=1}^{2} |E^i_{1,Q'}| B(u^i_{1,Q'}, v^i_{1,Q'}) + C \frac{|E_{1,Q'}| \left(\langle w \rangle_{E^1_{1,Q'}} - \langle w \rangle_{E^2_{1,Q'}} \right)^2}{\langle w \rangle^3_{E_{1,Q'}}} \tag{3.48}$$

$$= \sum_{j=2}^{3} |E_{j,Q'}| B(u_{j,Q'}, v_{j,Q'}) + C \frac{|E_{1,Q'}| \left(\langle w \rangle_{E^1_{1,Q'}} - \langle w \rangle_{E^2_{1,Q'}} \right)^2}{\langle w \rangle^3_{E_{1,Q'}}} .$$

Using the size condition (3.40) and the convexity property (3.42) allow the inequalities (3.47) and (3.48) respectively. Iterating this process $n - 1$ times more, we get

$$|E_{1,Q'}| \langle w^{-1} \rangle_{E_{1,Q'}} \ge \sum_{j=2^n}^{2^{n+1}-1} |E_{j,Q'}| B(u_{j,Q'}, v_{j,Q'}) + C \sum_{j=1}^{2^n-1} \frac{|E_{j,Q'}| \left(\langle w \rangle_{E^1_{j,Q'}} - \langle w \rangle_{E^2_{j,Q'}} \right)^2}{\langle w \rangle^3_{E_{j,Q'}}} .$$

Due to our construction of Haar system, for all $j = 2^n, 2^n + 1, ..., 2^{n+1} - 1$, $|E_{j,Q'}|$'s are mutually disjoint and $|E_{j,Q'}| = |Q'|/2^n$ i.e. $\{E_{j,Q'}\}_{j=2^n,...,2^{n+1}-1}$ is a set of dyadic sub-cubes of Q', $\mathcal{D}^n_1(Q')$. Thus,

$$|E_{1,Q'}| \langle w^{-1} \rangle_{E_{1,Q'}} \ge \sum_{k=1}^{2^n} |Q'_k| B(u_{Q'_k}, v_{Q'_k}) + C \sum_{j=1}^{2^n-1} \frac{|E_{j,Q'}| \left(\langle w \rangle_{E^1_{j,Q'}} - \langle w \rangle_{E^2_{j,Q'}} \right)^2}{\langle w \rangle^3_{E_{j,Q'}}} ,$$

where Q'_k's are enumerations of 2^n dyadic sub-cubes of Q'. Iterating this procedure and using fact $B \geq 0$ yield that

$$\left|E_{1,Q'}\right|\langle w^{-1}\rangle_{E_{1,Q'}} \geq C \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j=1}^{2^n-1} \frac{\left(\langle w\rangle_{E^1_{j,Q'}} - \langle w\rangle_{E^2_{j,Q'}}\right)^2}{\langle w\rangle^3_{E_{j,Q'}}} \left|E_{j,Q'}\right|,$$

which completes the proof of (3.43). The similar observations in the end of the proof of Proposition 3.2.9 yields (3.44) and (3.45). □

The next lemma appeared in [Be].

Lemma 3.2.10. *The following function*

$$B(u,v) = \sqrt[4]{uv}$$

is defined on domain $\mathfrak{D}$ which is given by

$$\mathfrak{D} = \left\{(u,v) \in \mathbb{R}^2 \middle| u,v > 0 \text{ and } uv \geq 1\right\},$$

and B satisfies the following size and differential condition in $\mathfrak{D}$:

$$0 \leq B(u,v) \leq \sqrt[4]{uv}\,, \tag{3.49}$$

and

$$-\left(du,dv\right) d^2B(u,v)(du,dv)^t \geq \frac{1}{8} v^{1/4}u^{-7/4}|du|^2\,. \tag{3.50}$$

Furthermore, (3.50) implies the following convexity property. For all (u,v), (u_1,v_1) and $(u_2,v_2) \in \mathfrak{D}$, where $(u,v) = \big((u_1,v_1)/2, (v_1,v_2)/2\big)$:

$$B(u,v) - \frac{B(u_1,v_1) + B(u_2,v_2)}{2} \geq Cv^{1/4}u^{-7/4}(u_1-u_2)^2\,. \tag{3.51}$$

The following generalizes the result that appeared in [Be] to the multidimensional regular and anisotropic cases.

Proposition 3.2.11. *There exist a positive constant C so that for all weight w and w^{-1} and for all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, ..., 2^n - 1$:*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w \rangle^{1/4}_{E_{j,Q}} \langle w^{-1} \rangle^{1/4}_{E_{j,Q}} \leq C \langle w \rangle^{1/4}_{E_{i,Q'}} \langle w^{-1} \rangle^{1/4}_{E_{i,Q'}} \tag{3.52}$$

and, if $w \in A^d_2$, the following inequality holds for all dyadic cubes $Q' \in \mathcal{D}^n$:

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w \rangle_{E_{j,Q}} \langle w^{-1} \rangle_{E_{j,Q}} \leq C 2^{2(n-1)} [w]_{A^d_2} . \tag{3.53}$$

Moreover, if $w \in A^R_2$, the following inequality holds for all dyadic cubes $Q' \in \mathcal{D}^n$:

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j: E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w \rangle_{E_{j,Q}} \langle w^{-1} \rangle_{E_{j,Q}} \leq C [w]_{A^R_2} . \tag{3.54}$$

Proof. Similarly with Proposition 3.2.9, we only prove for $i = 1$. For fixed dyadic cube Q', using the size condition (3.49), convexity property (3.51) and notation (3.46),

$$\begin{aligned} |E_{1,Q'}| \sqrt[4]{u_{1,Q'} v_{1,Q'}} &\geq |E_{1,Q'}| B(u_{1,Q'}, v_{1,Q'}) \\ &\geq \sum_{k=1}^{2} |E^k_{1,Q'}| B(u^k_{1,Q'}, v^k_{1,Q'}) + C \frac{\langle w^{-1} \rangle_{E_{1,Q'}}}{\langle w \rangle^{7/4}_{E_{1,Q'}}} \left(\langle w \rangle_{E^1_{1,Q'}} - \langle w \rangle_{E^2_{1,Q'}} \right)^2 \\ &= \sum_{j=2}^{3} |E_{j,Q'}| B(u_{j,Q'}, v_{j,Q'}) + C \frac{\langle w^{-1} \rangle_{E_{1,Q'}}}{\langle w \rangle^{7/4}_{E_{1,Q'}}} \left(\langle w \rangle_{E^1_{1,Q'}} - \langle w \rangle_{E^2_{1,Q'}} \right)^2 \\ &\geq \sum_{k=1}^{2^n} |R_k| B(u_{Q'_k}, v_{Q'_k}) + C \sum_{j=1}^{2^n-1} \frac{\langle w^{-1} \rangle_{E_{j,Q'}}}{\langle w \rangle^{7/4}_{E_{j,Q'}}} \left(\langle w \rangle_{E^1_{j,Q'}} - \langle w \rangle_{E^2_{j,Q'}} \right)^2 . \end{aligned} \tag{3.55}$$

Similarly with Proposition 3.2.9, for the inequality (3.55), we iterate the process $n-1$ times and use the notation Q'_k's for numbering of 2^n sub-cubes of Q'. We can finish the proof by iterating this progression and using the fact $B \geq 0$. In other words, we get:

$$\left|E_{1,Q'}\right| \sqrt[4]{\langle w\rangle_{E_{1,Q'}} \langle w^{-1}\rangle_{E_{1,Q'}}} \geq C \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j=1}^{2^n-1} \left|E_{j,Q'}\right| \frac{\langle w^{-1}\rangle_{E_{j,Q'}}^{1/4}}{\langle w\rangle_{E_{j,Q'}}^{7/4}} \left(\langle w\rangle_{E^1_{j,Q'}} - \langle w\rangle_{E^2_{j,Q'}}\right)^2.$$

This proves (3.52). If $w \in A_2^R$, then $\langle w\rangle_{E_{j,Q}} \langle w^{-1}\rangle_{E_{j,Q}} \leq [w]_{A_2^R}$ for all $Q \in \mathcal{D}^n$ and $j = 1, ..., 2^n - 1$. Thus, (3.52) yields immediately (3.54). Since for all $w \in A_2^d$, $\langle w\rangle_{E_{j,Q}} \langle w^{-1}\rangle_{E_{j,Q}} \leq 2^{2(n-1)}[w]_{A_2^d}$, we can easily get (3.53) from (3.52). □

The similar version of Bellman function with the following lemma was appeared in [Be].

Lemma 3.2.12. *The following function*

$$B(u,v) = u\left(2Q - \frac{2Q}{uv} - \frac{4}{3}\ln(uv)\right)$$

is defined on domain $\mathfrak{D}$ *which is given by*

$$\mathfrak{D} = \left\{(u,v) \in \mathbb{R}^2 \middle| u,v > 0 \text{ and } 1 \leq uv \leq Q\right\},$$

and B *satisfies the following size and differential condition in* $\mathfrak{D}$*:*

$$0 \leq B(u,v) \leq 2Qu\,, \tag{3.56}$$

and

$$-(du,dv)\, d^2B(u,v)(du,dv)^t \geq \frac{2(du)^2}{3u}\,. \tag{3.57}$$

Furthermore, (3.57) implies the following convexity property. For all (u,v), (u_1+u_2) *and* $(u_2+v_2) \in \mathfrak{D}$, *where* $(u,v) = \big((u_1,v_1)/2, (u_2,v_2)/2\big)$, *there is a constant* C *such that*

$$B(u,v) - \frac{B(u_1,v_1)+B(u_2,v_2)}{2} \geq \frac{C}{u}(u_1-u_2)^2\,. \tag{3.58}$$

Proof. Since $uv > 0$ on $\mathfrak{D}$, the size condition (3.56) holds clearly. We also check easily the differential condition (3.57) as follows.

$$\begin{aligned}
-(du,dv)\, d^2B(u,v)(du,dv)^t &= -\frac{\partial^2 B}{\partial u^2}(du)^2 - 2\frac{\partial^2 B}{\partial u \partial v}dudv - \frac{\partial^2 B}{\partial v^2}(dv)^2 \\
&= \frac{4}{3u}(du)^2 + \frac{8}{3v}dudv + \left(\frac{4Q}{v^3} - \frac{4u}{3v^2}\right)(dv)^2 \\
&= \frac{2}{3u}(du)^2 + \left(\sqrt{\frac{2}{3u}}du + \sqrt{\frac{8u}{3v^2}}dv\right)^2 + \left(\frac{4Q}{v^3} - \frac{4u}{v^2}\right)(dv)^2
\end{aligned}$$

Using the domain of B, we can easily see that

$$\frac{4Q}{v^3} - \frac{4u}{v^2} \geq \frac{4uv}{v^3} - \frac{4u}{v^2} = 0\,,$$

thus discarding non-negative terms provide the differential condition (3.57). Set $\Delta u = (u_1 - u_2)/2$ and $(\Delta v = (v_1 - v_2)/2$, then $u_1 = u + \Delta u$, $u_2 = u - \Delta u$, $v_1 = v + \Delta v$ and $v_2 = v - \Delta v$. Note that $|u \pm s\Delta u| \leq |u| + |\Delta u| \leq 2u$, for any number $s \in [0,1]$. We now using Taylor's theorem and the differential condition (3.57) to see the convexity condition (3.58):

$$\begin{aligned}
&B(u,v) - \frac{B(u+\Delta u, v+\Delta v) - B(u-\Delta u, v-\Delta u)}{2} \\
&\quad = B(u,v) - \frac{1}{2}\Big(B(u,v) + \nabla B(u,v)(\Delta u, \Delta v)^t \\
&\qquad\qquad + \int_0^1 (1-s)\,(\Delta u, \Delta v)\, d^2B(u+s\Delta u, v+s\Delta v)(\Delta u,\Delta v)^t ds \\
&\qquad\qquad + B(u,v) + \nabla B(u,v)(-\Delta u, -\Delta v)^t \\
&\qquad\qquad + \int_0^1 (1-s)\,(-\Delta u, -\Delta v)\, d^2B(u-s\Delta u, v-s\Delta v)(-\Delta u,-\Delta v)^t ds\Big) \\
&\quad \geq \int_0^1 \frac{(1-s)(\Delta u)^2}{3(u+s\Delta u)}ds + \int_0^1 \frac{(1-s)(\Delta u)^2}{3(u+s\Delta u)}ds \\
&\quad \geq \frac{(\Delta u)^2}{6u}\int_0^1 (1-s)\,ds = \frac{(\Delta u)^2}{12u} = \frac{(u_1-u_2)^2}{48u}\,.
\end{aligned}$$

□

We are now ready to prove the sharp version of Buckley's inequality in a multivariable setting. The single variable version of the following proposition first appeared in [Wi1]. In [Per2], one can also find a Bellman function proof of a similar result which can be extended to the doubling measure case.

Proposition 3.2.13 (Wittwer's sharp version of Buckley's inequality). *There exist a positive constant C so that for all weight $w \in A_2^d$ and all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, ..., 2^n - 1$:*

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w \rangle_{E_{j,Q}} \leq C 2^{2(n-1)} [w]_{A_2^d} \langle w \rangle_{E_{i,Q'}} , \tag{3.59}$$

and for all weight $w \in A_2^R$ and all dyadic cubes $Q' \in \mathcal{D}^n$ and $i = 1, ..., 2^n - 1$:

$$\frac{1}{|E_{i,Q'}|} \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j : E_{j,Q} \subseteq E_{i,Q'}} \left(\frac{\langle w \rangle_{E^1_{j,Q}} - \langle w \rangle_{E^2_{j,Q}}}{\langle w \rangle_{E_{j,Q}}} \right)^2 |E_{j,Q}| \langle w \rangle_{E_{j,Q}} \leq C [w]_{A_2^R} \langle w \rangle_{E_{i,Q'}} . \tag{3.60}$$

Proof. we only prove only for $i = 1$. For any $E_{j,Q'}$ which is non-empty unions from $\mathcal{D}^n(Q)$ and $l = 1, 2$, set

$$u_{j,Q'} = \langle w \rangle_{E_{j,Q'}}, v_{j,Q'} = \langle w^{-1} \rangle_{E_{j,Q'}}, u^l_{j,Q'} = \langle w \rangle_{E^l_{j,Q'}} \text{ and } v^l_{j,Q'} = \langle w^{-1} \rangle_{E^l_{j,Q'}} .$$

By Hölder's inequality and A_2^d condition, $(u_{j,Q'}, v_{j,Q'})$ and $(u^l_{j,Q'}, v^l_{j,Q'})$ belong to the domain $\mathfrak{D} = \{(u, v) \in \mathbb{R}^2 \,|\, u, v > 0 \text{ and } 1 \leq uv \leq 2^{2(n-1)} [w]_{A_2^d}\}$ of the function B defined in Lemma 3.2.12. Thus, by using the size condition (3.56) and convexity property (3.58), we have

$$\begin{aligned} 2 \cdot 2^{2(n-1)} [w]_{A_2^d} |E_{1,Q'}| \langle w \rangle_{E_{1,Q'}} &\geq |E_{1,Q'}| B(u_{1,Q'}, v_{1,Q'}) \\ &\geq \sum_{l=1}^{2} |E^l_{1,Q'}| B(u^l_{1,Q'}, v^l_{1,Q'}) + C \frac{\left(\langle w \rangle_{E^1_{1,Q'}} - \langle w \rangle_{E^2_{1,Q'}}\right)^2}{\langle w \rangle_{E_{1,Q'}}} . \end{aligned}$$

Applying convexity property (3.58) $n - 1$ times more, we get

$$2 \cdot 2^{2(n-1)} [w]_{A_2^d} |E_{1,Q'}| \langle w \rangle_{E_{1,Q'}} \geq \sum_{k=1}^{2^n} |R_k| B(u_{Q'_k}, v_{Q'_k}) + C \sum_{j=1}^{2^n - 1} \frac{\left(\langle w \rangle_{E^1_{j,Q'}} - \langle w \rangle_{E^2_{j,Q'}}\right)^2}{\langle w \rangle_{E_{j,Q'}}} .$$

Since B is positive iterating the above process will yield that

$$2 \cdot 2^{2(n-1)} [w]_{A_2^d} \left| E_{1,Q'} \right| \langle w \rangle_{E_{1,Q'}} \geq C \sum_{Q \in \mathcal{D}^n(Q')} \sum_{j=1}^{2^n - 1} \frac{\left(\langle w \rangle_{E_{j,Q}^1} - \langle w \rangle_{E_{j,Q}^2} \right)^2}{\langle w \rangle_{E_{j,Q}}} .$$

This proves (3.59). The inequality (3.60) can be seen by using the domain $\mathfrak{D} = \left\{ (u,v) \in \mathbb{R}^2 \,|\, u,v > 0 \text{ and } 1 \leq uv \leq [w]_{A_2^R} \right\}$. □

Chapter 4

Commutator of the Hilbert transform

In this chapter, we will show our result about the commutator of the Hilbert transform:

$$\|[b,H]f\|_{L^2(w)} \leq C\|b\|_{BMO}[w]^2_{A_2}\|f\|_{L^2(w)}\,. \tag{4.1}$$

Due to (2.17), we will prove (4.1) with the dyadic shift operator S instead of the Hilbert transform H. In the decomposition

$$[b,S] = [\pi_b,S] + [\pi_b^*,S] + [\lambda_b,S]\,,$$

both π_b and S obey linear bounds. Therefore, we have

$$\|[\pi_b,S]\|_{L^2(w)\to L^2(w)} + \|[\pi_b^*,S]\|_{L^2(w)\to L^2(w)} \leq C\|b\|_{BMO^d}[w]^2_{A_2^d}\,.$$

To finish the proof of (4.1), it suffices to show that

$$\|[\lambda_b,S]\|_{L^2(w)} \leq C\|b\|_{BMO^d}[w]^2_{A_2^d}\,.$$

In fact, we will get a better result on this term. In Section 4.1 we will start our discussion on how to find the linear bound for the term $[\lambda_b,S]$, most of which will be very similar to calculations performed in [Pet2]. In Section 4.2 we will finish the linear estimate for the term $[\lambda_b, S]$, and prove Theorem 2.3.2. In Section 4.3 we prove the linear bound for $\pi_b^* S$. In Section 4.4 we reduce the proof of the linear bound for $S\pi_b$ to verifying three embedding conditions, two are proved in this section, the third is proved in Section 4.5 using a Bellman function argument.

4.1 Linear bound for $[\lambda_b, S]$ part I

In general, when we analyse commutator operators, a subtle cancelation delivers the result one wants to find. In the analysis of the commutator $[b, S]$, the part $[\lambda_b, S]$ will allow for certain cancelation. First, let us rewrite $[\lambda_b, S]$.

$$\begin{aligned}
[\lambda_b, S](f) &= \lambda_b(Sf) - S(\lambda_b f) \\
&= \sum_{I\in\mathcal{D}} \langle b\rangle_I \langle Sf, h_I\rangle h_I - \sum_{J\in\mathcal{D}} \langle \lambda_b f, h_J\rangle (h_{J_-} - h_{J_+}) \\
&= \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{D}} \langle b\rangle_I \langle f, h_J\rangle \langle h_{J_-} - h_{J_+}, h_I\rangle h_I \\
&\qquad - \sum_{J\in\mathcal{D}} \sum_{I\in\mathcal{D}} \langle b\rangle_I \langle f, h_I\rangle \langle h_I, h_J\rangle (h_{J_-} - h_{J_+}) \,.
\end{aligned}$$

From the orthogonality of Haar system, both double sums collapse to just one,

$$\begin{aligned}
[\lambda_b, S](f) &= \sum_{J\in\mathcal{D}} \langle b\rangle_{J_-} \langle f, h_J\rangle h_{J_-} - \sum_{J\in\mathcal{D}} \langle b\rangle_{J_+} \langle f, h_J\rangle h_{J_+} \\
&\qquad - \sum_{J\in\mathcal{D}} \frac{\langle b\rangle_{J_+} + \langle b\rangle_{J_-}}{2} \langle f, h_J\rangle (h_{J_-} - h_{J_+}) \\
&= \sum_{J\in\mathcal{D}} \frac{\langle b\rangle_{J_-} - \langle b\rangle_{J_+}}{2} \langle f, h_J\rangle h_{J_-} - \sum_{J\in\mathcal{D}} \frac{\langle b\rangle_{J_+} - \langle b\rangle_{J_-}}{2} \langle f, h_J\rangle h_{J_+} \\
&= - \sum_{J\in\mathcal{D}} \Delta_J b \,\langle f, h_J\rangle (h_{J_+} + h_{J_-}) \,,
\end{aligned}$$

recall the notation $\Delta_J b = (\langle b\rangle_{J_+} - \langle b\rangle_{J_-})/2$. To find the $L^2(w)$ operator norm of $[\lambda_b, S]$, it is enough to deal with the linear operator

$$S_b(f) = \sum_{I\in\mathcal{D}} \Delta_I b\,\langle f, h_I\rangle h_{I_-}\,.$$

Recall (3.11) that if $b \in BMO^d$, $|\Delta_I b| \le \|b\|_{BMO^d}$. We shall state the weighted operator norm of S_b as a Theorem and give a detailed proof. Theorem 1.0.2 is a direct consequence of the following Theorem. We will prove the following Theorem by the technique used in [Pet2].

Theorem 4.1.1. *There exists a constant $C > 0$, such that*

$$\|S_b\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2^d}\|b\|_{BMO^d} \tag{4.2}$$

for all $b \in BMO^d$ and $w \in A_2^d$ for all $f \in L^2(w)$.

Inequality (4.2) is equivalent to the following inequality for any positive functions $f \in L^2(w^{-1})$ and $g \in L^2(w)$,

$$|\langle S_{b,w^{-1}}f, g\rangle_w| \le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,, \tag{4.3}$$

where $S_{b,w^{-1}}(f) = S_b(w^{-1}f)$, since $f \in L^2(w^{-1})$ if and only if $w^{-1}f \in L^2(w)$. Expanding f and g in the disbalanced Haar systems respectively for $L^2(w^{-1})$ and $L^2(w)$ yields for (4.3),

$$\begin{aligned}
|\langle S_{b,w^{-1}}f, g\rangle_w| &= \left|\int S_{b,w^{-1}}\Big(\sum_{I\in\mathcal{D}}\langle f, h_I^{w^{-1}}\rangle_{w^{-1}} h_I^{w^{-1}}\Big)\Big(\sum_{J\in\mathcal{D}}\langle g, h_J^w\rangle_w h_J^w\Big) w\,dx\right| \\
&= \left|\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}\langle f, h_I^{w^{-1}}\rangle_{w^{-1}}\langle g, h_J^w\rangle_w \int h_J^w S_{b,w^{-1}}(h_I^{w^{-1}}) w\,dx\right| \\
&= \left|\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}\langle f, h_I^{w^{-1}}\rangle_{w^{-1}}\langle g, h_J^w\rangle_w \langle S_{b,w^{-1}}h_I^{w^{-1}}, h_J^w\rangle_w\right|.
\end{aligned} \tag{4.4}$$

In (4.4),

$$\begin{aligned}
\langle S_{b,w^{-1}}h_I^{w^{-1}}, h_J^w\rangle_w &= \Big\langle \sum_{L\in\mathcal{D}} \Delta_L b\,\langle h_L, w^{-1}h_I^{w^{-1}}\rangle\, h_{L^-}, h_J^w\Big\rangle_w \\
&= \sum_{L\in\mathcal{D}} \Delta_L b\,\langle h_L, h_I^{w^{-1}}\rangle_{w^{-1}}\langle h_{L^-}, h_J^w\rangle_w\,.
\end{aligned}$$

Since $\langle h_L, h_I^{w^{-1}}\rangle_{w^{-1}} \neq 0$, only when $L \subseteq I$ and $\langle h_{L^-}, h_J^w\rangle_w \neq 0$ only when $L_- \subseteq J$, then we have non-zero terms if $I \subseteq J$ or $\hat{J} \subseteq I$ in the sum of (4.4). Thus we can split the sum into four parts, $\sum_{I=J}$, $\sum_{I=\hat{J}}$, $\sum_{\hat{J}\subsetneq I}$, and $\sum_{I\subsetneq J}$. Let us now introduce the truncated shift operator

$$S_b^I(f) := \sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle f, h_L\rangle h_{L_-}\,,$$

and its composition with multiplication by w^{-1},

$$S_{b,w^{-1}}^I(f) := \sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}f, h_L\rangle h_{L_-}\,.$$

We will see that the weighted norm $\|S_{b,w^{-1}}^I\chi_I\|_{L^2(w)}$, proved in Chapter 3, plays a main role in our estimate for $\langle S_{b,w^{-1}}h_I^{w^{-1}}, h_J^w\rangle_w$. More specifically, we proved in Lemma 3.1.4 that

$$\|S_{b,w^{-1}}^I\chi_I\|_{L^2(w)} \leq c\|b\|_{BMO^d}[w]_{A_2^d} w^{-1}(I)^{1/2} \tag{4.5}$$

for all intervals I and weights $w \in A_2^d$.

4.2 Linear bound for $[\lambda_b, S]$ part II

We will continue to estimate the sum (4.4) in four parts.

4.2.1 $\sum_{I=\hat{J}}$

For this case, it is sufficient to show that

$$|\langle S_{b,w^{-1}}h_{\hat{J}}^{w^{-1}}, h_J^w\rangle_w| \leq c\|b\|_{BMO^d}[w]_{A_2^d}\,,$$

because then one can use Cauchy-Schwarz inequality and Plancherel in the part of (4.4) corresponding $\sum_{I=\hat{J}}$ to get estimate (4.3). Since $\langle h_k, h_I^w\rangle_w$ could be non-zero only if $K \subseteq I$,

$$|\langle S_{b,w^{-1}} h_{\hat{J}}^{w^{-1}}, h_J^w \rangle_w| = \left| \left\langle \sum_{L\in\mathcal{D}} \Delta_L b \, \langle w^{-1} h_L^{w^{-1}}, h_L \rangle h_{L_-}, h_J^w \right\rangle_w \right|$$
$$= \left| \sum_{L\in\mathcal{D}} \Delta_L b \, \langle h_{\hat{J}}^{w^{-1}}, h_L \rangle_{w^{-1}} \langle h_{L_-}, h_J^w \rangle_w \right|$$

has non-zero term only when $L \subseteq \hat{J}$. Thus

$$|\langle S_{b,w^{-1}} h_L^{w^{-1}}, h_J^w \rangle_w| = \left| \sum_{L\in\mathcal{D}(\hat{J})} \Delta_L b \, \langle h_{\hat{J}}^{w^{-1}}, h_L \rangle_{w^{-1}} \langle h_{L_-}, h_J^w \rangle_w \right|$$
$$= \left| \sum_{L\in\mathcal{D}(J)} \Delta_L b \, \langle h_{\hat{J}}^{w^{-1}}, h_L \rangle_{w^{-1}} \langle h_{L_-}, h_J^w \rangle_w \right| + \left| \sum_{L\in\mathcal{D}(J^s)} \Delta_L b \, \langle h_{\hat{J}}^{w^{-1}}, h_L \rangle_{w^{-1}} \langle h_{L_-}, h_J^w \rangle_w \right|$$
$$+ |\Delta_{\hat{J}} b \, \langle h_{\hat{J}}^{w^{-1}}, h_{\hat{J}} \rangle_{w^{-1}} \langle h_{\hat{J}_-}, h_J^w \rangle_w|$$
$$\leq |\langle S_{b,w^{-1}}^J h_{\hat{J}}^{w^{-1}}, h_J^w \rangle_w| + |\Delta_{\hat{J}} b \, \langle h_{\hat{J}}^{w^{-1}}, h_{\hat{J}} \rangle_{w^{-1}} \langle h_{\hat{J}_-}, h_J^w \rangle_w| \,,$$

in the second equality, J^s denotes the sibling of J, so for all $L \subseteq J^s$, $\langle h_{L_-}, h_J^w \rangle_w = 0$. Then, by (2.4),

$$|\Delta_{\hat{J}} b \, \langle h_{\hat{J}}^{w^{-1}}, h_{\hat{J}} \rangle_{w^{-1}} \langle h_{\hat{J}_-}, h_J^w \rangle_w| \leq \sqrt{2} \|b\|_{BMO^d} [w]_{A_2^d}^{1/2} \,. \tag{4.6}$$

So for the remaining part:

$$|\langle S_{b,w^{-1}}^J h_{\hat{J}}^{w^{-1}}, h_J^w \rangle_w| = |h_{\hat{J}}^{w^{-1}}(J) \langle S_{b,w^{-1}}^J \chi_J, h_J^w \rangle_w| \leq c \|b\|_{BMO^d} [w]_{A_2^d} \,, \tag{4.7}$$

here the last inequality uses Cauchy-Schwarz inequality, (2.6), and Lemma 3.1.4, that is estimate (4.5).

4.2.2 $\sum_{I=J}$

In this case, the argument is similar to the argument in Section 4.2.1. We have

$$|\langle S_{b,w^{-1}} h_J^{w^{-1}}, h_J^w \rangle_w| = |\sum_{L\in\mathcal{D}} \Delta_L b \, \langle h_J^{w^{-1}}, h_L \rangle_{w^{-1}} \langle h_{L_-}, h_J^w \rangle_w|$$

here we have zero summands, unless $L \subseteq J$. Thus,

$$
\begin{aligned}
|\langle S_{b,w^{-1}} h_J^{w^{-1}}, h_J^w \rangle_w| &= |\langle S_{b,w^{-1}}^J h_J^{w^{-1}}, h_J^w \rangle_w| \\
&\leq |\langle S_{b,w^{-1}}^{J_+} h_J^{w^{-1}}, h_J^w \rangle_w| + |\langle S_{b,w^{-1}}^{J_-} h_J^{w^{-1}}, h_J^w \rangle_w| + |\Delta_J\, b\, \langle h_J^{w^{-1}}, h_J \rangle_{w^{-1}} \langle h_{J_-}, h_J^w \rangle_w| \\
&\leq c\, \|b\|_{BMO^d} [w]_{A_2^d}\,.
\end{aligned}
$$

In the last inequality, we use same arguments as in (4.7) for the first two terms, and (2.5) for the last term.

4.2.3 $\sum_{\hat{J} \subsetneq I}$ and $\sum_{I \subsetneq J}$

To obtain our desired results, we need to understand the supports of $S_b(w^{-1} h_I^{w^{-1}})$ and $S_b^*(w h_J^w)$. Since

$$
S_b(w^{-1} h_I^{w^{-1}}) = \sum_{L \in \mathcal{D}} \Delta_L b\, \langle w^{-1} h_I^{w^{-1}}, h_L \rangle h_{L_-} = \sum_{L \in \mathcal{D}} \Delta_L b\, \langle h_I^{w^{-1}}, h_L \rangle_{w^{-1}} h_{L_-}\,,
$$

and $\langle h_I^{w^{-1}}, h_L \rangle_{w^{-1}}$ can be non-zero only when $L \subseteq I$, therefore $S_b(w^{-1} h_I^{w^{-1}})$ is supported by I. Also,

$$
\langle S_b(w^{-1} h_I^{w^{-1}}), h_J^w \rangle_w = \langle h_I^{w^{-1}}, S_b^*(w h_J^w) \rangle_{w^{-1}}\,, \tag{4.8}
$$

yield that $S_b^*(wh_J^w) = \sum_{L\in\mathcal{D}} \Delta_L b\langle wh_J^w, h_{L_-}\rangle h_L$ is supported by $\hat{J}$. Let us now consider the sum $\hat{J} \subsetneq I$. Then

$$\begin{aligned}
&\left| \sum_{I,J:\hat{J}\subsetneq I} \langle f, h_I^{w^{-1}}\rangle_{w^{-1}} \langle g, h_J^w\rangle_w \langle S_{b,w^{-1}} h_I^{w^{-1}}, h_J^w\rangle_w \right| \\
&\qquad = \left| \sum_{I,J:\hat{J}\subsetneq I} \langle f, h_I^{w^{-1}}\rangle_{w^{-1}} \langle g, h_J^w\rangle_w \langle h_I^{w^{-1}}, S_b^*(wh_J^w)\rangle_{w^{-1}} \right| \\
&\qquad = \left| \sum_{J\in\mathcal{D}} \sum_{I:I\supsetneq\hat{J}} \langle f, h_I^{w^{-1}}\rangle_{w^{-1}} \langle g, h_J^w\rangle_w h_I^{w^{-1}}(\hat{J}) \langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w \right| \qquad (4.9)\\
&\qquad = \left| \sum_{J\in\mathcal{D}} \langle f\rangle_{\hat{J},w^{-1}} \langle g, h_J^w\rangle_w \langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w \right| \qquad (4.10)\\
&\qquad \leq \|g\|_{L^2(w)} \left(\sum_{J\in\mathcal{D}} \langle f\rangle_{\hat{J},w^{-1}}^2 \langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w^2 \right)^{1/2}, \qquad (4.11)
\end{aligned}$$

here (4.8) and the fact that $S_b^*(wh_J^w)$ is supported by $\hat{J}$ are used for equality (4.9), and (4.10) uses (2.7) and (4.8). If we show that

$$\sum_{J\in\mathcal{D}} \langle f\rangle_{\hat{J},w^{-1}}^2 \langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w^2 \leq c\|b\|_{BMO^d}^2 [w]_{A_2^d}^2 \|f\|_{L^2(w^{-1})}^2 , \qquad (4.12)$$

then we have

$$\left| \sum_{I,J:\hat{J}\subsetneq I} \langle f, h_I^{w^{-1}}\rangle_{w^{-1}} \langle g, h_J^w\rangle_w \langle S_{b,w^{-1}} h_I^{w^{-1}}, h_J^w\rangle_w \right| \leq C\|b\|_{BMO^d}[w]_{A_2^d}\|f\|_{L(w^{-1})}^2 \|g\|_{L^2(w)} .$$

To prove the inequality (4.12), we apply Theorem 3.1.5. The embedding condition becomes

$$\sum_{J\in\mathcal{D}:J\subsetneq I} \langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w^2 \leq c\|b\|_{BMO^d}^2 [w]_{A_2^d}^2 w^{-1}(I)$$

after shifting the indices. Since $\langle h_{L_-}, h_J^w\rangle_w = 0$ unless $L \subseteq \hat{J}$, and we will sum over J such that $I \supseteq \hat{J}$, we can write

$$\begin{aligned}
\langle S_{b,w^{-1}}\chi_{\hat{J}}, h_J^w\rangle_w &= \sum_{L\in\mathcal{D}} \Delta_L b\,\langle w^{-1}\chi_{\hat{J}}, h_L\rangle \langle h_{L_-}, h_J^w\rangle_w = \sum_{L\in\mathcal{D}(\hat{J})} \Delta_L b\,\langle w^{-1}\chi_{\hat{J}}, h_L\rangle \langle h_{L_-}, h_J^w\rangle_w \\
&= \sum_{L\in\mathcal{D}(I)} \Delta_L b\,\langle w^{-1}\chi_I, h_L\rangle \langle h_{L_-}, h_J^w\rangle_w = \langle S_{b,w^{-1}}^I \chi_I, h_J^w\rangle_w .
\end{aligned}$$

Thus,

$$\sum_{J\in\mathcal{D}:J\subsetneq I}\langle S_{b,w^{-1}}\chi_{\hat{J}},h_J^w\rangle_w^2=\sum_{J\in\mathcal{D}:J\subsetneq I}\langle S^I_{b,w^{-1}}\chi_I,h_J^w\rangle_w^2\le\|S^I_{b,w^{-1}}\chi_I\|^2_{L^2(w)}\,,$$

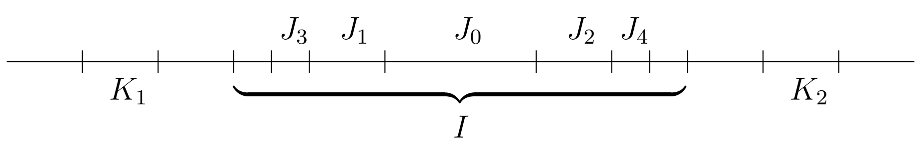
last inequality due to (2.2). By Lemma 3.1.4, the embedding condition holds. Hence we are done for the sum $\hat{J}\subsetneq I$. The part $\sum_{I\subsetneq J}$ is similar to $\sum_{\hat{J}\subsetneq I}$. One uses that $S_b(w^{-1}h_I^{w^{-1}})$ is supported by I and Theorem 3.1.5.

4.2.4 Proof of Theorem 2.3.2

To break $[b,S]$ into three parts, as in (2.20), we assumed that $b\in BMO^d$ is compactly supported. However, we need to replace such a b with a general BMO^d function. In order to pass from a compactly supported b to general $b\in BMO^d$, we need the following lemma which is suggested in [Ga] .

Lemma 4.2.1. *Suppose $\phi\in BMO$. Let $\widetilde{I}$ be the interval concentric with I having length $|\widetilde{I}|=3|I|$. Then there is $\psi\in BMO$ such that $\psi=\phi$ on I, $\psi=0$ on $\mathbb{R}\setminus\widetilde{I}$ and $\|\psi\|_{BMO}\le c\|\phi\|_{BMO}$.*

Proof. Without loss of generality, we assume $\langle\phi\rangle_I=0$. Write $I=\bigcup_{n=0}^\infty J_n$ where $dist(J_n,\partial I)=|J_n|$, as in following figure.



Then J_0 is the middle third of I . For $n>0$, let K_n be the reflection of J_n across the nearest endpoint of I and set

$$\psi(x)=\begin{cases}\phi(x)\,, & \text{if } x\in I\\ \langle\phi\rangle_{J_n}\,, & \text{if } x\in K_n\\ 0\,, & \text{otherwise}\,.\end{cases}$$

This construction of ψ satisfies Lemma 4.2.1.

□

By Theorem 4.1.1, Corollary 1.0.2, Theorem 2.4.1, Lemma 4.2.1 and using the fact $\|\pi_b\| = \|\pi_b^*\|$, we can prove Theorem 2.3.2.

Proof of Theorem 2.3.2. For any compactly supported $b \in BMO$,

$$\begin{aligned}
\|[b,H]\|_{L^2(w)\to L^2(w)} &\leq C \sup_{\alpha,r} \|[b, S^{\alpha,r}]\|_{L^2(w)\to L^2(w)} \\
&\leq C \sup_{\alpha,r} \big(\|[\pi_b, S^{\alpha,r}]\|_{L^2(w)\to L^2(w)} + \|[\pi_b^*, S^{\alpha,r}]\|_{L^2(w)\to L^2(w)} \\
&\qquad + \|[\lambda_b, S^{\alpha,r}]\|_{L^2(w)\to L^2(w)} \big) \\
&\leq C\big(4\|\pi_b\|_{L^2(w)\to L^2(w)} \sup_{\alpha,r} \|S^{\alpha,r}\|_{L^2(w)\to L^2(w)} + C[w]_{A_2}\|b\|_{BMO}\big) \\
&\leq C[w]_{A_2}^2 \|b\|_{BMO} .
\end{aligned}$$

For fixed b, we consider the sequence of intervals $I_k = [-k, k]$ and the sequence of BMO functions b_k which are constructed as in Lemma 4.2.1. Then, there is a constant c, which does not depend on k, such that $\|b_k\|_{BMO} \leq c\|b\|_{BMO}$. Furthermore, there is a uniform constant C such that

$$\|[b_k, H]\|_{L^2(w)\to L^2(w)} \leq C[w]_{A_2}^2 \|b\|_{BMO} . \tag{4.13}$$

Therefore, for some subsequence of integers k_j and $f \in L^2(w)$, $[b_{k_j}, H](f)$ converges to $[b,H](f)$ almost everywhere. Letting $j \to \infty$ and using Fatou's lemma, we deduce that (4.13) holds for all $b \in BMO$. □

4.3 Linear bound for $\pi_b^* S$

It might be useful to know what is the adjoint operator of S. Let us define

$$\operatorname{sgn}(I) := \pm 1 , \text{ if } I = \hat{I}_{\mp} .$$

Then, for any function $f, g \in L^2$,

$$\begin{aligned}
\langle Sf, g\rangle &= \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}\langle f, h_I\rangle\langle g, h_J\rangle\langle h_{I_-} - h_{I_+}, h_J\rangle \\
&= \sum_{I\in\mathcal{D}}\langle f, h_I\rangle\langle g, h_{I_-}\rangle - \sum_{I\in\mathcal{D}}\langle f, h_I\rangle\langle g, h_{I_+}\rangle \\
&= \sum_{I\in\mathcal{D}}\langle f, h_I\rangle\big(\operatorname{sgn}(I_-)\langle g, h_{I_-}\rangle + \operatorname{sgn}(I_+)\langle g, h_{I_+}\rangle\big) \\
&= \sum_{I\in\mathcal{D}}\langle f, h_{\hat{I}}\rangle \operatorname{sgn}(I)\langle g, h_I\rangle \\
&= \left\langle f, \sum_{I\in\mathcal{D}}\operatorname{sgn}(I)\langle g, h_I\rangle h_{\hat{I}}\right\rangle = \langle f, S^* g\rangle\,.
\end{aligned}$$

Now, we see the adjoint operator of dyadic shift operator S is

$$S^* f(x) = \sum_{I\in\mathcal{D}}\operatorname{sgn}(I)\langle f, h_I\rangle h_{\hat{I}}(x)\,.$$

In particular we see that $S^* h_J = \operatorname{sgn}(J) h_{\hat{J}}$. The following lemma provides the bound we are looking for the term $\pi_b^* S$.

Lemma 4.3.1. *Let $w \in A_2^d$ and $b \in BMO^d$. Then, there exists C so that*

$$\|\pi_b^* S\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2^d}\|b\|_{BMO^d}\,.$$

Proof. In order to prove Lemma 4.3.1 it is enough to show that for any positive square integrable function f, g

$$\langle \pi_b^* S(f w^{-1/2}), g w^{1/2}\rangle \le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2}\|g\|_{L^2}\,. \tag{4.14}$$

Using the system of functions $\{H_I^w\}_{I\in\mathcal{D}}$ defined in (2.8), we can rewrite the left hand side of (4.14)

$$\begin{aligned}\langle \pi_b^* S(fw^{-1/2}), gw^{1/2}\rangle &= \langle S(fw^{-1/2}), \pi_b(gw^{1/2})\rangle \\ &= \sum_{I\in\mathcal{D}} \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle S(fw^{-1/2}), h_I\rangle \\ &= \sum_{I\in\mathcal{D}} \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \operatorname{sgn}(I) \langle fw^{-1/2}, h_{\hat{I}}\rangle \\ &= \sum_{I\in\mathcal{D}} \operatorname{sgn}(I) \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle fw^{-1/2}, H_{\hat{I}}^{w^{-1}}\rangle \frac{1}{\sqrt{|\hat{I}|}} \\ &\quad + \sum_{I\in\mathcal{D}} \operatorname{sgn}(I) \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle fw^{-1/2}, A_{\hat{I}}^{w^{-1}} \chi_{\hat{I}}\rangle \frac{1}{\sqrt{|\hat{I}|}} . \end{aligned} \tag{4.15}$$

Our claim is that both sums in (4.15) are bounded by $[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2} \|g\|_{L^2}$, i.e.

$$\left| \sum_{I\in\mathcal{D}} \operatorname{sgn}(I) \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle fw^{-1/2}, H_{\hat{I}}^{w^{-1}}\rangle \frac{1}{\sqrt{|\hat{I}|}} \right| \le C[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2} \|g\|_{L^2} \tag{4.16}$$

and

$$\left| \sum_{I\in\mathcal{D}} \operatorname{sgn}(I) \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle fw^{-1/2}, A_{\hat{I}}^{w^{-1}} \chi_{\hat{I}}\rangle \frac{1}{\sqrt{|\hat{I}|}} \right| \le C[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2} \|g\|_{L^2} . \tag{4.17}$$

First let us verify the bound for (4.16). Using Cauchy-Schwarz inequality,

$$\begin{aligned} &\left| \sum_{I\in\mathcal{D}} \operatorname{sgn}(I) \langle gw^{1/2}\rangle_I \langle b, h_I\rangle \langle fw^{-1/2}, H_{\hat{I}}^{w^{-1}}\rangle \frac{1}{\sqrt{|\hat{I}|}} \right| \\ &\le \left(\sum_{I\in\mathcal{D}} \langle gw^{1/2}\rangle_I^2 \langle b, h_I\rangle^2 \langle w^{-1}\rangle_{\hat{I}} \right)^{1/2} \left(\sum_{I\in\mathcal{D}} \frac{1}{|\hat{I}|\langle w^{-1}\rangle_{\hat{I}}} \langle f, w^{-1/2} H_{\hat{I}}^{w^{-1}}\rangle^2 \right)^{1/2} \\ &\le \|f\|_{L^2} \left(\sum_{I\in\mathcal{D}} \langle gw^{1/2}\rangle_I^2 \langle b, h_I\rangle^2 \langle w^{-1}\rangle_{\hat{I}} \right)^{1/2} . \end{aligned} \tag{4.18}$$

Thus, for (4.16), it is enough to show that

$$\sum_{I\in\mathcal{D}} \langle gw^{1/2}\rangle_I^2 \langle b, h_I\rangle^2 \langle w^{-1}\rangle_{\hat{I}} \le C[w]_{A_2^d}^2 \|b\|_{BMO^d}^2 \|g\|_{L^2}^2 . \tag{4.19}$$

It is clear that $2\langle w\rangle_{\hat{I}} \geq \langle w\rangle_I$, thus

$$\sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_I^2\langle b,h_I\rangle^2\langle w^{-1}\rangle_{\hat{I}} = \sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_I^2\langle b,h_I\rangle^2\langle w^{-1}\rangle_{\hat{I}}\langle w\rangle_I\langle w\rangle_I^{-1}$$
$$\leq 2[w]_{A_2^d}\sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_I^2\langle b,h_I\rangle^2\langle w\rangle_I^{-1}\,.$$

If we show for all $J\in\mathcal{D}$,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w\rangle_I^{-1}\langle w\rangle_I^2 = \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w\rangle_I \leq [w]_{A_2^d}\|b\|^2_{BMO^d}\langle w\rangle_J\,, \quad (4.20)$$

then by Weighted Carleson Embedding Theorem 3.1.5 with w instead of w^{-1} , we will have (4.19) . Since $b\in BMO^d$, $\{\langle b,h_I\rangle^2\}_{I\in\mathcal{D}}$ is a Carleson sequence with constant $\|b\|^2_{BMO^d}$ that is

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2 \leq \|b\|^2_{BMO^d}\,.$$

Applying Lemma 3.1.8 with $\alpha_I = \langle b,h_I\rangle$ we have inequality (4.20). We now concentrate on the estimate (4.17), we can estimate the left hand side of (4.17) as follows.

$$\left|\sum_{I\in\mathcal{D}}\operatorname{sgn}(I)\langle gw^{1/2}\rangle_I\langle b,h_I\rangle\langle fw^{-1/2},A_{\hat{I}}^{w^{-1}}\chi_{\hat{I}}\rangle\frac{1}{\sqrt{|\hat{I}|}}\right|$$
$$= \left|\sum_{I\in\mathcal{D}}\operatorname{sgn}(I)\langle gw^{1/2}\rangle_I\langle b,h_I\rangle\langle fw^{-1/2}\rangle_{\hat{I}}A_{\hat{I}}^{w^{-1}}\sqrt{|\hat{I}|}\right|$$
$$\leq \sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_I\,|\langle b,h_I\rangle|\,\langle fw^{-1/2}\rangle_{\hat{I}}\,|A_{\hat{I}}^{w^{-1}}|\sqrt{|\hat{I}|}$$
$$\leq 2\sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_{\hat{I}}\,|\langle b,h_I\rangle|\,\langle fw^{-1/2}\rangle_{\hat{I}}\,|A_{\hat{I}}^{w^{-1}}|\sqrt{|\hat{I}|}$$
$$= 2\sum_{I\in\mathcal{D}}\langle gw^{1/2}\rangle_I\left(|\langle b,h_{I_-}\rangle|+|\langle b,h_{I_+}\rangle|\right)\langle fw^{-1/2}\rangle_I\,|A_I^{w^{-1}}|\sqrt{|I|}\,.$$

By Bilinear Embedding Theorem (Corollary 3.1.3 with $v = w^{-1}$), inequality (4.17) holds provided the following three inequalities hold,

$$\forall\, J \in \mathcal{D}, \quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \big(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|\big)|A_I^{w^{-1}}|\sqrt{|I|}\langle w^{-1}\rangle_I\langle w\rangle_I \leq C\|b\|_{BMO^d}\,[w^{-1}]_{A_2^d}\,, \tag{4.21}$$

$$\forall\, J \in \mathcal{D}, \quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \big(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|\big)|A_I^{w^{-1}}|\sqrt{|I|}\langle w^{-1}\rangle_I \leq C\|b\|_{BMO^d}\,[w^{-1}]_{A_2^d}\langle w^{-1}\rangle_J\,, \tag{4.22}$$

$$\forall\, J \in \mathcal{D}, \quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \big(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|\big)|A_I^{w^{-1}}|\sqrt{|I|}\langle w\rangle_I \leq C\|b\|_{BMO^d}\,[w^{-1}]_{A_2^d}\langle w\rangle_J\,. \tag{4.23}$$

For (4.21), by Cauchy-Schwarz inequality

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} &\big(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|\big)|A_I^{w^{-1}}|\sqrt{|I|}\langle w^{-1}\rangle_I\langle w\rangle_I \\ &\leq \left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} \big(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|\big)^2\langle w^{-1}\rangle_I\langle w\rangle_I\right)^{1/2} \\ &\qquad \times \left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} (A_I^{w^{-1}})^2|I|\langle w^{-1}\rangle_I\langle w\rangle_I\right)^{1/2}.\end{aligned}$$

Since

$$\sum_{I\in\mathcal{D}}(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|)^2 \leq 3\sum_{I\in\mathcal{D}}\langle b, h_I\rangle^2\,, \tag{4.24}$$

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(|\langle b, h_{I_-}\rangle| + |\langle b, h_{I_+}\rangle|)^2\langle w^{-1}\rangle_I\langle w\rangle_I &\leq C[w^{-1}]_{A_2^d}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b, h_I\rangle^2 \\ &\leq C[w^{-1}]_{A_2^d}\|b\|^2_{BMO^d}\,,\end{aligned}$$

and by Lemma 3.1.9,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w^{-1}\rangle_I\langle w\rangle_I \leq C\,[w^{-1}]_{A_2^d}\,.$$

Thus embedding condition (4.21) holds. For (4.22), by Cauchy-Schwarz inequality and (4.24) we have

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(|\langle b,h_{I_-}\rangle|+|\langle b,h_{I_+}\rangle|\right)|A_I^{w^{-1}}|\sqrt{|I|}\langle w^{-1}\rangle_I$$
$$\leq C\Big(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w^{-1}\rangle_I\Big)^{1/2}\Big(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w^{-1}\rangle_I\Big)^{1/2}.$$

By Theorem 3.1.6,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w^{-1}\rangle_I=\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{\langle w^{-1}\rangle_{I_+}-\langle w^{-1}\rangle_{I_-}}{2\langle w^{-1}\rangle_I}\right)^2|I|\langle w^{-1}\rangle_I$$
$$\leq C[w^{-1}]_{A_2^d}\langle w^{-1}\rangle_J\,.$$

Similarly with (4.20), we have

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w^{-1}\rangle_I\leq[w^{-1}]_{A_2}\|b\|^2_{BMO^d}\langle w^{-1}\rangle_J\,.$$

To finish, we must estimate (4.23). In a similar way with (4.22), we need to estimate

$$\Big(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w\rangle_I\Big)^{1/2}\Big(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w\rangle_I\Big)^{1/2}.$$

By Lemma 3.1.7, applied to w^{-1} instead of w, we have

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w\rangle_I\leq[w^{-1}]_{A_2^d}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}(A_I^{w^{-1}})^2|I|\langle w^{-1}\rangle_I^{-1}$$
$$=[w^{-1}]_{A_2^d}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{\langle w^{-1}\rangle_{I_+}-\langle w^{-1}\rangle_{I_-}}{\langle w^{-1}\rangle_I^3}\right)^2|I|$$
$$\leq C[w^{-1}]_{A_2^d}\langle w\rangle_J\,.$$

This completes the proof of Lemma 4.3.1. □

Due to the almost self adjoint property of the Hilbert transform, a certain bound for π_b^*H immediately returns the same bound for $H\pi_b$. However we have to prove the boundedness of $S\pi_b$ independently because S is not self adjoint.

4.4 Linear bound for $S\pi_b$

Lemma 4.4.1. *Let $w \in A_2^d$ and $b \in BMO^d$. Then, there exists C so that*

$$\|S\pi_b\|_{L^2(w)\to L^2(w)} \le C[w]_{A_2^d}\|b\|_{BMO^d}\,.$$

Proof. We are going to prove Lemma 4.4.1 by showing

$$\langle S\pi_b(w^{-1}f), g\rangle_w \le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,, \tag{4.25}$$

for any positive function $f, g \in L^2$. Since

$$\langle S\pi_b(f), h_I\rangle = \mathrm{sgn}(I)\langle \pi_b(f), h_{\hat{I}}\rangle = \mathrm{sgn}(I)\langle f\rangle_{\hat{I}}\langle b, h_{\hat{I}}\rangle\,,$$

we have

$$S\pi_b(f) = \sum_{I\in\mathcal{D}} \mathrm{sgn}(I)\langle f\rangle_{\hat{I}}\langle b, h_{\hat{I}}\rangle h_I\,.$$

By expanding g in the disbalanced Haar system for $L^2(w)$,

$$\begin{aligned}\langle S\pi_b(w^{-1}f), g\rangle_w &= \sum_{I\in\mathcal{D}} \langle w^{-1}f\rangle_{\hat{I}}\langle b, h_{\hat{I}}\rangle \mathrm{sgn}(I)\langle h_I, g\rangle_w \\ &= \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}} \mathrm{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_J^w\rangle_w\langle h_I, h_J^w\rangle_w\,.\end{aligned}$$

Since $\langle h_I, h_J^w\rangle_w$ could be non zero only if $J \supseteq I$, we can split above sum into three parts,

$$\sum_{I\in\mathcal{D}} \mathrm{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_I^w\rangle_w\langle h_I, h_I^w\rangle_w, \tag{4.26}$$

$$\sum_{I\in\mathcal{D}} \mathrm{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_{\hat{I}}^w\rangle_w\langle h_I, h_{\hat{I}}^w\rangle_w, \tag{4.27}$$

and

$$\sum_{I\in\mathcal{D}}\sum_{J:J\supsetneq\hat{I}} \mathrm{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_J^w\rangle_w\langle h_I, h_J^w\rangle_w\,. \tag{4.28}$$

We claim that all sums, (4.26), (4.27), and (4.28), can be bounded with a bound that depends on $[w]_{A_2^d}\|b\|_{BMO^d}$ at most linearly. Since $|\langle h_I, h_I^w\rangle_w| \le \langle w\rangle_I^{1/2}$, we can estimate (4.26)

$$\begin{aligned}
&\left|\sum_{I\in\mathcal{D}} \operatorname{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_I^w\rangle_w\langle h_I, h_I^w\rangle_w\right| \\
&\qquad \le \left(\sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_{\hat{I}}^2\langle f\rangle_{\hat{I},w^{-1}}^2\langle b, h_{\hat{I}}\rangle^2\langle w\rangle_I\right)^{1/2}\left(\sum_{I\in\mathcal{D}}\langle g, h_I^w\rangle^2\right)^{1/2} \\
&\qquad \le C\|g\|_{L^2(w)}[w]_{A_2^d}^{1/2}\left(\sum_{I\in\mathcal{D}}\langle f\rangle_{I,w^{-1}}^2\langle b, h_I\rangle^2\langle w^{-1}\rangle_I\right)^{1/2}.
\end{aligned}$$

By Weighted Carleson Embedding Theorem 3.1.5,

$$\sum_{I\in\mathcal{D}}\langle f\rangle_{I,w^{-1}}^2\langle b, h_I\rangle^2\langle w^{-1}\rangle_I \le C[w]_{A_2^d}\|b\|_{BMO^d}^2\|f\|_{L^2(w^{-1})}^2$$

holds provided that the following Carleson condition hold,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b, h_I\rangle^2\langle w^{-1}\rangle_I \le [w]_{A_2^d}\|b\|_{BMO^d}^2\langle w^{-1}\rangle_J$$

which we already have in (4.20). Thus, we have

$$\left|\sum_{I\in\mathcal{D}} \operatorname{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_I^w\rangle_w\langle h_I, h_I^w\rangle_w\right| \le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}. \tag{4.29}$$

Similarly to (4.26), we can estimate (4.27) using $|\langle h_I, h_{\hat{I}}^w\rangle_w| \le \langle w\rangle_I^{1/2} \le \sqrt{2}\langle w\rangle_{\hat{I}}^{1/2}$

$$\begin{aligned}\Big|\sum_{I\in\mathcal{D}} \operatorname{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g, h_{\hat{I}}^w\rangle_w\langle h_I, h_{\hat{I}}^w\rangle_w\Big| \\ &\le \sqrt{2}\sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\,|\langle b, h_{\hat{I}}\rangle|\,\langle g, h_{\hat{I}}^w\rangle_w\langle w\rangle_{\hat{I}}^{1/2} \\ &= 2\sqrt{2}\sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_I\langle f\rangle_{I,w^{-1}}\,|\langle b, h_I\rangle|\,\langle g, h_I^w\rangle_w\langle w\rangle_I^{1/2} \\ &\le 2\sqrt{2}\Big(\sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_I^2\langle f\rangle_{I,w^{-1}}^2\langle b, h_I\rangle^2\langle w\rangle_I\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}}\langle g, h_I^w\rangle^2\Big)^{1/2} \\ &\le C\|g\|_{L^2(w)}[w]_{A_2^d}^{1/2}\Big(\sum_{I\in\mathcal{D}}\langle f\rangle_{I,w^{-1}}^2\langle b, h_I\rangle^2\langle w^{-1}\rangle_I\Big)^{1/2} \\ &\le C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}\,.\end{aligned}$$

Since h_J^w is constant on $\hat{I}$, for $J \supsetneq \hat{I}$ and we denote this constant by $h_J^w(\hat{I})$. Then we know by (2.7),

$$\sum_{J:J\supsetneq\hat{I}}\langle g, h_J^w\rangle_w\langle h_I, h_J^w\rangle_w = \sum_{J:J\supsetneq\hat{I}}\langle g, h_J^w\rangle_w h_J^w(\hat{I})\langle h_I, w\rangle = \langle g\rangle_{\hat{I},w}\langle h_I, w\rangle\,.$$

Thus, we can rewrite (4.28)

$$\Big|\sum_{I\in\mathcal{D}}\operatorname{sgn}(I)\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\langle b, h_{\hat{I}}\rangle\langle g\rangle_{\hat{I},w}\langle h_I, w\rangle\Big|$$

$$\le \sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_{\hat{I}}\langle f\rangle_{\hat{I},w^{-1}}\,|\langle b, h_{\hat{I}}\rangle|\,\langle g\rangle_{\hat{I},w}|\langle h_I, w\rangle| \tag{4.30}$$

$$= \sum_{I\in\mathcal{D}}\langle w^{-1}\rangle_I\,|\langle b, h_I\rangle|\,(|\langle h_{I_-}, w\rangle| + |\langle h_{I_+}, w\rangle|)\langle f\rangle_{I,w^{-1}}\langle g\rangle_{I,w}\,. \tag{4.31}$$

We claim the sum (4.31) is bounded by $[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}$. We are going to prove it using Petermichl's Bilinear Embedding Theorem 3.1.2. Thus, we need to show that the following three embedding conditions hold,

$$\forall\ J\in\mathcal{D},\quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I(|\langle h_{I_-},w\rangle|+|\langle h_{I_+},w\rangle|)\frac{1}{\langle w\rangle_I}$$
$$\leq C[w]_{A_2^d}\|b\|_{BMO^d}\,\langle w^{-1}\rangle_J\,, \tag{4.32}$$
$$\forall\ J\in\mathcal{D},\quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I(|\langle h_{I_-},w\rangle|+|\langle h_{I_+},w\rangle|)\frac{1}{\langle w^{-1}\rangle_I}$$
$$\leq C[w]_{A_2^d}\|b\|_{BMO^d}\,\langle w\rangle_J\,, \tag{4.33}$$
$$\forall\ J\in\mathcal{D},\quad \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I(|\langle h_{I_-},w\rangle|+|\langle h_{I_+},w\rangle|)\leq C[w]_{A_2^d}\|b\|_{BMO^d}\,. \tag{4.34}$$

After we split the sum in (4.32):

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I|\langle h_{I_-},w\rangle|\frac{1}{\langle w\rangle_I}+\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I|\langle h_{I_+},w\rangle|\frac{1}{\langle w\rangle_I}\,,$$

we start with Cauchy-Schwarz inequality to estimate the first sum of embedding condition (4.32),

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I|\langle h_{I_-},w\rangle|\frac{1}{\langle w\rangle_I}&=\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\,\langle w^{-1}\rangle_I\frac{\sqrt{|I_-|}|\Delta_{I_-}w|}{\langle w\rangle_I}\\
&\leq\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w^{-1}\rangle_I^2\langle w\rangle_I\right)^{1/2}\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I_-||\Delta_{I_-}w|^2\frac{1}{\langle w\rangle_I^3}\right)^{1/2}\\
&\leq C[w]_{A_2}^{1/2}\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w^{-1}\rangle\right)^{1/2}\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I_-||\Delta_{I_-}w|^2\frac{1}{\langle w\rangle_{I_-}^3}\right)^{1/2}\\
&\leq C[w]_{A_2^d}\|b\|_{BMO^d}\langle w^{-1}\rangle_J\,.\end{aligned}\tag{4.35}$$

Inequality (4.35) due to Lemma 3.1.7 and (4.20) . Also, the other sum can be estimated by exactly the same method. Thus we have the embedding condition (4.32). To see the embedding condition (4.33), it is enough to show

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle\,\langle h_{I_-},w\rangle|\leq C[w]_{A_2^d}\|b\|_{BMO^d}\langle w\rangle_J\,,$$

as we did above. We use Cauchy-Schwarz inequality for embedding condition (4.33), then

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle\langle h_{I_-},w\rangle| &= \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|\langle b,h_I\rangle|\sqrt{|I_-|}|\Delta_{I_-}w| \\ &\le \left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\frac{1}{\langle w^{-1}\rangle_I}\right)^{1/2}\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I_-||\Delta_{I_-}w|^2\langle w^{-1}\rangle_I\right)^{1/2} \\ &\le C\|b\|_{BMO^d}\langle w\rangle_J^{1/2}\left(\frac{[w]_{A_2}}{|J|}\sum_{I\in\mathcal{D}(J)}|I_-||\Delta_{I_-}w|^2\frac{1}{\langle w\rangle_{I_-}}\right)^{1/2} \qquad (4.36) \\ &\le C[w]_{A_2^d}\|b\|_{BMO^d}\langle w\rangle_J\,. \qquad (4.37)\end{aligned}$$

Here inequality (4.36) uses Lemma 3.1.8, and inequality (4.37) uses the fact that $\langle w\rangle_I^{-1}\le 2\langle w\rangle_{I_-}^{-1}$ and Theorem 3.1.6 after shifting the indices.

If we show the embedding condition (4.34), then we can immediately finish the estimate for (4.31) with bound $C[w]_{A_2^d}\|b\|_{BMO^d}\|f\|_{L^2(w^{-1})}\|g\|_{L^2(w)}$. Combining this and (4.29) will give us our desired result. □

4.5 Proof for embedding condition (4.34)

The following lemma lies at the heart of the matter for the proof of the embedding condition (4.34).

Lemma 4.5.1. *There is a positive constant C so that for all dyadic interval $J\in\mathcal{D}$*

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I|\langle w\rangle_I^{1/4}\langle w^{-1}\rangle_I^{1/4}\left(\frac{|\Delta_{I_+}w|+|\Delta_{I_-}w|}{\langle w\rangle_I}\right)^2\le C\langle w\rangle_J^{1/4}\langle w^{-1}\rangle_J^{1/4}, \qquad (4.38)$$

whenever w is a weight. Moreover, if $w\in A_2^d$ then for all $J\in\mathcal{D}$

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I|\langle w\rangle_I\langle w^{-1}\rangle_I\left(\frac{|\Delta_{I_+}w|+|\Delta_{I_-}w|}{\langle w\rangle_I}\right)^2\le C\,[w]_{A_2^d}\,.$$

Proof of condition (4.34). By using Cauchy-Schwarz inequality and Lemma 4.5.1, we have:

$$
\begin{aligned}
&\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle\langle w^{-1}\rangle_I(|\langle h_{I_-},w\rangle|+|\langle h_{I_+},w\rangle|)\\
&\qquad=\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle\langle w^{-1}\rangle_I\sqrt{\frac{|I|}{2}}\left(|\Delta_{I_+}w|+|\Delta_{I_-}w|\right)\\
&\qquad\leq\frac{1}{\sqrt{2}}\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\langle b,h_I\rangle^2\langle w^{-1}\rangle_I\langle w\rangle_I\right)^{1/2}\\
&\qquad\qquad\times\left(\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I\langle w\rangle_I^{-1}(|\Delta_{I_+}w|+|\Delta_{I_-}w|)^2\right)^{1/2}\\
&\qquad\leq C[w]_{A_2^d}\|b\|_{BMO^d}\,.
\end{aligned}
$$

□

We turn to the proof of Lemma 4.5.1. In the first place, we need to revisit some properties of the function $B(u,v):=\sqrt[4]{uv}$ on the domain $\mathfrak{D}_0$ which is given by

$$\{(u,v)\in\mathbb{R}^2_+\;:\;uv\geq 1/2\}\,.$$

It is known, we refer to [Be], that $B(u,v)$ satisfies the following differential inequality in $\mathfrak{D}_0$

$$-(du,dv)d^2B(u,v)(du,dv)^t\geq\frac{1}{8}\frac{v^{1/4}}{u^{7/4}}|du|^2\,. \tag{4.39}$$

Furthermore, this implies the following convexity condition. For all (u,v), $(u_\pm,v_\pm)\in\mathfrak{D}_0$,

$$B(u,v)-\frac{B(u_+,v_+)+B(u_-,v_-)}{2}\geq C_1\frac{v^{1/4}}{u^{7/4}}(u_+-u_-)^2\,, \tag{4.40}$$

where $u=(u_++u_-)/2$ and $v=(v_++v_-)/2$.

Lemma 4.5.2. *Let us define*

$$A(u,v,\Delta u):=aB(u,v)+B(u+\Delta u,v)+B(u-\Delta u,v)\,,$$

on the domain $\mathfrak{D}_1$ *with some positive constant* $a > 0$. *Here* $(u, v, \Delta u) \in \mathfrak{D}_1$ *means all pairs* (u, v), $(u + \Delta u, v)$, $(u - \Delta u, v) \in \mathfrak{D}_0$. *Then* A *has the size property,*

$$\text{if} \quad (u, v, \Delta u) \in \mathfrak{D}_1, \quad \text{then} \quad 0 \le A(u, v, \Delta u) \le (a + 2)\sqrt[4]{uv}\,, \tag{4.41}$$

and the convexity property,

$$A(u, v, \Delta u) - \frac{1}{2}\big[A(u_+, v_+, \Delta u_1) + A(u_-, v_-, \Delta u_2)\big] \ge C_2 \frac{v^{1/4}}{u^{7/4}}(\Delta u_1^2 + \Delta u_2^2)\,, \tag{4.42}$$

where $u = (u_+ + u_-)/2\,, v = (v_+ + v_-)/2,$ *and* $\Delta u = (u_+ - u_+)/2$.

Proof. The property (4.42) is directly from the definition of function $B(u, v)$. At the end, Δu will play the role of $\Delta_I w$, Δu_1 is $\Delta_{I_+} w$, and Δu_2 is $\Delta_{I_-} w$. We can rewrite the left hand side of the inequality (4.42) as follows

$$\begin{aligned}
A(u,&v, \Delta u) - \frac{1}{2}\big[A(u_+, v_+, \Delta u_1) + A(u_-, v_-, \Delta u_2)\big] \\
&= aB(u, v) + B(u + \Delta u, v) + B(u - \Delta u, v) \\
&\quad - \frac{1}{2}\big[aB(u_+, v_+) + B(u_+ + \Delta u_1, v_+) + B(u_+ - \Delta u_1, v_+) \\
&\qquad + aB(u_-, v_-) + B(u_- + \Delta u_1, \Delta u_-) + B(u_- - \Delta u_1, v_-)\big] \\
&= aB(u, v) - \frac{a}{2}(B(u_+, v_+) + B(u_-, v_-)) + B(u_+, v) + B(u_-, v) \\
&\quad - \frac{1}{2}\Big[B(u + \Delta u + \Delta u_1, v + \Delta v) + B(u + \Delta u - \Delta u_1, v + \Delta v) \\
&\qquad + B(u - \Delta u + \Delta u_2, v - \Delta v) + B(u - \Delta u - \Delta u_2, v - \Delta v)\Big]\,.
\end{aligned} \tag{4.43}$$

Using Taylor's theorem:

$$\begin{aligned}
B(u + u_0, v + v_0) =& B(u, v) + \nabla B(u, v)(u_0, v_0)^t \\
&+ \int_0^1 (1 - s)(u_0, v_0) d^2 B(u + su_0, v + sv_0)(u_0, v_0)^t ds\,,
\end{aligned}$$

and the differential convexity condition (4.38) of $B(u, v)$, we are going to estimate the lower bounds for (4.42).

$$
\begin{aligned}
&-\frac{1}{2}B(u+\Delta u+\Delta u_1, v+\Delta v)\\
&=-\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u+\Delta u_1,\Delta v)^t\Big)\\
&\quad-\frac{1}{2}\int_0^1(1-s)(\Delta u+\Delta u_1,\Delta v)d^2B(u+s(\Delta u+\Delta u_1),v+s\Delta v)(\Delta u+\Delta u_1,\Delta v)^t ds\\
&\geq-\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u+\Delta u_1,\Delta v)^t\Big)\\
&\qquad\qquad+\frac{1}{16}\int_0^1(1-s)\frac{(v+s\Delta v)^{1/4}}{(u+s(\Delta u+\Delta u_1))^{7/4}}(\Delta u+\Delta u_1)^2 ds\\
&\geq-\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u+\Delta u_1,\Delta v)^t\Big)\\
&\qquad\qquad+\frac{(\Delta u+\Delta u_1)^2}{16(4u)^{7/4}}\int_0^1(1-s)(v+s\Delta v)^{1/4}ds \qquad (4.44)\\
&\geq-\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u+\Delta u_1,\Delta v)^t\Big)\\
&\qquad\qquad+\frac{(\Delta u+\Delta u_1)^2 v^{1/4}}{16(4u)^{7/4}}\int_0^1(1-s)(1+s\frac{\Delta v}{v})^{1/4}ds\\
&\geq-\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u+\Delta u_1,\Delta v)^t\Big)+\frac{1}{144\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(\Delta u+\Delta u_1)^2\,. \qquad (4.45)
\end{aligned}
$$

Inequality (4.44) is due to the following inequalities

$$|\Delta u|=\frac{|u_+-u_-|}{2}\leq\frac{|u_++u_-|}{2}=u \text{ and } |\Delta u_1|=\frac{|u_{++}-u_{+-}|}{2}\leq u_+\leq 2u\,.$$

Since $(1-s)^{1/4}\leq(1-|\beta|s)^{1/4}\leq(1+\beta s)^{1/4}$ for any $|\beta|<1$, it is clear that

$$\int_0^1(1-s)(1+\beta s)^{1/4}ds\geq\int_0^1(1-s)^{5/4}ds=\frac{4}{9}\,,$$

and this allows the inequality (4.45). With the same arguments, we also estimate the following lower bounds:

$$\begin{aligned}
&-\frac{1}{2}\Big[B(u+\Delta u-\Delta u_1, v+\Delta v)+B(u-\Delta u+\Delta u_2, v-\Delta v)\\
&\qquad\qquad +B(u-\Delta u-\Delta u_2, v-\Delta v)\Big]\\
&\quad\geq -\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(\Delta u-\Delta u_1,\Delta v)^t\Big)+\frac{1}{144\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(\Delta u-\Delta u_1)^2\\
&\qquad -\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(-\Delta u+\Delta u_2,-\Delta v)^t\Big)+\frac{1}{144\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(-\Delta u+\Delta u_2)^2\\
&\qquad -\frac{1}{2}\Big(B(u,v)+\nabla B(u,v)(-\Delta u-\Delta u_2,-\Delta v)^t\Big)+\frac{1}{144\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(\Delta u+\Delta u_2)^2\,.
\end{aligned} \tag{4.46}$$

We can have the following inequality by combining (4.45), (4.46) and (4.43),

$$\begin{aligned}
A(u,v,\Delta u)&-\frac{1}{2}\big[A(u_+,v_+,\Delta u_1)+A(u_-,v_-,\Delta u_2)\big]\\
&\geq\ (a-2)B(u,v)-\frac{a}{2}(B(u_+,v_+)+B(u_-,v_-))+B(u_+,v)+B(u_-,v)\\
&\qquad\quad +\frac{1}{72\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(2\Delta u^2+\Delta u_1^2+\Delta u_2^2)\\
&\geq\ \frac{1}{72\cdot 4^{3/4}}\frac{v^{1/4}}{u^{7/4}}(\Delta u_1^2+\Delta u_2^2)\,.
\end{aligned} \tag{4.47}$$

To see the inequality (4.47), using convexity condition (4.39) of $B(u,v)=\sqrt[4]{uv}$ and inequality: $(1-s)u\leq u-s\Delta u\leq u+s\Delta u\,,$

$$\begin{aligned}
(a-2)B(u,v)&-\frac{a}{2}\Big(B(u_+,v_+)+B(u_-,v_-)\Big)+B(u_+,v)+B(u_-,v)\\
&=a\bigg(B(u,v)-\frac{1}{2}(B(u_+,v_+)+B(u_-,v_-)\bigg)\\
&\qquad-\bigg(\frac{3}{16}\Delta u^2\int_0^1(1-s)v^{1/4}(u+s\Delta u)^{-7/4}ds\\
&\qquad\qquad+\frac{3}{16}\Delta u^2\int_0^1(1-s)v^{1/4}(u-s\Delta u)^{-7/4}ds\bigg)\\
&\geq aC_1\frac{v^{1/4}}{u^{7/4}}\Delta u^2-\frac{6}{16}\Delta u^2\frac{v^{1/4}}{u^{7/4}}\int_0^1(1-s)^{-3/4}ds\\
&=aC_1\frac{v^{1/4}}{u^{7/4}}\Delta u^2-\frac{3}{2}\Delta u^2\frac{v^{1/4}}{u^{7/4}}=\bigg(aC_1-\frac{3}{2}\bigg)\frac{v^{1/4}}{u^{7/4}}\Delta u^2\,.
\end{aligned} \tag{4.48}$$

Choosing a constant a sufficiently large so that $aC_1 > 3/2$, quantity in (4.48) remains positive. This observation and discarding nonnegative terms yield inequality (4.47). Choosing the constant $C_2 = 1/(36 \cdot 4^{3/4})$ in (4.42) completes the proof of the concavity property of $A(u, v, \Delta u)$. □

We now turn to the proof of Lemma 4.5.1.

Proof of Lemma 4.5.1. Let $u_I := \langle w \rangle_I$, $v_I := \langle w^{-1} \rangle_I$, $u_\pm = u_{I_\pm}$, $v_\pm = v_{I_\pm}$, $\Delta u_I = \Delta_I w$, $\Delta u_1 = \Delta u_{I_+}$, and $\Delta u_2 = \Delta u_{I_-}$. Then by Hölder's inequality $(u, v, \Delta u)$, $(u_+, v_+, \Delta u_1)$, and $(u_-, v_-, \Delta u_2)$ belong to $\mathfrak{D}_1$. Fix $J \in \mathcal{D}$, by properties (4.41) and (4.42)

$$\begin{aligned} &(a+2)|J|\sqrt[4]{\langle w \rangle_J \langle w^{-1} \rangle_J} \geq |J| A(u_J, v_J, \Delta u_J) \\ &\geq \frac{1}{2}\Big(|J_+|A(u_+, v_+, \Delta u_1) + |J_-|A(u_-, v_-, \Delta u_2)\Big) + |J| C \frac{\langle w^{-1} \rangle_J}{\langle w \rangle_J^{7/4}} (|\Delta_{J_+} u|^2 + |\Delta_{J_-} u|^2). \end{aligned}$$

Since $A(u, v, \Delta u) \geq 0$, iterating the above process will yield

$$|J|\sqrt[4]{\langle w \rangle_J \langle w^{-1} \rangle_J} \geq C \sum_{I \in \mathcal{D}(J)} |I| \langle w^{-1} \rangle_I^{1/4} \langle w \rangle_I^{-7/4} (|\Delta_{I_+} w|^2 + |\Delta_{I_-} w|^2). \tag{4.49}$$

Also, one can easily have

$$|J|\sqrt[4]{\langle w \rangle_J \langle w^{-1} \rangle_J} \geq C \sum_{I \in \mathcal{D}(J)} |I| \langle w^{-1} \rangle_I^{1/4} \langle w \rangle_I^{-7/4} \, \Delta_{I_+} w^2, \tag{4.50}$$

and

$$|J|\sqrt[4]{\langle w \rangle_J \langle w^{-1} \rangle_J} \geq C \sum_{I \in \mathcal{D}(J)} |I| \langle w^{-1} \rangle_I^{1/4} \langle w \rangle_I^{-7/4} \, \Delta_{I_-} w^2. \tag{4.51}$$

Then,

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}(|\Delta_{I_+}w|+|\Delta_{I_-}w|)^2$$

$$= \frac{1}{|J|}\Bigg(\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}(|\Delta_{I_+}w|^2 + |\Delta_{I_-}w|^2) + 2\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}(|\Delta_{I_+}w|\,|\Delta_{I_-}w|)\Bigg)$$

$$\leq \frac{1}{|J|}\Bigg(\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}(|\Delta_{I_+}w|^2+|\Delta_{I_-}w|^2) + 2\Big(\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}\,\Delta_{I_+}w^2\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}(J)}|I|\langle w^{-1}\rangle_I^{1/4}\langle w\rangle_I^{-7/4}\,\Delta_{I_-}w^2\Big)^{1/2}\Bigg)$$

$$\leq \frac{3}{C}\sqrt[4]{\langle w\rangle_I\langle w^{-1}\rangle_I}\,.$$

□

Remark 4.5.3. The linear bounds in $L^2(w)$ for $[\lambda_b, S]$, $\pi_b^* S$ and $S\pi_b$ can be deduced form the results in [HyLReVa] once it is observed that all three operators are Haar shift operators of the second class, see Section 5.1.

Chapter 5

Commutators of Riesz transforms and the Beurling-Ahlfors operator and sharp bounds

In this chapter, we are going to introduce two more general classes of dyadic shift operators, the convex hull of which now includes one dimensional Calderón-Zygmund convolution operators with sufficiently smooth kernel (See [Va]). By showing that the commutator of the first class of dyadic shift operators with λ_b also belongs to the same class, we will extend our result to more general class of commutators. Among the convolution operators that fit this theory are the Riesz transforms and the Beurling-Ahlfors operator. We also prove in this chapter that the bounds in $L^p(w)$ obtained for the commutator of the Hilbert and Riesz transforms and the Beurling-Ahlfors operator are sharp in terms of their dependence on the A_p-characteristic of the weight.

5.1 Dyadic shift operators

The dyadic shift operator was first introduced in [Pet1] to study the weighted norm estimate for the Hilbert transform. It was also encountered in [PetTV], so Riesz transforms can be obtained as the result of averaging some dyadic shift operator. Recently, in [LPetRe] and [CrMP], a more general class of dyadic shift operators, so called the Haar shift operators were introduced. The Hilbert transform, Riesz transforms, and Beurling-Ahlfors operator are in the convex hull of this class, as they can be written as appropriate averages of Haar shift operators. Let $\mathcal{D}^n$ denote the collection of dyadic cubes in $\mathbb{R}^n$, $\mathcal{D}^n(Q)$ denotes dyadic subcubes of Q , and $|Q|$ denotes the volume of the dyadic cube Q . We start with some definitions.

Definition 5.1.1. A Haar function on a cube $Q \subset \mathbb{R}^n$ is a function H_Q such that

(1) H_Q is supported on Q, and is constant on $\mathcal{D}^n(Q)$.

(2) $\|H_Q\|_\infty \leq |Q|^{-1/2}$.

(3) H_Q has a mean zero.

Examples of such a Haar function are the standard Haar functions $\{h^s_{j,Q}\}$, and the Haar functions $\{h_{j,Q}\}$ introduced in Section 2.2, for each $j = 1, ..., 2^n - 1$.

Definition 5.1.2. Given an integer $\tau > 0$, we say an operator of the following form is in *the first class of Haar shift operators of index* τ

$$T_\tau f(x) = \sum_{Q \in \mathcal{D}^n} \sum_{\substack{Q',Q'' \in \mathcal{D}^n(Q) \\ 2^{-\tau n}|Q| \leq |Q'|,|Q''|}} a_{Q',Q''} \langle f, H_{Q'} \rangle H_{Q''}(x) ,$$

where the constant $a_{Q',Q''}$ satisfy the following size condition:

$$|\, a_{Q',Q''}| \leq C \left(\frac{|Q'|}{|Q|} \cdot \frac{|Q''|}{|Q|} \right)^{1/2} . \qquad (5.1)$$

Note that once a choice of Haar functions has been made $\{H_Q\}_{Q\in\mathcal{D}^n}$, then this is an orthogonal family, such that $\|H_Q\|_{L^2}\leq 1$, so one could normalize in L^2 . Note that one can easily see that the dyadic shift operator S belongs to the first class of a Haar shift operator of index $\tau=1$ with

$$a_{I',I''}=\begin{cases}\pm 1 & \text{for } I'=I, \quad I''=I_{\mp}\\ 0 & \text{otherwise}\ .\end{cases}$$

One of the main result in [LPetRe] and [CrMP] is the following

Theorem 5.1.3 ([LPetRe], [CrMP]). *Let T be in the first class of Haar shift operators of index τ . Then for all $w\in A_2^d$, there exists $C(\tau,n)$ which only depends on τ and n such that*

$$\|T\|_{L^2(w)\to L^2(w)}\leq C(\tau,n)[w]_{A_2^d}\,.$$

As a consequence of this Theorem, linear bounds for the Hilbert transform, Riesz transforms, and the Beurling-Ahlfors operator are recovered. There are now two different proofs of Theorem (5.1.3) in [LPetRe] and [CrMP]. The commutator $[\lambda_b,S]$ is also in the first class of Haar shift operators of index $\tau=1$. Recall the observation in Section 4.1

$$[\lambda_b,S](f)=-\sum_{I\in\mathcal{D}}\Delta_I b\langle f,h_I\rangle(h_{I_+}+h_{I_-})\,.$$

Then we can see

$$a_{I',I''}=\begin{cases}-\Delta_I b & \text{for } I'=I, \quad I''=I_{\pm}\\ 0 & \text{otherwise}\ ,\end{cases}$$

moreover $|\,a_{I',I''}|=|\Delta_I b|\leq 2\|b\|_{BMO^d}$, this means the constant $a_{I',I''}$ satisfy the size condition (5.1) with $C=2\sqrt{2}\|b\|_{BMO^d}$. These observations, Theorem 2.4.1, and Theorem 5.1.3 immediately recover the quadratic bound for the commutator of the Hilbert transform which was proved in Chapter 4. We now define the second class of Haar shift operators of index τ .

Definition 5.1.4. Given an integer $\tau > 0$, we say an operator T of the form in Definition 5.1.2 is in *the second class of Haar shift operators of index* τ, if T is bounded on L^2 and the function H_Q satisfy the condition (1) and (2) in Definition 5.1.1.

The second class of Haar shift operators is more general than the first class. One can easily observe that the operators π_b, $S\pi_b$ and $\pi_b^* S$ do not satisfy the condition (c) on Definition 5.1.1, however these operators satisfy the conditions of Definition 5.1.4. Note that the n-variable paraproduct is a sum of $2^n - 1$ operators in the second class of Haar shift operator of index 1, the restricted n-variable dyadic paraproduct

$$\pi_b f = \sum_{Q \in \mathcal{D}^n} \langle f \rangle_Q \langle b, H_Q \rangle H_Q \,.$$

Similarly, with Haar functions defined in Section 2.2, the restricted n-variable dyadic paraproduct will be

$$\pi_b^j f = \sum_{Q \in \mathcal{D}^n} \langle f \rangle_{E_{j,Q}} \langle b, h_{j,Q} \rangle h_{j,Q} \,,$$

for $j = 1, ..., 2^n - 1$. In [HyLReVa], the linear estimate for the maximal truncations of these operators is presented. This also recovers our linear bound estimates for $S\pi_b$ and $\pi_b^* S$. On the other hand, authors in [CrMP] also reproduce the linear estimate for the dyadic paraproduct with a different technique.

Lemma 5.1.5. *Let T_τ be a Haar shift operator of the first class, then $[\lambda_b, T_\tau]$ is an operator of the same class.*

Proof. We are going to use the restricted multi-variable λ_b operator which is

$$\lambda_b f = \sum_{Q \in \mathcal{D}^n} \langle b \rangle_Q \langle f, H_Q \rangle H_Q \,.$$

One can get the n-variable λ_b operator by summing over $2^n - 1$ of restricted λ_b

operator. Observe that,

$$\begin{aligned}[\lambda_b, T_\tau]f &= \lambda_b(T_\tau f) - T_\tau(\lambda_b f) \\ &= \sum_{Q\in\mathcal{D}^n} \sum_{\substack{Q',Q''\in\mathcal{D}^n(Q)\\ 2^{-\tau n}|Q|\le|Q'|,|Q''|}} a_{Q',Q''}\langle b\rangle_{Q''}\langle f, H_{Q'}\rangle H_{Q''} \\ &\qquad - \sum_{Q\in\mathcal{D}^n} \sum_{\substack{Q',Q''\in\mathcal{D}^n(Q)\\ 2^{-\tau n}|Q|\le|Q'|,|Q''|}} a_{Q',Q''}\langle b\rangle_{Q'}\langle f, H_{Q'}\rangle H_{Q''} \\ &= \sum_{Q\in\mathcal{D}^n} \sum_{\substack{Q',Q''\in\mathcal{D}^n(Q)\\ 2^{-\tau n}|Q|\le|Q'|,|Q''|}} a_{Q',Q''}(\langle b\rangle_{Q''} - \langle b\rangle_{Q'})\langle f, H_{Q'}\rangle H_{Q''}\,.\end{aligned}$$

Since

$$\big|\, a_{Q',Q''}(\langle b\rangle_{Q''} - \langle b\rangle_{Q'})\big| \le C(\tau)\|b\|_{BMO}|\, a_{Q',Q''}|\,,$$

$[\lambda_b, T_\tau]$ remains in the same class of T_τ . □

Theorem 5.1.3 and Lemma 5.1.5 allow to extend our result to more general class of commutators including the Riesz transforms and the Beurling-Ahlfors operator as in Theorem 1.0.3.

Remark 5.1.6. By Theorem 5.1.3, Lemma 5.1.5 and the result of [Va], we now know that the $L^2(w)$-norm of the commutators of one dimensional Calderón-Zygmund convolution operators with sufficiently smooth kernel depends quadratically on the A_2-characteristic.

Remark 5.1.7. Most recently, authors in [CPerP] presented a more general result about this subject with completely different but more classical and elegant methods. More precisely, they prove that if any linear operator bounded on $L^2(w)$ for any $w \in A_2$ with

$$\|T\|_{L^2(w)\to L^2(w)} \le \phi([w]_{A_2})$$

for a some increasing function ϕ then there are constants $c(n)$ and $C(n)$ independent of $[w]_{A_2}$ such that

$$\|[b,T]\|_{L^2(w)\to L^2(w)} \le C(n)\phi(c(n)[w]_{A_2})[w]_{A_2}\|b\|_{BMO}\,.$$

5.2 Sharpness of the results

In this section, we start proving that the quadratic estimate in Theorem 2.3.2 is sharp, by showing an example which returns quadratic bound. This example was discovered by C. Peréz [P3] who is kindly allowing us to reproduce it in this dissertation. The same calculations show that the bounds in Theorem 1.0.1 are also sharp for $p \neq 2$ and $1 < p < \infty$. Variations over this example will then show that the bounds in Theorem 1.0.3 are sharp for the Riesz transforms and the Beurling-Ahlfors operator as well.

5.2.1 The Hilbert transform

Consider the weight, for $0 < \delta < 1$:

$$w(x) = |x|^{1-\delta}.$$

It is well known that w is an A_2 weight and

$$[w]_{A^2} \sim \frac{1}{\delta}.$$

We now consider the function $f(x) = x^{-1+\delta}\chi_{(0,1)}(x)$ and BMO function $b(x) = \log|x|$. We claim that

$$\|[b, H]f(x)| \geq \frac{1}{\delta^2} f(x).$$

For $0 < x < 1$, we have

$$[b, H]f(x) = \int_0^1 \frac{\log x - \log y}{x - y} y^{-1+\delta} dy = \int_0^1 \frac{\log(x/y)}{x - y} y^{-1+\delta} dy$$
$$= x^{-1+\delta} \int_0^{1/x} \frac{\log(1/t)}{1 - t} t^{-1+\delta} dt.$$

Now,

$$\int_0^{1/x} \frac{\log(1/t)}{1-t} t^{-1+\delta} dt = \int_0^1 \frac{\log(1/t)}{1-t} t^{-1+\delta} dt + \int_1^{1/x} \frac{\log(1/t)}{1-t} t^{-1+\delta} dt,$$

and since $\frac{\log(1/t)}{1-t}$ is positive for $(0,1)\cup(1,\infty)$ we have for $0<x<1$

$$|[b,H]f(x)| > x^{-1+\delta}\int_0^1 \frac{\log(1/t)}{1-t}t^{-1+\delta}dt\,. \tag{5.2}$$

But since

$$\int_0^1 \frac{\log(1/t)}{1-t}t^{-1+\delta}dt > \int_0^1 \log(1/t)t^{-1+\delta}dt = \int_0^\infty se^{-s\delta}ds = \frac{1}{\delta^2}\,, \tag{5.3}$$

our claim follows and

$$\|[b,H]f\|_{L^2(w)} \geq \frac{1}{\delta^2}\|f\|_{L^2(w)} \sim [w]_{A_2}^2\|f\|_{L^2(w)}\,.$$

A first approximation of what the bounds in $L^p(w)$ is given by an application of the sharp extrapolation theorem for the upper bound, paired with the knowledge of the sharp bound on $L^2(w)$ to obtain a lower bound.

Proposition 5.2.1. *For $1<p<\infty$ there exist constants c and C only depending on p such that*

$$c[w]_{A_p}^{2\min\{1,\frac{1}{p-1}\}}\|b\|_{BMO} \leq \|[b,H]\|_{L^p(w)\to L^p(w)} \leq C[w]_{A_p}^{2\max\{1,\frac{1}{p-1}\}}\|b\|_{BMO}\,, \tag{5.4}$$

for all $b\in BMO$.

Proof. Because the upper bound in (5.4) is the direct consequence of the quadratic bound in the Theorem 2.3.2 and sharp extrapolation theorem, we will only prove the lower bound. Let us assume that, for $1<r<2$ and $\alpha<1$,

$$\|[b,H]\|_{L^r(w)\to L^r(w)} \leq C[w]_{A_r}^{2\alpha}\|b\|_{BMO}\,.$$

This and the sharp extrapolation theorem return

$$\|[b,H]\|_{L^2(w)\to L^2(w)} \leq C[w]_{A_2}^{2\alpha}\|b\|_{BMO}\,.$$

This contradicts to the sharpness $(p=2)$. Similarly, one can conclude for $p>2$. □

We now consider the weight $w(x) = |x|^{(1-\delta)(p-1)}$ then w is an A_p weight with $[w]_{A_p} \sim \delta^{1-p}$. By (5.2) and (5.3) we have

$$\|[b,H]f\|_{L^p(w)} \geq \frac{1}{\delta^2}\|f\|_{L^p(w)} = (\delta^{1-p})^{\frac{2}{p-1}}\|f\|_{L^p(w)} \sim [w]_{A_p}^{\frac{2}{p-1}}\|f\|_{L^p(w)} \,.$$

This shows the upper bound in (5.4) is sharp for $1 < p \leq 2$. We use the duality argument to see the sharpness of the quadratic estimate for $p > 2$. Note that the commutator is a self-adjoint operator:

$$\langle bH(f) - H(bf), g\rangle = \langle f, H^*(bg)\rangle - \langle f, bH^*(g)\rangle = \langle f, bH(g) - H(bg)\rangle \,.$$

Consider $1 < p \leq 2$ and set $u = w^{1-p'}$, then

$$\begin{aligned} \|[b,H]\|_{L^{p'}(u)\to L^{p'}(u)} &= \|[b,H]\|_{L^{p'}(w^{1-p'})\to L^{p'}(w^{1-p'})} = \|[b,H]^*\|_{L^{p'}(w^{1-p'})\to L^{p'}(w^{1-p'})} \\ &= \|[b,H]\|_{L^p(w)\to L^p(w)} \leq C\|b\|_{BMO^d}[w]_{A_p}^{\frac{2}{1-p}} \\ &= C\|b\|_{BMO}[w^{1-p'}]_{A_{p'}}^2 = C\|b\|_{BMO}[u]_{A_{p'}}^2 \,. \end{aligned} \tag{5.5}$$

Since the inequality in (5.5) is sharp, we can conclude that the result of Theorem 1.0.1 is also sharp for $p > 2$.

5.2.2 Beurling-Ahlfors operator

Recall the Beurling-Ahlfors operator $\mathcal{B}$ is given by convolution with the distributional kernel $p.v.1/z^2$:

$$\mathcal{B}f(x,y) = p.v.\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{f(x-u,y-v)}{(u+iv)^2}\,dudv \,.$$

Then the commutator of the Beurling-Ahlfors operator can be written:

$$[b,\mathcal{B}]f(x,y) = p.v.\frac{1}{\pi}\int_{\mathbb{R}^2}\frac{b(x,y)-b(s,t)}{((x-s)+i(y-t))^2}\,f(s,t)\,dsdt \,.$$

It was observed, in [DV], that the linear bound for the Beurling-Ahlfors operator is sharp in $L^2(w)$, with weights $w(z) = |z|^\alpha$ and functions $f(z) = |z|^{-\alpha}$ where

$|\alpha| < 2$. Similarly, we consider weights $w(z) = |z|^{2-\delta}$ where $0 < \delta < 1$. Note that $w(z) = |z|^{2-\delta} : \mathbb{C} \to [0, \infty)$ is a A_2-weight with $[w]_{A_2} \sim \delta^{-1}$. We also consider a BMO function $b(x) = \log|z|$. Let $E = \{(r, \theta) \,|\, 0 < r < 1,\, 0 < \theta < \pi/2\}$ and $\Omega = \{(r, \theta) \,|\, 1 < r < \infty,\, \pi < \theta < 3\pi/2\}$ We are going to estimate $|[b, \mathcal{B}]f(z)|$ for $z \in \Omega$ with a function $f(z) = |z|^{\delta-2}\chi_E(z)$. Let $z = x + iy$ and $\zeta = s + ti$. Then, for $z \in \Omega$,

$$
\begin{aligned}
|[b, \mathcal{B}]f(z)| &= \frac{1}{\pi}\left| \int_E \frac{(b(z) - b(\zeta))f(\zeta)}{(z - \zeta)^2}\, d\zeta \right| \\
&= \frac{1}{\pi}\left| \int_E \frac{b(x, y) - b(s, t)}{((x - s) + i(y - t))^2} f(s, t) ds dt \right| \\
&= \frac{1}{\pi}\left| \int_E \frac{(\log|z| - \log|\zeta|)|\zeta|^{\delta-2}((x - s)^2 - (y - t)^2)}{((x - s)^2 + (y - t)^2)^2}\, dsdt \right. \\
&\qquad \left. + i \int_E \frac{(\log|z| - \log|\zeta|)|\zeta|^{\delta-2}(2(x - s)(y - t))}{((x - s)^2 + (y - t)^2)^2}\, dsdt \right|.
\end{aligned}
$$

For $z \in \Omega$ and $\zeta \in E$, we have $(x - s)(y - t) \geq xy$ and by triangle inequality $((x - s)^2 + (y - t)^2)^2 = |z - \zeta|^4 \leq (|z| + |\zeta|)^4$. After neglecting the positive term (real part), we get

$$
\begin{aligned}
|[b, \mathcal{B}]f(z)|^2 &\geq \frac{4}{\pi^2}\left(xy \int_E \frac{\log(|z|/|\zeta|)|\zeta|^{\delta-2}}{(|z| + |\zeta|)^4}\, dsdt \right)^2 \\
&= \frac{4}{\pi^2}\left(xy \int_0^{\pi/2}\int_0^1 \frac{\log(|z|/r) r^{\delta-2} r}{(|z| + r)^4}\, drd\theta \right)^2 \\
&= x^2y^2\left(\frac{1}{|z|^4}\int_0^1 \frac{\log(|z|/r) r^{\delta-1}}{(1 + r/|z|)^4} dr \right)^2 \\
&= x^2y^2\left(\frac{1}{|z|^4}\int_0^{1/|z|} \frac{\log(1/t)(|z|t)^{\delta-1}}{(1 + t)^4}\, |z| dt \right)^2 \\
&= x^2y^2\left(\frac{1}{|z|^{4-\delta}}\int_0^{1/|z|} \frac{\log(1/t) t^{\delta-1}}{(1 + t)^4}\, dt \right)^2.
\end{aligned}
$$

Since $|z|/(|z|+1) \leq 1/(1+t)$, for $t < 1/|z|$, we have

$$\begin{aligned}
|[b,\mathcal{B}]f(z)|^2 &\geq x^2y^2\left(\frac{1}{|z|^{-\delta}(|z|+1)^4}\int_0^{1/|z|}\log(1/t)t^{\delta-1}\,dt\right)^2\\
&= \frac{x^2y^2}{|z|^{-2\delta}(|z|+1)^8}\left(\frac{|z|^{-\delta}(1+\delta\log|z|)}{\delta^2}\right)^2\\
&= \frac{x^2y^2}{(|z|+1)^8}\frac{(1+\delta\log|z|)^2}{\delta^4}\,.
\end{aligned}$$

Then, we can estimate the $L^2(w)$-norm as follows.

$$\begin{aligned}
\|[b,\mathcal{B}]f\|^2_{L^2(w)} &\geq \frac{1}{\delta^4}\int_\Omega \frac{x^2y^2(1+\delta\log|z|)^2}{(|z|+1)^8}\,|z|^{2-\delta}\,dxdy\\
&= \frac{1}{\delta^4}\int_1^\infty\int_\pi^{3\pi/2}\frac{r^4\cos^2\theta\sin^2\theta(1+\delta\log r)^2}{(r+1)^8}\,r^{3-\delta}drd\theta\\
&= \frac{\pi}{\delta^4 16}\int_1^\infty \frac{r^{7-\delta}(1+\delta\log r)^2}{(r+1)^8}\,dr \geq \frac{\pi}{\delta^4 16}\int_1^\infty \frac{r^{7-\delta}(1+\delta\log r)^2}{(2r)^8}\,dr\\
&= \frac{\pi}{\delta^4 2^{12}}\int_1^\infty (1+\delta\log r)^2 r^{-1-\delta}\,dr\\
&= \frac{\pi}{\delta^4 2^{12}}\left(\frac{1}{\delta}+\frac{2\delta}{\delta^2}+\frac{2\delta^2}{\delta^3}\right) = \frac{5\pi}{2^{12}}\cdot\frac{1}{\delta^5}\,.
\end{aligned}$$

Combining with $\|f\|^2_{L^2(w)} = \pi/2\delta$, we have that $\|[b,\mathcal{B}]f\|_{L^2(w)}/\|f\|_{L^2(w)} \sim \delta^{-2}$, which allows to conclude that the quadratic bound for the commutator with the Beurling-Ahlfors operators is sharp in $L^2(w)$. Same calculations with weights $w(z) = |z|^{(2-\delta)(p-1)}$ and functions $f(z) = |z|^{(\delta-2)(p-1)}$ will provide the sharpness for $1 < p \leq 2$, and it is sufficient to conclude for all $1 < p < \infty$ because the Beurling-Ahlfors operator is essentially self adjoint operator ($\mathcal{B}^* = e^{i\phi}\mathcal{B}$), so the commutator of the Beurling-Ahlfors operator is also self adjoint.

5.2.3 Riesz transforms

Consider weights $w(x) = |\,x|^{n-\delta}$ and functions $f(x) = x^{\delta-n}\chi_E(x)$ where $E = \{x\,|\,x \in (0,1)^n \cap B(0,1)\}$, and a BMO function $b(x) = \log|\,x|$. It was observed that $|\,x|^{n-\delta}$ is an A_2-weight in $\mathbb{R}^n$ with $[w]_{A^2} \sim \delta^{-1}$. We are going to estimate $[b,R_j]f$ over the set $\Omega = \{y \in B(0,1)^c \,|\, y_i < 0 \text{ for all } i = 1, 2, ..., n\}$, where R_j stands for the j-th direction Riesz transform on $\mathbb{R}^n$ and is defined as follows:

$$R_j f(x) = c_n p.v. \int_{\mathbb{R}^n} \frac{y_j}{|\,y|^{n+1}} f(x-y)\, dy\,, \qquad 1 \le j \le n\,,$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$. One can observe that, for all $x \in E$ and fixed $y \in \Omega$,

$$|\,y_j - x_j| \ge |\,y_j| \text{ and } |\,y - x| \le |\,y| + |\,x|\,.$$

Then,

$$\begin{aligned}
|[b, R_j]f(y)| &= \left| \int_E \frac{(y_j - x_j)(\log|\,y| - \log|\,x|)|\,x|^{\delta-n}}{|\,y - x|^{n+1}}\, dx \right| \\
&\ge |\,y_j| \int_E \frac{\log(|\,y|/|\,x|)|\,x|^{\delta-n}}{(|\,y| + |\,x|)^{n+1}}\, dx \\
&\ge |\,y_j| \int_{E\cap S^{n-1}} \int_0^1 \frac{\log(|\,y|/r) r^{\delta-n} r^{n-1}}{(|\,y| + r)^{n+1}}\, dr d\sigma \\
&= C(n)|\,y_j| \int_0^{1/|y|} \frac{\log(1/t)(t|\,y|)^{\delta-1}|\,y|}{(|\,y| + |\,y|t)^{n+1}}\, dt \\
&= \frac{C(n)|\,y_j|}{|\,y|^{n+1-\delta}} \int_0^{1/|y|} \frac{\log(1/t) t^{\delta-1}}{(1+t)^{n+1}}\, dt \\
&\ge \frac{C(n)|\,y_j|}{|y|^{n+1-\delta}} \left(\frac{|\,y|}{|\,y| + 1} \right)^{n+1} \int_0^{1/|y|} \log(1/t) t^{\delta-1}\, dt \\
&= \frac{C(n)|\,y_j|}{|\,y|^{-\delta}(|\,y| + 1)^{n+1}} \left(\frac{|\,y|^{-\delta}(1 + \delta \log|\,y|)}{\delta^2} \right) \\
&= \frac{C(n)|\,y_j|}{(|\,y| + 1)^{n+1}} \frac{1 + \delta \log|\,y|}{\delta^2}\,.
\end{aligned}$$

We now can bound from below the $L^2(w)$-norm as follows.

$$\begin{aligned}
\|[b, R_j]f(x)\|^2_{L^2(w)} &> \frac{C(n)}{\delta^4} \int_\Omega \frac{y_j^2 (1 + \delta \log|\,y|)^2}{(|\,y| + 1)^{2n+2}} |\,y|^{n-\delta} dy \\
&\ge \frac{C(n)}{\delta^4} \int_{\Omega\cap S^{n-1}} \int_1^\infty \frac{\gamma_j^2 r^2 (1 + \delta \log r)^2 r^{n-\delta} r^{n-1}}{(r+1)^{2n+2}}\, dr d\sigma(\gamma) \\
&\ge \frac{C(n)}{\delta^4} \int_1^\infty \frac{(1 + \delta \log r)^2 r^{2n-\delta+1}}{r^{2n+2}}\, dr \\
&= \frac{C(n)}{\delta^4} \int_1^\infty (1 + \delta \log r)^2 r^{-\delta-1}\, dr = \frac{C(n)}{\delta^5}\,,
\end{aligned}$$

which establishes sharpness for the commutator of the Riesz transform when $p = 2$. Since $R_j^* = -R_j$, one can easily check that the commutator of Riesz transforms are also self-adjoint operators. Furthermore, choosing weight $w(x) = |\, x|^{(n-\delta)(p-1)}$, we will obtain the sharpness for $1 < p < \infty$ by the same argument we used in the case of the Hilbert transform.

As a consequence of [CPerP], bounds for the k-th order commutator with H and R_j's, defined recursively by

$$T_b^k = [b, T_b^{k-1}]\,,$$

are bounded in $L^p(w)$, for $1 < p < \infty$, with a bound $[w]_{A_p}^{(1+k)\max\{1,\frac{1}{p-1}\}} \|b\|_{BMO}^k$. Similar examples can be constructed to obtain lower bounds for the k-th order commutator with H and R_j's, furthermore that is the sharp bound for those operators. Those results are recorded in [CPerP].

Chapter 6

Multivariable dyadic paraproduct

In Section 7.1 we will prove Theorem 2.4.4 and Theorem 2.4.5 which provide the linear bounds for the dyadic paraproduct in $L^2_{\mathbb{R}^n}(w)$, and dimension free estimates. In Section 7.2 we compare anisotropic weights and classical A_2 weights.

6.1 Proof of Theorem 2.4.4 and Theorem 2.4.5

We are going to prove Theorem 2.4.4 only when $p = 2$, and following the one-dimenstional proof discovered by Beznosova [Be]. The sharp extrapolation theorem [DGPerPet] returns immediately the other cases ($1 < p < \infty$). For the case $p = 2$ we use the duality arguments. Precisely, it is sufficient to prove the inequality

$$\langle \pi_b(fw^{-1/2}), gw^{1/2} \rangle \leq C(n)[w]_{A_2^d} \|b\|_{BMO^d_{\mathbb{R}^n}} \|f\|_{L^2_{\mathbb{R}^n}} \|g\|_{L^2_{\mathbb{R}^n}} . \tag{6.1}$$

Proof. Using the orthogonal Haar system (2.14), we can split the left hand side of

(6.1) as follows.

$$\langle \pi_b(fw^{-1/2}), gw^{1/2}\rangle = \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, h_{j,Q}\rangle$$

$$= \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, H^w_{j,Q}\rangle\frac{1}{\sqrt{|E_{j,Q}|}} \tag{6.2}$$

$$+ \sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, A^w_{j,Q}\chi_{E_{j,Q}}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}\ . \tag{6.3}$$

We are going to prove that both sum (6.2) and (6.3) are bounded with a bound that depends linearly on both $[w]_{A_2^d}$ and $\|b\|_{BMO^d_{\mathbb{R}^n}}$ which means

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, H^w_{j,Q}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}$$
$$\leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\|f\|_{L^2_{\mathbb{R}^n}}\|g\|_{L^2_{\mathbb{R}^n}} \tag{6.4}$$

and

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, A^w_{j,Q}\chi_{E_{j,Q}}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}$$
$$\leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\|f\|_{L^2_{\mathbb{R}^n}}\|g\|_{L^2_{\mathbb{R}^n}}\ . \tag{6.5}$$

We will estimate term (6.4) first using Cauchy-Schwarz inequality,

$$\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle\langle fw^{-1/2}\rangle_{E_{j,Q}}\langle gw^{1/2}, H^w_{j,Q}\rangle\frac{1}{\sqrt{|E_{j,Q}|}}$$
$$\leq \left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\frac{\langle gw^{1/2}, H^w_{j,Q}\rangle^2}{|E_{j,Q}|\langle w\rangle_{E_{j,Q}}}\right)^{1/2}\left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle^2\langle fw^{-1/2}\rangle^2_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}\right)^{1/2}$$
$$\leq \|g\|_{L^2_{\mathbb{R}^n}}\left(\sum_{Q\in\mathcal{D}^n}\sum_{j=1}^{2^n-1}\langle b, h_{j,Q}\rangle^2\langle fw^{-1/2}\rangle^2_{E_{j,Q}}\langle w\rangle_{E_{j,Q}}\right)^{1/2}, \tag{6.6}$$

Here the inequality (6.6) follows from (2.15). We now claim that the sum in (6.6) is bounded by $C[w]^2_{A_2^d}\|b\|^2_{BMO^d_{\mathbb{R}^n}}\|f\|^2_{L^2_{\mathbb{R}^n}}$, which will be provided by the Multivariable

Version of the Weighted Carleson Embedding Theorem 3.2.2. with the embedding condition:

$$\forall\, Q' \in \mathcal{D}^n, \quad \frac{1}{\left|E_{i,Q'}\right|} \sum_{Q\in\mathcal{D}^n(E_{i,Q'})} \sum_{j:E_{j,Q}\subseteq E_{i,Q}} \langle w^{-1}\rangle^2_{E_{j,Q}} \langle w\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle^2$$
$$\leq C[w]^2_{A_2^d} \|b\|^2_{BMO^d_{\mathbb{R}^n}} \langle w^{-1}\rangle_{E_{i,Q'}} . \qquad (6.7)$$

Since for all $Q \in \mathcal{D}^n$

$$2^{2(n-1)}[w]_{A_2^d} \geq \langle w\rangle_{E_{j,Q}} \langle w^{-1}\rangle_{E_{j,Q}} \,,$$

The Embedding condition (6.7) can be seen as follows.

$$\sum_{Q\in\mathcal{D}^n(E_{i,Q'})} \sum_{j:E_{j,Q}\subseteq E_{i,Q}} \langle w^{-1}\rangle^2_{E_{j,Q}} \langle w\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle^2 \qquad (6.8)$$
$$\leq\; 2^{2(n-1)}[w]_{A_2^d} \sum_{Q\in\mathcal{D}^n(Q')} \sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \langle w^{-1}\rangle_{E_{j,Q}} \langle b, h_{j,Q}\rangle^2$$
$$\leq 4\cdot 2^{7(n-1)}[w]^2_{A_2^d} \|b\|^2_{BMO^d_{\mathbb{R}^n}} w^{-1}(E_{i,Q'})\,. \qquad (6.9)$$

Here the inequality (6.9) follows from (2.22) and Proposition 3.2.7 applied to $\alpha_{j,Q} = \langle b, h_{j,Q}\rangle^2$, $A = 3^{n-1}\|b\|^2_{BMO^d_{\mathbb{R}^n}}$ and $v = w^{-1}$. This estimates finishes the proof of the inequality (6.4) with $C \approx 2^{7(n-1)/2}$.

We now turn to the proof of the inequality (6.5). In order to prove the inequality (6.5), we need to show that

$$\sum_{Q\in\mathcal{D}^n} \sum_{j=1}^{2^n-1} \langle b, h_{j,Q}\rangle \langle f w^{-1/2}\rangle_{E_{j,Q}} \langle g w^{1/2}\rangle_{E_{j,Q}}\, A^w_{j,Q} \sqrt{|E_{j,Q}|}$$
$$\leq C[w]_{A_2^d} \|b\|_{BMO^d_{\mathbb{R}^n}} \|f\|_{L^2_{\mathbb{R}^n}} \|g\|_{L^2_{\mathbb{R}^n}} \,, \qquad (6.10)$$

and this is provided by the following three embedding conditions due to the Multivariable Version of the Bilinear Embedding Theorem 3.2.5: For all $Q' \in \mathcal{D}^n$ and

$i = 1, ..., 2^n - 1$,

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\,, \tag{6.11}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}} \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\langle w\rangle_{E_{i,Q'}}\,, \tag{6.12}$$

$$\frac{1}{|E_{i,Q'}|}\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w^{-1}\rangle_{E_{j,Q}} \leq C(n)[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\langle w^{-1}\rangle_{E_{i,Q'}}\,. \tag{6.13}$$

Proposition 3.2.11 makes it easy to prove the embedding condition (6.11). Using Cauchy-Schwarz inequality,

$$\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}} \tag{6.14}$$

$$\leq \left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \tag{6.15}$$

$$\times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2}$$

$$\leq 2^{n-1}[w]^{1/2}_{A_2^d}\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\right)^{1/2} \tag{6.16}$$

$$\times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2}$$

$$\leq C\,2^{2(n-1)}[w]_{A_2^d}\left|E_{i,Q'}\right|^{1/2}\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}}\langle b,h_{j,Q}\rangle^2\right)^{1/2} \tag{6.17}$$

$$\leq C\,2^{5(n-1)/2}[w]_{A_2^d}\|b\|_{BMO^d_{\mathbb{R}^n}}\left|E_{i,Q'}\right|\,. \tag{6.18}$$

Here we use (3.53) for the inequality (6.17) and the fact that $b \in BMO^d_{\mathbb{R}^n}$ for the inequality (6.18). We also use Cauchy-Schwarz inequality for the inequality (6.12), then

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w\rangle_{E_{j,Q}} \\
&\quad\le \left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \langle b,h_{j,Q}\rangle^2\langle w\rangle_{E_{j,Q}}\right)^{1/2} \\
&\qquad\qquad \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2} \\
&\quad\le C2^{5(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]^{1/2}_{A^d_2}\langle w\rangle^{1/2}_{E_{i,Q'}} \\
&\qquad\qquad \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w\rangle_{E_{j,Q}}\right)^{1/2} \qquad (6.19)\\
&\quad\le C\,2^{7(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]_{A^d_2}\langle w\rangle_{E_{i,Q'}}\,. \qquad (6.20)
\end{aligned}
$$

Inequality (6.19) and (6.20) follow by (3.36) and Proposition 3.2.13 respectively. We can establish inequality (6.13) with Proposition 3.2.9 as follows,

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left|\langle b,h_{j,Q}\rangle\, A^w_{j,Q}\right|\sqrt{|E_{j,Q}|}\langle w^{-1}\rangle_{E_{j,Q}} \\
&\quad\le \left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \langle b,h_{j,Q}\rangle^2\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \\
&\qquad\qquad \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \\
&\quad\le C2^{5(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]^{1/2}_{A^d_2}\langle w\rangle^{1/2}_{E_{i,Q'}} \\
&\qquad\qquad \times\left(\sum_{Q\in\mathcal{D}^n(Q')}\sum_{j:E_{j,Q}\subseteq E_{i,Q'}} \left(A^w_{j,Q}\right)^2|E_{j,Q}|\langle w^{-1}\rangle_{E_{j,Q}}\right)^{1/2} \\
&\quad\le C2^{7(n-1)/2}\|b\|_{BMO^d_{\mathbb{R}^n}}[w]_{A^d_2}\langle w^{-1}\rangle_{E_{i,Q'}}\,.
\end{aligned}
$$

To sum up, we can establish the inequality (6.1) with a constant $C(n)\approx 2^{7(n-1)/2}$.

Furthermore, if we replace $[w]_{A_2^d}$ by $[w]_{A_2^R}$ and $\|b\|_{BMO^d}$ by $\|b\|_{BMO^R}$ then we can establish proof of the dimension free estimate in Theorem 2.4.5. □

6.2 Comparison of the A_2^R weight and the A_2 weight

Let us consider the weight $w_\alpha = |\,x|^\alpha$. Is is a well known fact that $[w]_{A_2} \sim \frac{1}{n^2-\alpha^2}$. This means $w_\alpha \in A_2$ if and only if $|\,\alpha| < n$. Let us now observe the $[w_\alpha]_{A_2^R}$ when $n = 2$ and $\alpha = 1$. Since the main singularity of w_α occurs at the origin, it is enough to observe the case of $R_t = [0, t] \times [0, 1]$ with sufficiently large t .

$$\begin{aligned}\int_{R_t} w_1(x,y)\,dxdy &= \int_0^1\int_0^t (x^2+y^2)^{1/2}\,dxdy\\ &= \left(\int_0^{\arctan(1/t)}\int_0^{t/\cos\theta} r^2\,drd\theta + \int_{\arctan(1/t)}^{\pi/2}\int_0^{1/\sin\theta} r^2\,drd\theta\right)\\ &= \frac{1}{3}\left(\int_0^{\arctan(1/t)} t^3\sec^3\theta\,d\theta + \int_{\arctan(1/t)}^{\pi/2}\csc^3\theta\,d\theta\right).\end{aligned}$$

Since the series expansion of $\sec^3\theta = 1 + 3\theta^2/2 + \mathcal{O}(\theta^4)$,

$$t\int_0^{\arctan(1/t)}\sec^3\theta\,d\theta \sim t\big(\arctan(1/t) + (\arctan^3(1/t))/3\big) \to 1 \text{ as } t \to \infty\,.$$

For the second integral, we use the series expansion of $\csc^3\theta = 1/\theta^3 + 1/2\theta + 17\theta/120 + \mathcal{O}(\theta^3)$,

$$\frac{1}{t^2}\int_{\arctan(1/t)}^{\pi/2}\csc^3\theta\,d\theta \sim \frac{1}{t^2}\left(\frac{1}{2\arctan^2(1/t)} + \frac{1}{2}\ln(\arctan(1/t)) + \frac{17\arctan(1/t)}{120}\right) \to \frac{1}{2}$$

as $t \to \infty$. For sufficiently large t, we see

$$\int_{R_t} w_1(x,y)dxdy \sim t^2\,.$$

And

$$\int_{R_t} w_1^{-1}(x,y)\,dxdy = \int_0^1 \int_0^t (x^2+y^2)^{-1/2}\,dxdy$$
$$= \int_0^{\arctan(1/t)} \int_0^{t/\cos\theta} drd\theta + \int_{\arctan(1/t)}^{\pi/2} \int_0^{1/\sin\theta} drd\theta\,.$$

For the first integral, we have

$$\int_0^{\arctan(1/t)} \int_0^{t/\cos\theta} drd\theta = t\int_0^{\arctan(1/t)} \sec\theta\,d\theta$$
$$= t\log|\sec(\arctan(1/t)) + \tan(\arctan(1/t))|$$
$$= t\log\left|\sqrt{\frac{1}{t^2}+1}+\frac{1}{t}\right| = t\log\left(\frac{\sqrt{1+t^2}+1}{t}\right) \to 1$$

as $t\to\infty$. Using series expansion of $\csc\theta = 1/\theta + \theta/6 + \mathcal{O}(\theta^3)$ the second integral can be estimated as follows

$$\int_{\arctan(1/t)}^{\pi/2} \int_0^{1/\sin\theta} drd\theta = \int_{\arctan(1/t)}^{\pi/2} \csc\theta\,d\theta$$
$$\sim -\log(\arctan(1/t)) - \frac{\arctan^2(1/t)}{12} \sim \log t\,.$$

Thus, for sufficiently large t,

$$\int_{R_t} w_1^{-1}(x,y)\,dxdy \sim \log t\,.$$

To sum up, we have

$$\langle w_1\rangle_{R_t}\langle w_1^{-1}\rangle_{R_t} \sim \log t\,.$$

These observations allow us to conclude that there is a weight which belongs to A_2 but not to A_2^R, when $n\geq 2$. Some dimension free bound on the weighted Lebesgue spaces via Poisson A_2-characteristics of weights are established for the Riesz transforms [Pet3] in $\mathbb{R}^n$, and the square function [PetWic] in the unit ball in $\mathbb{C}^n$. Here we say a weight belongs to the class of Poisson A_2. if

$$[w]_{P_2} := \sup_{t\in\mathbb{R}^+,y\in\mathbb{R}^n} \widetilde{w}(t,y)\widetilde{w^{-1}}(t,y) < \infty\,,$$

where $\widetilde{w}(t,y)$ stands for the Poisson extension to the upper half plane $\mathbb{R}^{n+1}_+$:

$$\widetilde{w}(t,y) := w * P_t(y) = \int_{\mathbb{R}^n} w(x)P_t(y-x)\,dx = \int_{\mathbb{R}^n} \frac{t}{(t^2+|\,y-x|^2)^{\frac{n+1}{2}}}\,w(x)\,dx\,.$$

It was also observed in [Pet3] that $\widetilde{w_\alpha}(t,0)$ diverges if and only if $\alpha \geq 1$. For the case $n=1$, the three weight classes are the same. However, for the case $n \geq 2$, we know there is a weight that belongs to A_2^R or P_2 but not to A_2, for instance $w(x) = |\,x|$. For the one dimensional case, since $[w]_{A_2} = [w]_{A_2^R}$, we have that there is constants such that [Huk]

$$c[w]_{A_2^R} \leq [w]_{P_2} \leq C[w]^2_{A_2^R}\,.$$

For $n \geq 2$, we don't know yet how these two weight classes A_2^R and P_2 are related. We may conjecture that the same relation as well as in the one dimensional case holds for $n \geq 2$.

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