

9-14-1995

# Static Output Feedback: A Survey

V.L. Syrmos

C.T. Abdallah

P. Dorato

K. Grigoriadis

Follow this and additional works at: [https://digitalrepository.unm.edu/ece\\_rpts](https://digitalrepository.unm.edu/ece_rpts)

---

## Recommended Citation

Syrmos, V.L.; C.T. Abdallah; P. Dorato; and K. Grigoriadis. "Static Output Feedback: A Survey." (1995).  
[https://digitalrepository.unm.edu/ece\\_rpts/10](https://digitalrepository.unm.edu/ece_rpts/10)

This Technical Report is brought to you for free and open access by the Engineering Publications at UNM Digital Repository. It has been accepted for inclusion in Electrical & Computer Engineering Technical Reports by an authorized administrator of UNM Digital Repository. For more information, please contact [disc@unm.edu](mailto:disc@unm.edu).

DEPARTMENT OF ELECTRICAL AND  
COMPUTER ENGINEERING  
SCHOOL OF ENGINEERING  
UNIVERSITY OF NEW MEXICO

**STATIC OUTPUT FEEDBACK: A SURVEY**

V.L. Syrmos  
EE Dpt.  
University of Hawaii  
Honolulu, HI 96822

C. Abdallah and P. Dorato  
EECE Dpt.  
The University of New Mexico  
Albuquerque, NM 87131

K. Grigoriadis  
ME Dpt.  
University of Houston  
Houston, TX 77204

UNM Technical Report: EECE95-008

Report Date: September 14, 1995

## **Abstract**

This paper reviews the static output feedback problem in the control of linear, time-invariant (LTI) systems. It includes analytical and computational methods and presents in a unified fashion, the knowledge gained in the decades of research into this most important problem.

## **Key Words**

Output Feedback, Linear Systems, Pole Placement, Linear Quadratic Regulator, Linear Matrix Inequalities, Decision Methods, Computational Complexity.

## **Acknowledgements**

A preliminary version of this paper appeared in the Proceedings of the 33rd IEEE CDC, 1994, Lake Buena Vista, FL, pp. 837–842 Corresponding Author: P. Dorato, peter@eece.unm.edu

# Contents

- 1 Introduction** **2**
  
- 2 Stabilizability By Static Output Feedback** **4**
  - 2.1 Necessary Conditions . . . . . 4
  - 2.2 Sufficient Conditions . . . . . 5
  - 2.3 Design Approaches and Limitations . . . . . 5
    - 2.3.1 Youla Parameterization Method . . . . . 5
    - 2.3.2 Inverse Linear Quadratic Approach . . . . . 6
    - 2.3.3 Covariance Assignability by Output Feedback . . . . . 6
    - 2.3.4 Output Structural Constraint Approach . . . . . 7
    - 2.3.5 Coupled Linear Matrix Inequality Formulation . . . . . 8
    - 2.3.6 Decision Methods . . . . . 9
  
- 3 Pole Placement With Static Output Feedback** **11**
  - 3.1 Necessary Conditions . . . . . 11
  - 3.2 Sufficient Conditions . . . . . 11
  - 3.3 Design Approaches and Limitations . . . . . 13
  
- 4 Eigenstructure Assignment With Static Output Feedback** **15**
  - 4.1 Necessary Conditions . . . . . 15
  - 4.2 Sufficient Conditions . . . . . 16
  - 4.3 Design Approaches and Limitations . . . . . 16
  
- 5 LQ Regulator With Static Output Feedback** **18**
  - 5.1 Design Approaches and Limitations . . . . . 18
  - 5.2 Inverse Problem . . . . . 19
  
- 6 How Hard is SOF?** **20**
  
- 7 Conclusion** **22**

# Chapter 1

## Introduction

The static output feedback (SOF) problem is probably the most important open question in control engineering, see for example the two recent surveys (Bernstein, 1992; Blondel et al., 1995). Simply stated, the problem is as follows: Given a linear, time-invariant system, find a static output feedback so that the closed-loop system has some desirable characteristics, or determine that such a feedback does not exist. The problem is important in its own right, but also because many other problems are reducible to some variation of it. This paper attempts to survey the state of knowledge concerning the output feedback problem. The survey will encompass both Single-Input-Single-Output (SISO) and Multi-Input-Multi-Output (MIMO) systems. Even though the SISO case may be efficiently resolved using graphical techniques, we include it here because the fundamental question of the existence of static output controllers is still open, even in the scalar case.

This survey paper has two main parts. The first involves the study of the time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) \quad (0.1)$$

under the influence of static output feedback of the form

$$u(t) = Ky(t) + v(t). \quad (0.2)$$

The closed-loop system is

$$\dot{x} = (A + BKC)x(t) + Bv(t) \equiv A_c x(t) + Bv(t). \quad (0.3)$$

The problem of output feedback evolves around the selection of a constant feedback gain matrix  $K$  to achieve various closed-loop properties. We take the state  $x(t) \in \mathbb{R}^n$ , the control input  $u(t) \in \mathbb{R}^m$ , and the output  $y(t) \in \mathbb{R}^p$ . The problem may also be studied in a transfer function setting where one is given a transfer matrix relationship between the input  $u(s)$  and the output  $y(s)$  ( $s$  denotes the Laplace transform variable) such that  $y(s) = C(sI - A)^{-1}Bu(s) \equiv H(s)u(s)$  and the objective is to find  $K$  in the feedback law (0.2) so that the closed-loop system,

$$\begin{aligned} y(s) &= C(sI - A - BKC)^{-1}Bv(s) \\ &= [I - H(s)K]^{-1}H(s)v(s) \end{aligned} \quad (0.4)$$

satisfies some performance objectives. The case where a dynamical output compensator of order  $q \leq n$  is used may be brought back to the static output feedback case as follows (see for example (Nett et al., 1989)): Suppose the dynamic compensator is given in state-space form as

$$\dot{x}_f(t) = A_f x_f(t) + B_f y(t) \quad (0.5)$$

$$u(t) = C_f x_f(t) + D_f y(t) + v(t) \quad (0.6)$$

Then, an augmented state-system is obtained when  $u_f(t) = \dot{x}_f(t)$ , and  $y_f(t) = x_f(t)$  by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_f(t) \\ y_f(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \\ 0 & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} u_f(t) \\ u(t) \end{bmatrix} \quad (0.7)$$

so that the feedback law is now static and given by

$$\begin{bmatrix} u_f(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \begin{bmatrix} y_f(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v(t) \quad (0.8)$$

or in a more compact description

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t); \quad \tilde{y}(t) = \tilde{C}\tilde{x}(t); \quad \tilde{u}(t) = \tilde{K}\tilde{y}(t) + \tilde{v}(t) \quad (0.9)$$

where

$$\begin{aligned} \tilde{x} &= \begin{bmatrix} x \\ x_f \end{bmatrix}; \quad \tilde{u} = \begin{bmatrix} u_f \\ u \end{bmatrix}; \quad \tilde{y} = \begin{bmatrix} y_f \\ y \end{bmatrix}; \quad \tilde{v} = \begin{bmatrix} 0 \\ v \end{bmatrix} \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}; \quad \tilde{K} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \end{aligned} \quad (0.10)$$

The second part of this paper involves the solution of various coupled matrix design equations of the sort obtained in pole-placement and LQ design using output feedback, game theory, and elsewhere. Such coupled systems of equations are currently “solved” using iterative numerical techniques. The computational difficulty or cost of such numerical techniques have not been investigated until recently (Blondel and Tsitsiklis, 1995; Toker and Özbay, 1995).

We recall here a few mathematical definitions which will be used in this paper. We say that a rational function  $H(s)$  is Bounded-Input-Bounded-Output-Stable (BIBO) stable or that it belongs to  $H^\infty$  if it is proper, with all its poles in the left-half-plane (LHP). We let  $\mathcal{S}$  denote the set of matrices whose entries are in  $H^\infty$ . A Unit in  $\mathcal{S}$  is a member of  $\mathcal{S}$  whose inverse is also in  $\mathcal{S}$ . A matrix is said to be epic if it has full row rank and monic if it has full column rank. In what follows,  $A^T$  denote the transpose of any matrix  $A$ , and the controller is  $u = Ky + v$ .

The paper is organized as follows: Section 2 contains a discussion of stabilizability using static output feedback, including (non-testable) necessary and sufficient conditions. The chapter also includes design procedures such as the covariance assignment and the decision methods. The pole placement problem is presented in Section 3 and the eigenstructure assignment is discussed in chapter 4. Section 5 is devoted to the Linear Quadratic Regulator problem with output feedback. Section 6 reviews some recent results on the computational complexity of the SOF problem and our conclusions are presented in chapter 7.

## Chapter 2

# Stabilizability By Static Output Feedback

In this chapter, we discuss the problem of stabilizing an open-loop unstable system with static output feedback. We present first some necessary conditions, then some sufficient ones for the solvability of this problem. We then discuss some approaches used to find a stabilizing gain  $K$ .

### 2.1 Necessary Conditions

We first identify the cases where static output feedback can not stabilize an open-loop unstable system. This will at least provide us with necessary conditions, which when violated, tell us that dynamic feedback compensation is required. In order to state these conditions, we recall the following theorems.

**Theorem 2.1.1** (Youla et al., 1974) **The Parity-Interlacing-Property (PIP)** A linear system  $H(s)$  is stabilizable with a stable compensator  $C(s)$  (i.e., *strongly* stabilizable) if and only if the number of real poles of  $H(s)$ , counted according to their McMillan degree, between any pair of real blocking zeros in the right-half-plane is even. A system which satisfies the pole-zero constraints is said to satisfy the PIP. ■

Note that in the SISO case, the PIP fails to hold for many real systems. On the other hand, as observed in (Hågander and Bernhardsson, 1990), (Vidyasagar, 1985) and (Youla et al., 1974), the PIP holds generically in the MIMO case.

**Theorem 2.1.2** (Wei, 1990) A linear system  $H(s)$  is stabilizable with a stable compensator  $C(s)$  which has no real unstable zeros if and only if: 1)  $H(s)$  satisfies the PIP, and 2) The number of real blocking zeros of  $H(s)$  between any two real poles of  $H(s)$  is even. In this case we say that  $H(s)$  satisfies the even PIP. ■

Using Theorem 2.1.2, the following necessary condition is obtained:

**Necessary Condition 1:** A necessary condition for static output stabilizability is that the plant  $H(s)$  satisfies the even PIP. ■

## 2.2 Sufficient Conditions

We start out by noting the simple case of SISO systems, of relative degree  $n^* \leq 1$ , and which are minimum phase (the finite zeros are stable). A simple root-locus argument then shows that such systems are stabilizable with a large enough static output feedback. In fact, The minimum-phase and the relative-degree conditions are necessary and sufficient to make square (i.e. same number of inputs and outputs) systems Strictly-Positive-Real (SPR) using static output feedback as described for example in (Gu, 1990b) and (Abdallah et al., 1991).

## 2.3 Design Approaches and Limitations

In the case of SISO systems, graphical approaches (root-locus, Nyquist) are used to answer both the existence and the design questions of stabilizing static output controllers. In addition, there exist some necessary and sufficient algebraic tests (Helmke and Anderson, 1992), (Pérez et al., 1993) for the existence of stabilizing output feedbacks. These tests however, require some preliminary derivations (finding roots, eigenvalues) which are just as complicated as the graphical methods. In addition, they are not easily extendable to the MIMO case, although some specialized cases may be resolved using the Multivariable Nyquist criterion (Brockett and Byrnes, 1981). The work in (Byrnes and Crouch, 1985), also presents a complete characterization of strictly-proper SISO systems related to each other with static output feedback. In fact, it states that such systems must share the same zeros and the same breakaway points. This then leads to the open question, of finding at least one stable transfer function having the same zeros and the same breakaway points as our open-loop system  $H(s)$ .

In this chapter, we list some parameterization results that are potentially useful in solving the static output feedback problem. The idea basically is that a stabilizing static output feedback must be a member of the family of all stabilizing output feedback compensators.

### 2.3.1 Youla Parameterization Method

The following result parameterizes all stabilizing controllers in terms of a matrix in  $\mathcal{S}$  (Vidyasagar, 1985).

**Theorem 2.3.1** A compensator  $C(s) = N_c(s)D_c(s)^{-1}$  where  $N_c(s)$  and  $D_c(s)$  are in  $\mathcal{S}$  internally stabilizes the plant  $H(s) = N_p(s)D_p(s)^{-1}$  if and only if  $N_c(s)N_p(s) + D_c(s)D_p(s)$  is a Unit of  $\mathcal{S}$ . Moreover, the set of all stabilizing compensators of  $H(s)$  is given by

$$\mathcal{C} = \{C(s); C(s) = [N_c(s) + D_p(s)Q(s)][D_c(s) - N_p(s)Q(s)]^{-1}\};$$

for any  $Q(s) \in \mathcal{S}$ . ■

It can then be argued that a necessary and sufficient condition for the static output stabilizability problem is that there exists a  $Q(s) \in \mathcal{S}$  such that

$$K = [N_c(s) + D_p(s)Q(s)][D_c(s) - N_p(s)Q(s)]^{-1} \quad (3.1)$$

is a constant matrix. In fact, such an approach is advocated in (Gu et al., 1993), where a search is conducted to find a  $Q(s)$  to reduce the order of the compensators. Unfortunately, this and other so-called necessary and sufficient conditions are non-testable and as such they cannot be used to answer the existence question.



### 2.3.2 Inverse Linear Quadratic Approach

In (Trofino-Neto and Kučera, 1993) another necessary and sufficient condition was found for the stabilizability of a linear system using output feedback in terms of the solvability of a modified LQR problem. In fact, the authors in (Trofino-Neto and Kučera, 1993), state and prove the following result.

**Theorem 2.3.2** Given the system (0.1), and let  $E_i = C^\dagger C$ , where superscript “ $\dagger$ ” denotes the Moore-Penrose inverse. Then, the system is stabilizable with static output feedback  $K = -R^{-1}(L + B^T P)E_i$  if and only if there exist matrices  $Q > 0$ ,  $R > 0$  and  $L$  of compatible dimensions such that the algebraic equation

$$A^T P + PA - E_i(PB + L^T)R^{-1}(B^T P + L)E_i + Q = 0 \quad (3.2)$$

has a unique solution  $P > 0$ . ■

The problem resides in the fact that one can not easily choose the matrices  $Q > 0$ ,  $R > 0$  and  $L$ , nor can we easily solve for  $P$  in (3.2). A related (non-testable) necessary and sufficient condition is given in (Kučera and de Souza, 1995).

### 2.3.3 Covariance Assignability by Output Feedback

The basic idea behind the covariance control theory is to provide a characterization of all assignable covariance matrices and, in addition, a parameterization of all controllers which assign a particular covariance (Hotz and Skelton, 1987), (Yasuda et al., 1993), (Skelton and Iwasaki, 1993). Given a stochastic system

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ y &= Cx \end{aligned} \quad (3.3)$$

where  $w(t)$  is a zero-mean white-noise disturbance of intensity  $W$ , the steady-state covariance matrix of the state vector  $x(t)$  is defined by

$$X = \lim_{t \rightarrow \infty} \mathcal{E}\{x(t)x(t)^T\} \quad (3.4)$$

where  $\mathcal{E}$  denotes the expectation operator. For a static output feedback control law  $u = Ky$  it is well known that  $X$  solves the Lyapunov equation

$$(A + BKC)X + X(A + BKC)^T + W, \quad X^T = 0. \quad (3.5)$$

A matrix  $X > 0$  is called an assignable covariance if there exists a controller gain  $K$  such that (3.5) is satisfied. If  $(A, B)$  is stabilizable and  $(A, C)$  is controllable then, from Lyapunov stability theory,  $X > 0$  is equivalent to stability of the closed-loop system. The following result parameterizes all assignable covariances by static output feedback (Yasuda et al., 1993).

**Theorem 2.3.3** A matrix  $X > 0$  is an assignable covariance by static output feedback if and only if  $X$  satisfies

$$(I - BB^\dagger)(AX + XA^T + W, X^T)(I - BB^\dagger) = 0 \quad (3.6)$$

$$(I - C^\dagger C)X^{-1}(AX + XA^T + W, X^T)X^{-1}(I - C^\dagger C) = 0 \quad (3.7)$$

$$(I - \Delta\Delta^\dagger)(I - C^\dagger C)X^{-1}(AX + XA^T + W, X^T) = 0 \quad (3.8)$$

where

$$\Delta = (I - C^\dagger C)X^{-1}BB^\dagger \quad (3.9)$$

■

A parameterization of all static output feedback gains that stabilize the system and assign a particular assignable covariance is obtained as follows (Yasuda et al., 1993).

**Theorem 2.3.4** Let  $X > 0$  be an assignable covariance matrix. Then all static output feedback gains that assign  $X$  to the closed loop system are parameterized by

$$K = -\frac{1}{2}B^\dagger(AX + XA^T + W, {}^T + \Sigma)X^{-1}C^\dagger + Z - B^\dagger BZCC^\dagger \quad (3.10)$$

where

$$\Sigma = [\Phi^\dagger \Psi + (I - \Phi^\dagger \Phi)S] (I - \Phi^\dagger \Phi) - (\Phi^\dagger \Psi)^T \quad (3.11)$$

$$\Phi = \begin{bmatrix} I - BB^\dagger \\ (I - C^\dagger C)X^{-1} \end{bmatrix} \quad (3.12)$$

$$\Psi = \begin{bmatrix} -(I - BB^\dagger) \\ (I - C^\dagger C)X^{-1} \end{bmatrix} (AX + XA^T + W, {}^T) \quad (3.13)$$

and  $Z$  is an arbitrary matrix and  $S$  is an arbitrary skew-symmetric matrix. ■

Conditions (3.6)-(3.13) can be interpreted as a state-space parameterization of all stabilizing static output feedback gains in terms of the state covariance matrix  $X$ . The major difficulty in covariance control theory is to test if the coupled covariance assignability equations (3.6)-(3.8) have a common solution  $X > 0$ , and to obtain such a solution if one exists. Once a common solution  $X > 0$  is found, the parameterization (3.10)-(3.13) provides all static gains that stabilize the system and assign  $X$  as a closed-loop covariance. A deterministic interpretation of the covariance control theory is given in (Yasuda et al., 1993).

### 2.3.4 Output Structural Constraint Approach

The static output feedback problem can be viewed as a state feedback problem where the feedback gain is subject to a structural constraint. In particular,  $A + BKC$  is stable if and only if  $A + BL$  is stable where  $LY = 0$  and  $Y$  is an orthonormal basis of the null space of  $C$ . Defining the augmented matrices

$$F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+m) \times (n+m)}, \quad G = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{(n+m) \times m} \quad (3.14)$$

and the functions

$$\Theta(W) = FW + WF^T \quad (3.15)$$

$$f(W) = \text{trace} [W_2^T W_1^{-1} W_2 - W_2^T C^T (CW_1 C^T)^{-1} CW_2] \quad (3.16)$$

where  $W$  is the  $(n + m) \times (n + m)$  symmetric matrix

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \quad (3.17)$$

with  $W_1 > 0$ , a necessary and sufficient condition for output stabilization can be expressed as follows (Peres et al., 1993).

**Theorem 2.3.5** There exists a stabilizing static output feedback gain if and only if

$$\mathcal{C} \cap \mathcal{W} \neq \emptyset \quad (3.18)$$

where

$$\mathcal{C} = \{W : v^T \Theta(W) v \leq 0 \quad \forall v \in \mathcal{N}(G)\} \quad (3.19)$$

$$\mathcal{W} = \{W : f(W) \leq 0\} \quad (3.20)$$

$$(3.21)$$

where  $\mathcal{N}(G)$  denotes the null space of  $G$ . The set of all stabilizing static output feedback gain is parameterized by

$$K = W_2^T C_2^T (C_2 W_1 C_2^T)^{-1} \quad (3.22)$$

where  $W \in \mathcal{C} \cap \mathcal{W}$ . ■

The set  $\mathcal{C}$  is convex, however  $\mathcal{W}$  is non-convex making the condition (3.18) difficult to check. Although  $f$  is not guaranteed to be convex, a supporting hyperplane to the epigraph of  $f$  can be calculated in a large subset of the domain of  $f$  (Geromel et al., 1993). Based on that, cutting plane algorithms have been proposed to obtain stabilizing output feedback gains, but convergence of these algorithms is not guaranteed (Geromel et al., 1993), (Peres et al., 1993).

### 2.3.5 Coupled Linear Matrix Inequality Formulation

Necessary and sufficient conditions for static output feedback can be obtained in terms of coupled Linear Matrix Inequalities following a quadratic Lyapunov function approach. From Lyapunov stability theory we know that the closed-loop system matrix  $A + BKC$  is stable if and only if  $K$  satisfies the following matrix inequality

$$(A + BKC)P + P(A + BKC)^T < 0 \quad (3.23)$$

for some  $P > 0$ . For a fixed  $P$ , the inequality (3.23) is a **Linear Matrix Inequality** (LMI) in the matrix  $K$ , see (Boyd et al., 1994). The LMI (3.23) is convex in  $K$  so that convex programming techniques can be used to numerically find a  $K$  whenever  $P > 0$  is given. Necessary and sufficient conditions for static output feedback stabilization are obtained by finding the solvability conditions of (3.23) in terms of  $K$  (Iwasaki and Skelton, 1995), (El-Ghaoui and Gahinet, 1993).

**Theorem 2.3.6** There exists a stabilizing static output feedback gain if and only if there exists  $P > 0$  such that

$$B^\perp (AP + PA^T)(B^\perp)^T < 0 \quad (3.24)$$

$$(C^T)^\perp (A^T P^{-1} + P^{-1} A)((C^T)^\perp)^T < 0 \quad (3.25)$$

where  $B^\perp$  and  $(C^T)^\perp$  are full-rank matrices, orthogonal to  $B$  and  $C^T$  respectively. ■

Inequality (3.24) follows from (3.23) by simple multiplication on the left by  $B^\perp$  and on the right by  $(B^\perp)^T$ . Inequality (3.25) follows from (3.23), by multiplying on the left and right by  $P^{-1}$  and then multiplying on the left by  $(C^\perp)^T$ , and on the right by  $((C^T)^\perp)^T$ . In (Iwasaki and Skelton, 1995) and (El-Ghaoui and Gahinet, 1993) it is shown that the converse is also true, that is if there exists a  $P > 0$  which satisfies inequalities (3.24) and (3.25), then there exists a stabilizing static output feedback  $K$ . A parameterization of all static output feedback gains that correspond to a feasible solution  $P$  of (3.24)-(3.25) is provided in (Iwasaki and Skelton, 1995).

**Theorem 2.3.7** All stabilizing static output feedback gains are parameterized by

$$K = -R^{-1} B^T P Q^{-1} C^T (C Q^{-1} C^T)^{-1} + S^{1/2} L (C Q^{-1} C^T)^{-1/2} \quad (3.26)$$

where

$$S = R^{-1} - R^{-1}B^T P Q^{-1} [Q - C^T (CQ^{-1}C^T)^{-1}C]^{-1} Q P B R^{-1} > 0 \quad (3.27)$$

$$Q = P B R^{-1} B^T P - P A - A^T P \quad (3.28)$$

$$R^{-1} > B^\dagger [\Phi - \Phi(B^T)^\perp (B^\perp \Phi(B^\perp)^T)^{-1} B^\perp \Phi] B^{\dagger T} \quad (3.29)$$

$$\Phi = P^{-1} A^T + A P^{-1} \quad (3.30)$$

where  $P$  is any positive definite matrix which satisfies (3.24) and (3.25), and  $L$  is any matrix with  $\|L\| < 1$ . ■

Notice that (3.24) is an LMI on  $P$  and (3.25) is an LMI on  $P^{-1}$ , but finding such a  $P > 0$  is a difficult task since the two inequalities are **not** convex in  $P$ . Computational methods based on iterative sequential solutions of the two convex LMI problems with respect to  $P$  and  $P^{-1}$  have been proposed to find stabilizing static output feedback gains, but convergence of the algorithms is not guaranteed (Iwasaki et al., 1994), (Geromel et al., 1994). Also, in (Grigoriadis and Skelton, 1996) alternating projection methods are suggested to solve fixed-order and output feedback control problems described by LMIs but with no guaranteed convergence. It is interesting to note that many other static output feedback control problems, such as suboptimal  $H^\infty$  control, suboptimal Linear Quadratic control and  $\mu$ -synthesis with constant scaling can be formulated in terms of coupled LMIs as in (3.24)-(3.25), see (El-Ghaoui and Gahinet, 1993), (Gahinet and Apkarian, 1994), (Iwasaki and Skelton, 1994), (Iwasaki and Skelton, 1995).

### 2.3.6 Decision Methods

In 1975, a paper (Anderson et al., 1975) introduced decision methods to study the output feedback stabilization problem. By using a stability criterion, such as Routh-Hurwitz, the output feedback stabilizability problem can be reduced to a system of multivariable polynomial inequalities in  $k_{ij}$ , which are the  $ij$ -th component of the feedback matrix  $K$ . Decision methods permit one to establish, in a finite number of algebraic steps, the existence of real variables  $k_{ij}$  such that all polynomial inequalities are satisfied. Decision methods can be extended to eliminate not only the “existence” quantifier  $\exists$ , but also complex combinations of existence and “universal”,  $\forall$  quantifiers. This permits us to study not only nominal stabilizability but also robust stabilizability (Abdallah et al., 1995). Decision methods are currently referred to as Quantifier Elimination or QE techniques (Basu et al., 1994; Tarski, 1951) and are briefly discussed next.

Given the set of polynomials with integer coefficients  $P_i(X, Y)$ ,  $1 \leq i \leq s$  where  $X$  represents a  $k$  dimensional vector of quantified real variables and  $Y$  represents an  $l$  dimensional vector of un-quantified real variables, let  $X^{[i]}$  be a block of  $k_i$  quantified variables,  $Q_i$  be one of the quantifiers  $\exists$  (there exists) or  $\forall$  (for all), and let  $\Phi(Y)$  be the quantified formula

$$\Phi(Y) = (Q_1 X^{[1]}, \dots, Q_w X^{[w]}) F(P_1, \dots, P_s), \quad (3.31)$$

where  $F(P_1, \dots, P_s)$  is a quantifier free *Boolean* formula, that is a formula containing the Boolean operators  $\wedge$  (and) and  $\vee$  (or), operating on *atomic predicates* of the form  $P_i(Y, X^{[1]}, \dots, X^{[w]}) \geq 0$  or  $P_i(Y, X^{[1]}, \dots, X^{[w]}) > 0$  or  $P_i(Y, X^{[1]}, \dots, X^{[w]}) = 0$ . We can now state the general quantifier elimination problem

**General Quantifier Elimination Problem:** Find a quantifier-free Boolean formula  $\Psi(Y)$  such that  $\Phi(Y)$  is true if and only if  $\Psi(Y)$  is true. ■

In control problems, the un-quantified variables are generally the compensator parameters, represented by the parameter vector  $Y = q$ , and the quantified variables are the plant parameters, represented by the plant parameter vector  $p$ , and the frequency variable  $\omega$ . Uncertainty in plant parameters are characterized by quantified formulas of the type  $\forall(p_i) \ [p_i \leq p_i \leq \bar{p}_i]$  where  $\underline{p}_i$  and  $\bar{p}_i$  are rational numbers. *The quantifier-free formula  $\Psi(q)$  then represents a characterization of the compensator design.* An important special problem is the QE problem with no un-quantified variables (free variables), i.e.  $l = 0$ . This problem is referred to as the *General Decision Problem*.

**General Decision Problem:** With no un-quantified variables, i.e.  $l = 0$ , determine if the quantified formula given in (3.31) is true or false. ■

The general decision problem may be applied to the problem of *existence* of compensators that meet given specifications, in which case an “existence” quantifier is applied to the compensator parameter  $q$ .

Algorithms for solving general QE problems were first given by Tarski (Tarski, 1951) and Seidenberg (Seidenberg, 1954), and are commonly called Seidenberg-Tarski decision procedures. Tarski showed that QE is solvable in a finite number of steps, but his algorithm and later modifications are exponential in the size of the problem. Researchers in Control Theory have been aware of Tarski’s results and their applicability to Control problems since the 1970’s but the tedious operations made the technique very limited (Anderson et al., 1975).

Recently, new algorithms have been developed for the QE problem and software packages have been introduced. A sample of these packages is the package QEPCAD *Quantifier Elimination by Partial Cylindrical Algebraic Decomposition* (Hong, 1990). This software has been used to solve different fixed-structure and output feedback problems in (Abdallah et al., 1995).

The basic limitation of QE methods is still the computational complexity of existing algorithms. Typically, these algorithms are doubly exponential in the number of blocks  $X^{[i]}$  (Basu et al., 1994). Thus, only modestly-sized problems can be solved by these methods. Even so, some (robust) stabilization problems can only be solved using QE methods. In particular, QE methods have the distinct advantage over deterministic and random discretization methods in that results (when obtained) have no “holes” in the parameter space and require no probabilistic qualifications.

## Chapter 3

# Pole Placement With Static Output Feedback

Here, it is desired to select the gain  $K$  to place the poles (or the eigenvalues) in the closed-loop system (0.3) at desired locations. However, in a historical context a pole is said to be assignable (by output feedback) if  $K$  may be selected such that (0.3) has a pole arbitrarily close to a desired value. We say that (0.1) is (output) pole assignable if all the poles may be assigned given a desired symmetric (i.e. closed under complex conjugation) set of  $n$  poles. This problem is the most developed of all output feedback applications and recent results have provided necessary and sufficient conditions for the generic pole placement assignability.

### 3.1 Necessary Conditions

In (Herman and Martin, 1977) a necessary and sufficient condition for generic pole assignability with a **complex** gain matrix  $K$  was established as

$$mp \geq n, \tag{1.1}$$

however, simple counter-examples show that this is only necessary for the case of **real**  $K$  (Willems and Hesselink, 1978). In (Giannakopoulos and Karcanias, 1985), the necessary condition was strengthened to (1.1) plus full rank of the so-called Plücker matrix. Reference (Kabamba and Longman, 1982) defined (0.1) as locally completely assignable (for a given  $K$ ) if, for every desired set of small changes  $\delta\sigma_i$  in the poles  $\sigma_i$  of  $(A + BKC)$ , there exists a  $\delta K$  such that  $[A + B(K + \delta K)C]$  has poles at  $(\sigma_i + \delta\sigma_i)$ . A necessary and sufficient (but non-testable) condition for this to occur was given in terms of the independence of the closed-loop Markov parameter matrices.

### 3.2 Sufficient Conditions

In (Brasch and Pearson, 1970) it was shown that if (0.1) is minimal (i.e. controllable and observable), then almost any  $K$  will yield a cyclic  $\bar{A} = (A + BKC)$ , i.e. one such that  $sI - \bar{A} - BK\bar{C}$  has only one non-unity invariant polynomial. Moreover, for almost any choice of a vector  $q$ , we make  $\{\bar{A}, Bq\}$  controllable. Then, we can apply the scalar design formulas to obtain a gain matrix  $k$  such that

$\det(sI - \bar{A} + Bqk)$  is the desired closed-loop polynomial. In (Davison, 1970; Davison, 1971; Davison and Chow, 1973), this approach was exploited to show that if  $(A, B, C)$  is minimal with  $B$  and  $C$  of full rank, then  $\max(m, p)$  poles are assignable. Davison and Wang (Davison and Wang, 1975) and Kimura (Kimura, 1975; Kimura, 1978) showed that indeed, under these conditions,  $\min(n, m + p - 1)$  poles are assignable generically (i.e. for almost all  $A, B$  and  $C$ ). This translates into the sufficient condition for generic pole assignability that

$$m + p \geq n + 1. \quad (2.2)$$

An alternate proof of this was offered in (Brockett and Byrnes, 1981; Schumacher, 1980) where the authors showed that the pole placement with SOF is equivalent to classical Schubert problem. Moreover, they showed that if

$$d(m, p) = \frac{1!2! \cdots (p-1)!(mp)!}{m!(m+1)! \cdots (m+p-1)!} \quad (2.3)$$

is odd, and whenever  $\min\{m, p\} = 1$  or  $\min\{m, p\} = 2$  and  $\max\{m, p\} = 2^k - 1$ , a real  $K$  exists to generically assign the closed-loop poles.

Another sufficient condition for generic pole assignability was given in (Kimura, 1977) as

$$m + p + \beta > n + 1; \quad m > \beta; \quad p \geq \alpha \quad (2.4)$$

with  $\alpha$  and  $\beta$  the controllability and observability indices respectively. If (0.1) is minimal with  $B$  of full rank and  $A_d$  is the desired closed-loop plant matrix, then another sufficient condition for pole assignability was given in (Vardulakis, 1975) as  $(A - A_d)(I - C^\dagger C) = 0$ , with superscript “†” again denoting the Moore-Penrose inverse. This may be interpreted as a condition that any differences between the actual and the desired plant matrices occur in the perpendicular of  $\mathcal{N}(C)$  (with  $\mathcal{N}(\cdot)$  representing the Null space). More recently, Wang (Wang, 1994) has shown that  $n < mp$  is sufficient for generic pole assignability. In fact, the result of Wang is described in the following theorem.

**Theorem 3.2.1** *If  $n < mp$ , then the pole placement map*

$$\begin{aligned} \chi &: \mathbb{R}^{mp} \longrightarrow \mathbb{R}^n \\ K &\longmapsto \det(sI - A - BKC) \end{aligned}$$

*is surjective (or onto) for a generic set of real matrices  $(A, B, C)$ .*

In this setting, a subset  $S$  of  $\mathbb{R}^k$  is generic if its complement is contained in the zero set of some nonzero polynomial  $p(x_1, x_2, \dots, x_k)$ . A simpler proof of Wang’s result appeared in (Rosenthal et al., 1994).

Finally, a more recent result (Rosenthal and Wang, 1995) shows that generic pole assignability is possible with a compensator of MacMillan degree  $q$  as soon as

$$\begin{aligned} n &< q(m + p - 1) + mp - \min\{r_m(p - 1), r_p(m - 1)\} \\ r_m &= q - m[q/m] \\ r_p &= q - p[q/p] \end{aligned}$$

i.e.  $r_m$  and  $r_p$  are the remainders of  $q$  divided by  $m$  and  $p$  respectively. More importantly, the authors provided an algorithmic procedure for obtaining the compensator when it exists. By letting  $q = 0$ , we obtain the condition  $mp > n$  again.

### 3.3 Design Approaches and Limitations

It is worth discussing briefly the techniques used in some of the cited references. In (Davison, 1970; Davison, 1971; Davison and Chow, 1973; Davison and Wang, 1975), an explicit formula was given for  $K$  in terms of various matrices constructed from  $(A, B, C)$  and the desired poles. It amounts to an ‘‘Ackermann-type’’ formula for output feedback. In (Kimura, 1975; Kimura, 1977; Kimura, 1978) a different approach which relates closely to the eigenstructure assignment techniques in the next chapter was used. References (Brockett and Byrnes, 1981) and (Giannakopoulos and Karcianas, 1985) used the Grassman space (i.e. exterior algebra). In (Misra and Patel, 1989) an algorithm was given to assign the eigenvalues arbitrarily close to desired values for the case  $m + p > n$ . The Hessenberg form was used to solve two single-input problems. First,  $p - 1$  poles were placed, then  $n - p + 1$  poles were placed without disturbing the first poles assigned (c.f. (Srinathkumar, 1978)). A discussion on the relation between the pole-assignment problem and transmission zeros is also given. A related algorithm was given in (Miminis, 1985) to assign  $\max(m, p)$  poles. If condition (2.2) fails to hold, then the techniques of this section generally allow the assignment of  $m + p - 1 < n$  poles. There are no guarantees however, on the locations of the remaining closed-loop poles, which may often be unstable. A nice geometric framework involving lattices is provided in (Champetier and Magni, 1992; Magni and Champetier, 1988; Magni and Champetier, 1991). It is however difficult to translate that framework into computational techniques.

In the following, we present yet another set of the so-called necessary and sufficient (but non-testable) conditions for pole placement using output feedback. For notational ease assume that  $B$  is monic,  $C$  is epic. We suppose  $p \geq m$ ; the other case is handled in a similar way. The open-loop input-coupling, output-coupling, and transfer relations are revealed in matrix-fraction description (MFD) form by

$$(sI - A)^{-1}B = N_1(s)D^{-1}(s) \quad (3.5)$$

$$C(sI - A)^{-1} = F^{-1}(s)G_1(s) \quad (3.6)$$

$$H(s) = CN_1(s)D^{-1}(s) = N(s)D^{-1}(s) \quad (3.7)$$

$$H(s) = F^{-1}(s)G_1(s)B = F^{-1}(s)G(s) \quad (3.8)$$

with (3.5, 3.7) normalized right MFDs (e.g. right coprime,  $D(s)$  column-reduced and column-degree ordered), and (3.6, 3.8) normalized left MFDs (e.g. left coprime,  $F(s)$  row-reduced and row-degree ordered). The next result was shown in (Syrmos et al., 1992).

**Theorem 3.3.1** Let  $[N(s), D(s)]$  be a normalized right MFD for  $H(s)$ . There exists a feedback  $K$  that assigns the invariant polynomials if and only if the equation

$$\begin{bmatrix} -Y_m & X_m \\ F(s) & G(s) \end{bmatrix} \begin{bmatrix} -N(s) & X_p \\ D(s) & Y_p \end{bmatrix} = \begin{bmatrix} R_m(s) & 0 \\ 0 & R_p(s) \end{bmatrix} \quad (3.9)$$

is satisfied for some  $R_m(s)$  and  $R_p(s)$ , both having the desired closed-loop nonunit invariant polynomials. The solution must satisfy the conditions:

1.  $[G(s), F(s)]$  left coprime,  $F(s)$  row reduced and row-degree ordered.
2.  $X_m, Y_m, X_p, Y_p$  constant matrices with  $X_m$  and  $X_p$  nonsingular.

Then the required output feedback is given by  $K = -X_m^{-1}Y_m = -Y_pX_p^{-1}$  ■

**Note 3.3.1** Equation (3.9) is equivalent to

$$\begin{aligned} Y_mN(s) + X_mD(s) &= R_m(s); & F(s)N(s) &= G(s)D(s) \\ F(s)X_p + G(s)Y_p &= R_p(s); & Y_mX_p &= X_mY_p. \end{aligned}$$



Therefore, the condition is in terms of coupled Diophantine equations, which should be contrasted with the coupled LMI equations in the previous chapter.

Note that a similar derivation may be obtained using matrices in  $\mathcal{S}$  rather than polynomial matrices. Finally, note that a common limitation to these pole placement approaches (except for (Rosenthal and Wang, 1995)), is that unless all poles can be placed in the stability region, no guarantee exists for the stability of the closed-loop system.

## Chapter 4

# Eigenstructure Assignment With Static Output Feedback

First, we review eigenstructure assignment by state-variable feedback  $u(t) = -Fx(t) + v(t)$  (i.e.  $C = I$ ). While the pole-placement problem for multivariable systems is fairly complicated, Moore (Moore, 1976) showed that the problem of assigning both eigenvalues and eigenvectors has a straightforward solution. Given a symmetric set of desired closed-loop poles  $\{\mu_i\}$ ,  $i = 1, \dots, q$ , vectors  $\{v_i\}$  and  $\{u_i\}$  are found such that

$$[\mu_i I - A \quad B] \begin{bmatrix} v_i \\ u_i \end{bmatrix} = 0 \quad (0.1)$$

Then a feedback gain  $F$  defined by

$$F [v_1 \cdots v_q] = [u_1 \cdots u_q] \quad (0.2)$$

results in the closed-loop structure

$$[\mu_i I - (A - BF)] v_i = 0 \quad (0.3)$$

so that the  $v_i$  are assigned as the closed-loop eigenvectors for eigenvalues  $\mu_i$ .

### 4.1 Necessary Conditions

There is a certain freedom in the choice of the  $v_i$ , but for a real  $F$  to exist they must satisfy

1.  $v_i \in (\mu_i I - A)^{-1} R(B)$
2.  $v_i = v_j^*$  when  $\mu_i = m\mu_j^*$ , (where “\*” means complex conjugation)
3.  $\{v_i\}$  is a linearly independent set.

The integer  $q$  may be taken equal to  $n$ , but any uncontrollable poles must be included in  $\{m\mu_i\}$ , with the associated  $v_i$  satisfying  $w_i^T v_i \neq 0$ , where  $w_i$  is the left eigenvector associated with  $\mu_i$ . Note that we may write (0.1) as the generalized Lyapunov equation

$$VJ - AV = -BU \quad (1.4)$$

with  $V = [v_1 \cdots v_q]$ ,  $U = [u_1 \cdots u_q]$ ,  $J = \text{diag}(\mu_i)$ . Then (0.2) reduces to  $FV = U$ . Turning to the case of output feedback (0.2), Reference (Bengtsson and Lindahl, 1974) assumes that a state-variable feedback  $F$  which places both eigenvalues and eigenvectors has been selected by some procedure. Then, a method is given to find an output feedback  $K$  that preserves some of the poles of  $(A - BF)$  in (0.3). Although eigenvector assignment was not specifically addressed, the technique involves in fact preserving the eigenvectors  $v_i$  associated with the modes  $\{\mu_i, i = 1, \dots, q\}$ . Indeed, although  $KC = -F$ , may have no solution  $K$ , the reduced equation  $KCV = -FV$  may have a solution, so that (0.3) becomes  $[\mu_i I - (A - BKC)]v_i = 0$ . In (Srinathkumar, 1978), the technique of (Moore, 1976) was extended to output feedback, essentially by replacing (0.2) with  $KCV = U$ . From that work, it is clear that  $\max(m, p)$  poles are assignable by this method. The algorithm given assigns  $p - 1$  poles, and an additional (interesting but fairly complicated) procedure was given to assign a total of  $\min(n, m + p - 1)$  poles generically. The case of constrained output feedback (i.e. where some of the entries of  $K$  are set to zero) was covered in (Calvo-Ramon, 1986).

## 4.2 Sufficient Conditions

A major breakthrough occurred in (Kwon and Youn, 1987) where some techniques of (Kimura, 1977) were extended to show that, in some cases,  $m + p$  poles may be assigned. This is a better result than those associated with (2.2). It was obtained by considering the closed-loop right and left eigenstructure. A design example demonstrates the assignment of  $m + p$  poles. However, it is not clear in the paper what is actually going on in terms of system structure. A somewhat streamlined description of the main result is as follows. Let the desired closed-loop structure be described by the (possibly non-simple) Jordan matrix  $J$ . If there exist a direct sum decomposition  $J = J_1 \oplus J_2$  and matrices  $V_1, W_2, U$ , and  $Z$  such that

$$V_1 J_1 - A V_1 = -B U \quad (2.5)$$

$$J_2 W_2^T - W_2^T A = -Z^T C \quad (2.6)$$

$$W_2^T V_1 = 0 \quad (2.7)$$

then  $K = U(CV_1)^{-1}$  makes  $J$  the Jordan matrix of  $(A + BKC)$ . Moreover, the right eigenvectors corresponding to the poles in  $J_1$  are the columns of  $V_1$ , and the left eigenvectors corresponding to the poles in  $J_2$  are the columns of  $W_2$ . It should now be noted that  $p$  poles may be placed by using equation of (2.5) (c.f. (1.4)), and possibly  $m$  by using the dual relation, equation of (2.6). The construction of the required matrices in (2.5)-(2.7) may be confronted by using the right Null space of  $[\mu_i I - A \ B]$  and the left Null space of

$$\begin{bmatrix} \mu_i I - A \\ C \end{bmatrix}$$

with  $\{\mu_i\}$  the desired poles (Kwon and Youn, 1987). Unfortunately, the proposed solution algorithm is derived from only a sufficient condition, and relies on selecting some vectors to guarantee various conditions, so that some artistic ability and intuition is needed, along with a bit of luck, to apply the technique. In the case where  $p + m > n$  a computationally efficient algorithm is proposed in (Syrmos and Lewis, 1993) for the solution of the coupled Sylvester equations (2.5)-(2.7). The fixed-order compensator problem was also studied using two Coupled Sylvester equations in (Syrmos and Syrmos, 1992).

## 4.3 Design Approaches and Limitations

Although a given number of poles is generically assignable by the above approaches, nothing is known of the remaining closed-loop poles, which may be unstable. In (Fletcher and Ho, 1986) a technique was

given for approximate pole assignment which gives some idea of the location of all of the closed-loop poles. Eigenstructure assignment with output feedback was treated for some special cases in (Fletcher and Magni, 1987; Magni, 1987). Note that the condition expressed in terms of (2.5)-(2.7) is sufficient only. A necessary and sufficient condition for eigenstructure assignment using output feedback was also given in (Kwon and Youn, 1987); however, it was not used as the basis of any design algorithm. Yet another necessary and sufficient condition was given in (Syrmos and Lewis, 1994) in terms of a Bilinear Sylvester equation. However, it was not used as the basis of any algorithm either.

## Chapter 5

# LQ Regulator With Static Output Feedback

It is desired here to select  $K$  to minimize, subject to the constraint (0.3), the performance index

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad (0.1)$$

with  $Q \geq 0$  and  $R > 0$ , while stabilizing the closed-loop system. In (Johnson and Athans, 1970; Levine and Athans, 1970; Levine et al., 1971; Moerder and Calise, 1985), necessary conditions for optimality were given as

$$0 = A_c^T S + S A_c + Q - C^T K^T R K C \quad (0.2)$$

$$0 = A_c P + P A_c^T + X \quad (0.3)$$

$$0 = -R K C P C^T - B^T S P C^T, \quad (0.4)$$

with  $X = x(0)x(0)^T$  and  $A_c = A + B K C$ . Generally, optimal control with reduced information results in such coupled nonlinear matrix equations. If it is desired to eliminate the dependence of (0.2)-(0.4) on the specific initial conditions, then expected values may be taken of the performance index (0.1) so that  $X = \mathcal{E}\{x(0)x(0)^T\}$  in (0.3). It is generally assumed that  $x(0)$  is uniformly distributed on the unit sphere so that  $X = I$  (Levine and Athans, 1970). The tracking problem with output feedback is considered in (Bernstein and Haddad, 1987).

### 5.1 Design Approaches and Limitations

Algorithms for the solution of (0.2)-(0.4) and their discrete counterparts were proposed in (Choi and Sirisena, 1974; Kreisselmeier, 1975; Levine and Athans, 1970; Moerder and Calise, 1985; O'Reilly, 1978; Söderström, 1978; Toivonen, 1985). These algorithms are all iterative in nature. Convergent iterative algorithms for the continuous case were finally presented in 1985 (Moerder and Calise, 1985; Toivonen, 1985). The algorithm in (Moerder and Calise, 1985) requires repetitive solution of (0.2) and (0.3) for fixed values of  $K$  so that they are considered as two Lyapunov (i.e. linear matrix) equations, and the form  $K = R^{-1} B^T S P C^T (C P C^T)^{-1}$  as a candidate for the next choice for  $K$ . Compare this expression with that in Section 2.3.2 when  $L = 0$ . Note however, that it guarantees only a local minimum. Unfortunately, iterative algorithms such as these require the selection of an initial stabilizing gain. A direct procedure for finding such a  $K$  is unknown as discussed in chapter 2.

Conditions for the existence and global uniqueness of solutions to (0.2)-(0.4) such that  $P$  and  $S$  are positive definite and (0.3) is stable are not known. It has been shown (Ermer and Vandelinde, 1973) that in the discrete case there exists a gain that minimizes (0.1) locally and also stabilizes the system if  $Q \geq 0$ ,  $R > 0$ ,  $\text{rank}(C) = p$ ,  $X > 0$ , and  $(A, B, C)$  is output stabilizable; that is, there exists a  $K$  such that  $A_c$  is stable. However, there may be more than one local minimum, so that solution of (0.2)-(0.4) may not yield the global minimum. Similar sufficient conditions were given in (Moerder and Calise, 1985).

Necessary and sufficient conditions for the existence of a solution to the suboptimal LQ problem with output feedback in terms of LMIs are given in (Iwasaki et al., 1994).  $H^2$  optimal control with output feedback is treated in (Peres et al., 1993) using the techniques of chapter 2.3.4. However, the above approaches suffer from the same drawbacks discussed in chapter 2.3.

## 5.2 Inverse Problem

It was shown in (Gu, 1990a) that for square open-loop transfer functions, a necessary and sufficient condition for the existence of an output feedback that will stabilize the closed-loop system and minimize (0.1) is given as follows;

**Theorem 5.2.1** Let the system (0.1) have no transmission zeros on the  $ju$  axis. Then, there exists a matrix  $K$  such that  $u = Ky$  will stabilize the closed-loop system and minimize (0.1) for some  $Q \geq 0$  and  $R > 0$ , with  $\{\sqrt{Q}, A\}$  observable, if and only if  $\det(CB) \neq 0$  and the open-loop function  $C(sI - A)^{-1}B$  is minimum phase. ■

Note that this necessary and sufficient condition is the same as that required to make the closed-loop system  $K(sI - A - BK)^{-1}B$  Strictly-Positive-Real (SPR).

## Chapter 6

# How Hard is SOF?

The question many researchers have begun asking is whether it is worth spending any more time looking for an analytical solution to the SOF problem. In fact, many of them have pointed out that algorithmic and numerical solutions may be called upon to solve the problem in many interesting cases. The hope is then that someone can come up with an algorithm that can solve most of the SOF problems encountered in practice. In this chapter we review results from Computational Complexity theory to suggest that such hope may not be realistic, at least for moderate and large size problems.

Recently, many control problems have been shown to be  $\mathcal{NP}$ -complete (or  $\mathcal{NP}$ -Hard), (Blondel and Tsitsiklis, 1995; Poljak and Rohn, 1993; Nemirovskii, 1993; Coxson, 1993; Toker and Özbay, 1995). For the SOF problem, the exponential-time Tarski-Seidenberg elimination method (Tarski, 1951) can theoretically be used to determine whether or not a solution to the multivariable polynomial inequalities (obtained from the Routh-Hurwitz test) exists (Anderson et al., 1975). This answers the question of the *decidability* of the problem, but it does not address the more practical problem of whether or not efficient (i.e., polynomial-time) methods exist for solving the problem. In the language of computational complexity theory, the SOF problem is formulated as follows:

### STATIC OUTPUT FEEDBACK

Instance: A LTI plant of the form  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t)$ , under the influence of static output feedback of the form  $u(t) = Ky(t) + v(t)$ .

*Question:* Does there exist a real gain matrix  $K$  which guarantee the closed-loop stability of the LTI plant? ■

A problem closely related to STATIC OUTPUT FEEDBACK was studied in (Blondel and Tsitsiklis, 1995):

### STABLE MATRIX IN UNIT INTERVAL FAMILY

Instance: A positive integer  $n$ , a partition of  $I = \{(i, j) \mid 1 \leq i, j \leq n\}$  into disjoint sets  $I_1$  and  $I_2$ , rational numbers  $a_{ij}^*$  for  $(i, j) \in I_1$ .

*Question:* Does the set  $\mathcal{A}$  of  $n \times n$  matrices defined by

$$\mathcal{A} = \{A = a_{ij} \mid a_{ij} = a_{ij}^* \text{ for } (i, j) \in I_1, a_{ij} \in [-1, 1] \text{ for } (i, j) \in I_2\}$$

contain at least one stable matrix? ■

Notice, however, that some constraints are placed on all of the elements of  $A$ —each element is either a fixed rational number, or a rational number in the interval  $[-1, 1]$ . In (Blondel and Tsitsiklis, 1995), it was shown that STABLE MATRIX IN UNIT INTERVAL FAMILY  $\in \mathcal{NP}$ -Hard and used the result to show that the SOF problem when the entries of  $K$  are constrained to lie in some intervals is  $\mathcal{NP}$ -Hard. In (Toker and Özbay, 1995) the problem of STABLE MATRIX remains  $\mathcal{NP}$ -Hard even if no bounds is placed on the variations of  $a_{ij}$ . Blondel and Tsitsiklis conjectured that the computational complexity of their problems remains the same even in the absence of constraints on  $K$  (Blondel and Tsitsiklis, 1995). To date however, no such result is available.

In effect, the introduction of computational complexity methods into the study of the static output problem is suggesting that general algorithms (such as those obtained from the Decision methods) are almost doomed to failure. Computational complexity however, does not necessarily leads to the conclusion that every (or even most) SOF problem is computationally intractable. On the contrary, and due to the genericity results discussed in Section 2, the SOF problems may be solved for many specific problems. The complexity methods do however suggest that every effort should be applied to exploit the particular structure of a given SOF problem.



## Chapter 7

# Conclusion

It is clear from the studies cited here that the problem of static output feedback is still open. Various unconnected necessary conditions, sufficient conditions, and ad hoc solution techniques abound. Except for the generic pole assignment problem, where an algorithm exists (Rosenthal and Wang, 1995), and the QE software, not much exists in terms of an organized design. Unfortunately, the generic pole assignment problem is too restrictive and the decision methods are computationally inefficient. The result is total confusion for all but the expert in mathematical system theory, and the failure to use analytical output-feedback design in many applications. The so-called necessary and sufficient conditions are not efficiently testable, and as such only succeed in transforming the problem into another unsolved problem or into a numerical search problem with no guarantee of convergence to a solution. A common thread throughout these methods however, is the fact that the problem is equivalent to obtaining the solution of a coupled set of matrix (Lyapunov, Riccati, LMI, Bezout, etc) equations. The recent indications that the output feedback problem may be  $\mathcal{NP}$ -Hard implies that moderately large problems are computationally intractable. Exploitation of the special structure of particular problems seems to be the only promising approach to follow.

**Acknowledgments:** V. Syrmos acknowledges the financial support of the National Science Foundation under contract NCR-9210408, by the Advanced Research Programs Agency under contract MDA-972-93-1-0032 and by the University of Hawaii Research Council Funds. C. Abdallah acknowledges the financial support of Boeing Computer Services under Contract W-300445, and fruitful discussions with G. Heileman and P. Goldberg. K. Grigoriadis acknowledges the financial support of a University of Houston PEER Award.

# Bibliography

- Abdallah, C., Dorato, P., and Karni, S. (1991). “SPR design using feedback”. *Proc. IEEE Conf. on Decision and Control*, pages 1742–1743.
- Abdallah, C., Dorato, P., Yang, W., Liska, R., and Steinberg, S. (1995). “Applications of quantifier elimination theory to control system design”. In *Submitted to: IFAC World Congress*, San Francisco, CA.
- Anderson, B., Bose, N., and Jury, E. (1975). “Output feedback stabilization and related problems—solution via decision methods”. *IEEE Trans. Autom. Control*, AC-20:53–65.
- Basu, S., Pollack, R., and Roy, M.-F. (1994). “On the combinatorial and algebraic complexity of quantifier elimination”. In *Proc. 35th Symposium on Foundations of Computer Science*, pages 632–641, Santa Fe, NM.
- Bengtsson, G. and Lindahl, S. (1974). “A design scheme for incomplete state or output feedback with applications to boiler and power system control”. *Automatica*, 10:15–30.
- Bernstein, D. (1992). “Some open problems in matrix theory arising in linear systems and control”. *Linear Algebra and Its Applications*, 162–164:409–432.
- Bernstein, D. and Haddad, W. (1987). “Optimal output feedback for nonzero set point regulations”. *IEEE Trans. Autom. Control*, AC-32:641–645.
- Blondel, V., Gevers, M., and Lindquist, A. (1995). “Survey on the state of systems and control”. *Preprint*.
- Blondel, V. and Tsitsiklis, J. (1995). “NP-hardness of some linear control design problems”. *The European Journal of Control*, 1(1):To appear.
- Boyd, S., El-Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in Systems and Control*. SIAM, Philadelphia, PA, 1st edition.
- Brasch, F. and Pearson, J. (1970). “Pole assignment using dynamic compensators”. *IEEE Trans. Autom. Control*, AC-16:348–351.
- Brockett, R. and Byrnes, C. (1981). “Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint”. *IEEE Trans. Autom. Control*, AC-26:271–284.
- Byrnes, C. and Crouch, P. (1985). “Geometrical methods for the classification of linear feedback systems”. *Syst. Control Lett.*, pages 239–245.
- Calvo-Ramon, J. (1986). “Eigenstructure assignment by output feedback and residue analysis”. *IEEE Trans. Autom. Control*, AC-31:247–249.

- Champetier, C. and Magni, J. (1992). “On eigenstructure assignment by output feedback”. *SIAM J. Control Optim.*, page to appear.
- Choi, S. and Sirisena, H. (1974). “Computation of optimal output feedback gains for linear multivariable systems”. *IEEE Trans. Autom. Control*, AC-19:254–258.
- Coxson, G. E. (1993). *Computational Complexity of Robust Stability and Regularity in Families of Linear Systems*. PhD thesis, The University of Wisconsin – Madison, WI.
- Davison, E. (1970). “On pole assignment in linear systems with incomplete state feedback”. *IEEE Trans. Autom. Control*, AC-15:348–351.
- Davison, E. (1971). “A note on pole assignment in linear systems with incomplete state feedback”. *IEEE Trans. Autom. Control*, AC-16:98–99.
- Davison, E. and Chow, S. (1973). “An algorithm for the assignment of closed-loop poles using output feedback in large linear multivariable systems”. *IEEE Trans. Autom. Control*, AC-18:74–75.
- Davison, E. and Wang, S. (1975). “On pole assignment in linear multivariable systems using output feedback”. *IEEE Trans. Autom. Control*, AC-20:516–518.
- El-Ghaoui, L. and Gahinet, P. (1993). “Rank minimization under LMI constraints: A framework for output feedback problems”. *European Control Conf.*, pages 1176–1179.
- Ermer, C. and Vandelinde, V. (1973). “Output feedback gains for a linear discrete stochastic control problem”. *IEEE Trans. Autom. Control*, AC-18:154–155.
- Fletcher, L. and Ho, W. (1986). “Approximation results in output feedback pole assignment”. *Proc. IEEE Conf. on Decision and Control*, pages 1786–1787.
- Fletcher, L. and Magni, J. (1987). “Exact pole assignment by output feedback: Part 1”. *Int. J. of Control*, pages 1995–2007.
- Gahinet, P. and Apkarian, P. (1994). “A linear matrix inequality approach to  $H_\infty$  control”. *Int. J. Robust Nonlin. Control*, pages 421–448.
- Geromel, J., de Souza, C., and Skelton, R. (1994). “LMI numerical solution for output feedback stabilization”. *Submitted to Int. J. Control*.
- Geromel, J., Peres, P., and Souza, S. (1993). “Convex analysis of output feedback structural constraints”. *Proc. IEEE Conf. on Decision and Control*, pages 1363–1364.
- Giannakopoulos, C. and Karcaniyas, N. (1985). “Pole assignment of strictly proper and proper linear systems by constant output feedback”. *Int. J. of Control*, pages 543–565.
- Grigoriadis, K. and Skelton, R. (1996). “Fixed-order control design for LMI control problems using alternating projection methods”. *Automatica*. to appear.
- Gu, D.-W., Choi, B., and Postlethwaite, I. (1993). “Low-order stabilizing compensators”. *IEEE Trans. Autom. Control*, AC-38:1713–1717.
- Gu, G. (1990a). “On the existence of linear optimal control with output feedback”. *SIAM J. Control Optim.*, pages 711–719.
- Gu, G. (1990b). “Stabilizability conditions of multivariable uncertain systems via output feedback”. *IEEE Trans. Autom. Control*, 35:925–927.

- Hagander, P. and Bernhardsson, B. (1990). “On the notion of strong stabilization”. *IEEE Trans. Autom. Control*, AC-35:927–929.
- Helmke, U. and Anderson, B. (1992). “Hermitian pencils and output feedback stabilization of scalar systems”. *Int. J. of Control*, pages 857–876.
- Herman, R. and Martin, C. (1977). “Applications of algebraic geometry to systems theory: Part 1”. *IEEE Trans. Autom. Control*, AC-22:19–25.
- Hong, H. (1990). *Improvements in CAD-based Quantifier Elimination*. PhD thesis, The Ohio State University.
- Hotz, A. and Skelton, R. (1987). “Covariance control theory”. *Int. J. of Control*, pages 13–32.
- Iwasaki, T. and Skelton, R. (1994). “All controllers for the general  $H_\infty$  control problem: LMI existence conditions and state space formulas”. *Automatica*, 30:1307–1317.
- Iwasaki, T. and Skelton, R. (1995). “Parametrization of all stabilizing controllers via quadratic Lyapunov functions”. *J. Optim. Theory Appl.*, pages 291–307.
- Iwasaki, T., Skelton, R., and Geromel, J. (1994). “Linear quadratic suboptimal control with static output feedback”. *Syst. Control Lett.*, pages 421–430.
- Johnson, T. and Athans, M. (1970). “On the design of optimal constrained dynamic compensators for linear constant systems”. *IEEE Trans. Autom. Control*, AC-15:658–660.
- Kabamba, P. and Longman, R. (1982). “Exact pole assignment using direct or dynamic output feedback”. *IEEE Trans. Autom. Control*, AC-27:1244–1246.
- Kimura, H. (1975). “On pole assignment by gain output feedback”. *IEEE Trans. Autom. Control*, AC-20:509–516.
- Kimura, H. (1977). “A further result on the problem of pole assignment by gain output feedback”. *IEEE Trans. Autom. Control*, AC-22:458–463.
- Kimura, H. (1978). “On pole assignment by output feedback”. *Int. J. of Control*, pages 11–22.
- Kreisselmeier, G. (1975). “Stabilization of linear systems by constant output feedback using the riccati equation”. *IEEE Trans. Autom. Control*, AC-20:556–557.
- Kučera, V. and de Souza, C. (1995). “A necessary and sufficient condition for output feedback stabilizability”. *Automatica*, page Scheduled for Publication.
- Kwon, B.-H. and Youn, M.-J. (1987). “Eigenvalue–eigenvector assignment by output feedback”. *IEEE Trans. Autom. Control*, AC-32:417–421.
- Levine, W. and Athans, M. (1970). “On the determination of the optimal constant output feedback gains for linear multivariable systems”. *IEEE Trans. Autom. Control*, AC-15:44–48.
- Levine, W., Johnson, and Athans, M. (1971). “Optimal limited state variable feedback controllers for linear systems”. *IEEE Trans. Autom. Control*, AC-16:785–793.
- Magni, J. (1987). “On exact pole placement by output feedback: Part 3”. *Int. J. of Control*, pages 2021–2033.
- Magni, J. and Champetier, C. (1988). “A general framework for pole assignment algorithms”. *Proc. IEEE Conf. on Decision and Control*, pages 2223–2229.

- Magni, J. and Champetier, C. (1991). “A geometric framework for pole assignment algorithms”. *IEEE Trans. Autom. Control*, AC-36:1105–1111.
- Miminis, G. (1985). “A direct algorithm for pole assignment of time invariant linear systems using output feedback”. *Proc. IEEE Conf. on Decision and Control*, pages 541–547.
- Misra, P. and Patel, R. (1989). “Numerical algorithms for eigenvalue assignment by constant and dynamic output feedback”. *IEEE Trans. Autom. Control*, AC-34:579–588.
- Moerder, D. and Calise, A. (1985). “Convergence of a numerical algorithm for calculating optimal output feedback gains”. *IEEE Trans. Autom. Control*, AC-30:900–903.
- Moore, B. (1976). “On the flexibility offered by state feedback in multivariable control”. *IEEE Trans. Autom. Control*, AC-21:689–692.
- Nemirovskii, A. (1993). “Several NP-hard problems arising in robust stability analysis”. *Mathematics of Control, Signals, and Systems*, 6:99–105.
- Nett, C., Bernstein, D., and Haddad, W. (1989). “Minimal complexity control law synthesis, part I: Problem formulation and reduction to optimal static output feedback”. *Proc. American Control Conf.*, pages 2056–2064.
- O’Reilly, J. (1978). “Optimal instantaneous output feedback controllers for discrete time systems with inaccessible state”. *Int. J. of Control*, pages 9–16.
- Peres, P., Geromel, J., and Souza, S. (1993). “Optimal  $H_2$  control by output feedback”. *Proc. IEEE Conf. on Decision and Control*, pages 102–1076.
- Pérez, F., Abdallah, C., Dorato, P., and Docampo, D. (1993). “Algebraic tests for output stabilizability”. *Proc. IEEE Conf. on Decision and Control*, pages 2385–2386.
- Poljak, S. and Rohn, J. (1993). “Checking robust nonsingularity is NP-hard”. *Mathematics of Control, Signals, and Systems*, 6:1–9.
- Rosenthal, J., Schumacher, J., and Willems, J. (1994). “Generic eigenvalue assignment by memoryless real output feedback”. Technical Report BS-R9438, CWI, P.O.Box 94079, 1090 GB Amsterdam, The Netherlands.
- Rosenthal, J. and Wang, X. (1995). “Output feedback pole placement with dynamic compensators”. Technical Report BS-R9516, CWI, P.O.Box 94079, 1090 GB Amsterdam, The Netherlands.
- Schumacher, J. (1980). “Compensator synthesis using  $(C, A, B)$ -pairs”. *IEEE Trans. Autom. Control*, AC-25:1133–1138.
- Seidenberg, A. (1954). “A new decision method for elementary algebra”. *Annals of Math.*, 60:365–374.
- Skelton, R. and Iwasaki, T. (1993). “Lyapunov and covariance controllers”. *Int. J. of Control*, 57(3):519–536.
- Söderström, T. (1978). “On some algorithms for design of optimal constrained regulators”. *IEEE Trans. Autom. Control*, AC-23:1100–1101.
- Srinathkumar, S. (1978). “Eigenvalue/eigenvector assignment using output feedback”. *IEEE Trans. Autom. Control*, AC-23:79–81.
- Syrmos, V. and Lewis, F. (1993). “Output feedback eigenstructure assignment using two Sylvester equations”. *IEEE Trans. Autom. Control*, AC-38:495–499.

- Syrmos, V. and Lewis, F. (1994). “A bilinear formulation for the output feedback problem in linear systems”. *IEEE Trans. Autom. Control*. to appear.
- Syrmos, V., Lewis, F., Zagalak, P., and Kučera, V. (1992). “A generalized Bezout equation in output feedback design”. *Proc. IEEE Conf. on Decision and Control*, pages 3590–3594.
- Syrmos, V. and Syrmos, G. (1992). “Computational design technique for reduced-order compensators”. *IEEE Trans. Autom. Control*, AC-37:1774–1778.
- Tarski, A. (1951). *A Decision Method for Elementary Algebra and Geometry*. Univ. of California Press, Berkeley. 2nd Ed.
- Toivonen, H. (1985). “A globally convergent algorithm for the optimal constant output feedback problem”. *Int. J. of Control*, pages 1589–1599.
- Toker, O. and Özbay, H. (1995). “On the NP-hardness of solving bilinear matrix inequalities and simultaneous stabilization with static output feedback”. *Proc. American Control Conf.*, pages 2056–2064.
- Trofino-Neto, A. and Kučera, V. (1993). “Stabilization via static output feedback”. *IEEE Trans. Autom. Control*, AC-38:764–765.
- Vardulakis, A. (1975). “A sufficient condition for  $n$  specified eigenvalues to be assigned under constant output feedback”. *IEEE Trans. Autom. Control*, AC-20:428–429.
- Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach*. The MIT Press, Cambridge, MA, 1st edition.
- Wang, X. (1994). “Grassmanian, central projection and output feedback pole assignment of linear systems. *Preprint*.
- Wei, K. (1990). “Stabilization of a linear plant via a stable compensator having no real unstable zeros”. *Syst. Control Lett.*, 15(3):259–264.
- Willemns, J. and Hesselink, W. (1978). “Generic properties of the pole placement problem”. In *Proc. 7th IFAC World Congress*, pages 1725–1729.
- Yasuda, K., Skelton, R., and Grigoriadis, K. (1993). “Covariance controllers: A new parametrization of the class of all stabilizing controllers”. *Automatica*, 29:785–788.
- Youla, D., Bongiorno, J., and Lu, C. (1974). “Single-loop feedback stabilization of linear multivariable dynamical systems”. *Automatica*, 10:159–173.