3-30-2012

Robust stability of time delay systems: Theory

Chaouki T. Abdallah

John Chiasson

ROBUST STABILITY OF TIME DELAY SYSTEMS: THEORY

John Chiasson * **Chaouki Abdallah** **,[1]

* *ECE Dept, University of Tennessee, Knoxville TN 37996, USA, chiasson@utk.edu*
** *ECE Dept, University of New Mexico, Alburquerque NM 87131-1356, USA, chaouki@eece.unm.edu*

Abstract: Given that a time-delay system is stable for some delay $h_0 > 0$, a procedure is given to find the stability interval $[h_1^*, h_2^*]$ such that $h_0 \in [h_1^*, h_2^*]$ and for all h satisfying $h_1^* < h < h_2^*$ the system is stable. Further, the system is shown to be unstable if $h = h_1^*$ or $h = h_2^*$. It is then shown how this can be applied to test the robust stability (with respect to delay values) of a Smith-Predictor based controller.

Keywords: Time Delay, Stability, Smith Predictor

1. INTRODUCTION

This paper considers the robust stability of time-delay systems of the form

$$\frac{d^n y}{dt^n} + \sum_{i=0}^{n-1}\sum_{j=0}^{m} a_{ij}\frac{d^i}{dt^i}y(t-jh) = 0. \qquad (1)$$

Following Kamen Kamen (1982), the characteristic equation for (1) with $z = e^{-sh}$ is

$$\begin{aligned} a(s,z) &= s^n + \sum_{i=0}^{n-1}\sum_{j=0}^{m} a_{ij}s^i z^j = 0 \qquad z = e^{-sh} \\ &= s^n + \sum_{i=0}^{n-1} a_i(z)s^i = 0 \qquad (2) \end{aligned}$$

$$a_i(z) \triangleq \sum_{j=0}^{m} a_{ij}z^j \text{ for } i = 0, 1, \ldots n-1.$$

This is the so-called commensurate delay case since all delays are integer multiples of $h > 0$. For a fixed delay $h > 0$, the polynomial $a(s, e^{-hs})$ is said to be *asymptotically stable* if and only if (iff)

$$a(s, e^{-hs}) \neq 0 \qquad \text{for } \operatorname{Re}(s) \geq 0. \qquad (3)$$

where $\operatorname{Re}(s)$ denotes the real part of s. Condition (3) implies asymptotic stability of (1) (see Bellman and Cooke (1963)Hale and Lunel (1993)Diekmann and Walther (1995)Nicolescu (2001)).

Kamen Kamen (1982) introduced the notion of a system being *asymptotically stable independent of delay.* In this formulation, the polynomial $a(s, z)$ is said to be asymptotically stable independent of delay (i.o.d.) iff Kamen (1982)

$$a(s, e^{-hs}) \neq 0 \qquad \text{for } \operatorname{Re}(s) \geq 0, h \geq 0. \qquad (4)$$

Condition (4) implies asymptotic stability for the solutions of (1). Algebraic tests exist to check for stability i.o.d. are given in Chiasson and Lee (1985)Kamen (1982). Given that the system is stable when $h = 0$, tests that determine the value h^* such that the system is stable for $0 \leq h < h^*$ and unstable when $h = h^*$ are given in Hertz and Zeheb (1984a)Hertz and Zeheb (1984b)Chiasson (1988).

In this paper, it is not assumed that the system is necessarily stable for $h = 0$. Specifically, given that a time-delay system is stable for some delay

[1] The research of C.T. Abdallah is partially supported by NSF INT-9818312

$h_0 > 0$, a procedure is given to find the stability interval (h_1^*, h_2^*) such that $h_0 \in (h_1^*, h_2^*)$ and for all $h \in (h_1^*, h_2^*)$ the system is stable. Further, it is shown that the system is unstable if $h = h_1^*$ or $h = h_2^*$. This is motivated by the fact that in some cases, the introduction of delay may have stabilizing effects (see for example Petterson and Werner (1991)Niculescu and Abdallah (1999)C.T. Abdallah and Byrne (1993)) or by the Smith predictor structure as described later in the paper.

The remainder of the paper is divided as follows: In section 2 we prove the main result and illustrate via an example taken from Petterson and Werner (1991). Section 3 discusses the robust stability of the Smith Predictor control structure using the results of section 2. Section 4 presents our conclusions and directions for future research.

2. MAIN RESULTS

Following Hertz and Zeheb (1984b)Chiasson and Lee (1985)Chiasson (1988), the idea is to note that if a system is stable for some $h_0 > 0$, then as the system transitions from a stable to an unstable regime (as the delay h is either increased or decreased), it must have poles that cross over to the $j\omega$ axis. In other words, at the point of transition, there must be roots of the form $s = j\omega, z = e^{-j\omega h}$ $(\mathrm{Re}(s) = 0, |z| = 1)$ of (2).

To exploit the above idea, an auxiliary polynomial $\tilde{a}(s, z)$ is defined as

$$\begin{aligned}\tilde{a}(s,z) &\triangleq z^m a(-s, 1/z) \\ &= z^m(-s)^n + z^m\left(\sum_{i=0}^{n-1}\sum_{j=0}^{m} a_{ij}(-s)^i(1/z)^j\right) \\ &= z^m(-1)^n s^n + \sum_{i=0}^{n-1}\sum_{j=0}^{m}(-1)^i a_{ij} s^i z^{m-j}\end{aligned}$$

Suppose at $h = h^*$, there is an $s^* = j\omega^*$ and a corresponding $z^* = e^{-j\omega^* h^*}$ such that

$$a(s^*, e^{-j\omega^* h^*}) = a(s^*, z^*) = 0$$

Then taking the complex conjugate of $a(s^*, e^{-j\omega^* h^*})$ and multiplying through by $(e^{-j\omega^* h^*})^m$ results in

$$\begin{aligned}(e^{-j\omega^* h^*})^m a(-s^*, e^{j\omega^* h^*}) &= (z^*)^m a(-s^*, 1/z^*) \\ &= \tilde{a}(s^*, z^*) = 0\end{aligned}$$

That is, for any value of delay h^* resulting in roots on the $j\omega$ axis, there is a $(s^*, z^*) \triangleq (j\omega^*, e^{-j\omega^* h^*})$ that must be a *common zero* of the two polynomials $a(s^*, z^*), \tilde{a}(s^*, z^*)$. In other words,

$$a(s^*, z^*) = \tilde{a}(s^*, z^*) = 0$$

for some (s^*, z^*) satisfying $\mathrm{Re}(s^*) = 0, |z^*| = 1$.

This observation will be combined with the following lemma to get the desired test.

Lemma (Datko) Let $\sigma(h) = \sup\{\mathrm{Re}(s) \mid a(s, e^{-hs}) = 0\}$. Then $\sigma(h)$ is a continuous function of h for $h \geq 0$.

Proof This is a result of the work of Datko Datko (1978). See also the more general results of Cooke and Ferreira Cooke and Ferreira (1983)

Given that the time-delay system (3) is stable for some delay value $h_0 > 0$, we now define h_1^*, h_2^* where $h_1^* < h_0 < h_2^*$, such that the systems remains stable for h satisfying $h_1^* < h < h_2^*$ and the system is unstable if $h = h_1^*$ or $h = h_2^*$. Note that (3) being stable for some $h = h_0 > 0$ implies $a(0, 1) \neq 0$ as seen by setting $s = 0$ in (3).

Definition Let $\{(s_i, z_i)$ for $i = 1, \cdots k\}$ be the common zeros of $\{a(s, z), \tilde{a}(s, z)\}$ for which $\mathrm{Re}(s_i) = 0, s_i \neq 0$, and $|z_i| = 1$. For each such pair (s_i, z_i), for a given h_0, let $h_{1i}^* = \max\{h \in \Re \mid h < h_0, z_i = e^{-s_i h}\}$ and $h_{2i}^* = \min\{h \in \Re \mid h > h_0, z_i = e^{-s_i h}\}$. Then define

$$\begin{aligned}h_1^* &= \min\{h_{1i}^*\} \\ h_2^* &= \min\{h_{2i}^*\}.\end{aligned}$$

With this definition, the main result is now stated and proven.

Theorem It is assumed that the time-delay system (1) is stable for some $h_0 > 0$, that is,

$$a(s, e^{-h_0 s}) \neq 0 \qquad \text{for } \mathrm{Re}(s) \geq 0. \tag{5}$$

Then

$$a(s, e^{-hs}) \neq 0 \qquad \text{for } \mathrm{Re}(s) \geq 0 \text{ and}$$

$h_1^* < h < h_2^*$

Given that the system is stable for $h = h_0$, it follows that $\sigma(h_0) < 0$. By the above lemma, increasing h above h_0, there is an h_2 which is the first value of the delay $h > h_0$ satisfying $\sigma(h_2) = 0$. For this value of h_2, there is an s_2 such that

$$a(s_2, e^{-s_2 h_2}) = 0, \ \ \mathrm{Re}(s_2) = 0.$$

That is, $s_2 = j\omega_2, z_2 = e^{-j\omega_2 h_2}(|z_2| = 1)$. Further, $s_2 \neq 0$ as $s_2 = 0$ would imply that $a(0, 1) = 0$ contradicting the assumption (5). Clearly, (s_2, z_2) also satisfies $\tilde{a}(s_2, z_2) = 0$ and thus $h_2 = h_2^*$. On the other hand, by definition, $\sigma(h_2^*) = 0$ so that the system is unstable when $h = h_2^*$. Similar arguments show there is an h_1^* such that the system is stable for $h_1^* < h \leq h_0$ and unstable when $h = h_1^*$.

Remark A simple variation of the above proof shows that if the time-delay system (3) is unstable for some delay value $h_0 > 0$, then the system remains unstable for h satisfying $h_1^* \leq h \leq h_2^*$ where h_1^*, h_2^* are defined with respect to h_0 as in Definition 2.

Remark Consider the special case where the characteristic polynomial is of the form

$$a(s,z) = a_0(s) + a_1(s)z$$

Then

$$\tilde{a}(s,z) = za_0(-s) + a_1(-s)$$

and *any* common zero of $\{a(s,z), \tilde{a}(s,z)\}$ of the form $(j\omega_0, z_0)$ satisfies

$$\begin{aligned} a(j\omega_0, z_0) &= a_0(j\omega_0) + a_1(j\omega_0)z_0 = 0 \\ &\implies z_0 = -\frac{a_0(j\omega_0)}{a_1(j\omega_0)} \\ \tilde{a}(j\omega_0, z_0) &= z_0 a_0(-j\omega_0) + a_1(-j\omega_0) = 0 \\ &\implies z_0 = -\frac{a_1(-j\omega_0)}{a_0(-j\omega_0)} \end{aligned}$$

This implies $a_0(j\omega_0)a_0(-j\omega_0) = a_1(j\omega_0)a_1(-j\omega_0)$ resulting in

$$|z_0|^2 = z_0\bar{z}_0 = \left(-\frac{a_0(j\omega_0)}{a_1(j\omega_0)}\right)\left(-\frac{a_0(-j\omega_0)}{a_1(-j\omega_0)}\right) = 1.$$

That is, *any* common zero of $\{a(s,z), \tilde{a}(s,z)\}$ of $\{a(s,z), \tilde{a}(s,z)\}$ of the form $(j\omega_0, z_0)$ must automatically satisfy $|z_0| = 1$.

Example Consider the system given by

$$G(s, e^{-hs}) = \frac{b(s, e^{-hs})}{a(s, e^{-hs})} = \frac{1 - e^{-hs}}{s + 1 - e^{-hs}}.$$

Clearly $a(s, e^{-hs})\big|_{s=0} = a(0,1) = 0$ so that $a(s, e^{-hs})$ is *unstable independent of delay*. However, the numerator of $G(s, e^{-hs})$ also has a zero at $s = 0$ so that

$$G(0,1) = \lim_{s\to 0}\frac{b(s, e^{-hs})}{a(s, e^{-hs})} = \lim_{s\to 0}\frac{1 - e^{-hs}}{s + 1 - e^{-hs}} = 1.$$

Consequently, $s = 0$ is not a pole of $G(s)$ [2]. Next, it is shown that $c(s, e^{-hs}) \triangleq a(s, e^{-hs})/b(s, e^{-hs})$ has no zeros in the *open* right-half plane for $h \geq 0$. Proceeding, define

$$\begin{aligned} c(s,z) &= \frac{a(s,z)}{b(s,z)} = \frac{s+1-z}{1-z} \\ \tilde{c}(s,z) &= zc(-s, 1/z) = \frac{z(-s+1)-1}{z-1}. \end{aligned}$$

So if $G(s, e^{-hs})$ has a zero in the rhp for some $h > 0$ then there must be a pair (s^*, z^*) $c(s^*, z^*) = \tilde{c}(s^*, z^*) = 0$ with $\mathrm{Re}(s^*) = 0, |z^*| = 1$. Eliminating s from $c(s,z) = \tilde{c}(s,z) = 0$ shows that any such z^* must satisfy

$$r(z) = 1 - z = 0.$$

However, $z = 1$ requires $s = 0$ and the pair $(0,1)$ is not a zero $G(s, e^{-hs})$. Consequently, $G(s, e^{-hs})$ is stable independent of delay.

Theorem Any linear time-invariant finite-dimensional (i.e., delay free) which is both controllable and observable can be stabilized by delayed output feedback.

Proof For ease of presentation, consider the SISO case.

$$\begin{aligned} \frac{dx}{dt} &= Ax + bu \\ y &= cx \end{aligned}$$

As (A, b) is controllable, there is a $k \in \Re^n$ such that $A - bk^T$ is stable. The expression

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ cbu \\ cAbu + cb\dot{u} \\ \vdots \\ cA^{n-1}bu + cA^{n-1}b\dot{u} + \cdots + cbu^{(n-1)} \end{bmatrix}$$

can be solved for x so that the state feedback may be written as

$$u = -k^T x = -k^T \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}^{-1} \times \left(\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \begin{bmatrix} 0 \\ cbu \\ cAbu + cb\dot{u} \\ \vdots \\ cA^{n-1}bu + \cdots + cbu^{(n-1)} \end{bmatrix} \right).$$

the system is stable (the observability hypothesis implies the invertibility of the given matrix). Now

$$\begin{aligned} \dot{y} &= \lim_{h\to 0}\frac{y(t) - y(t-h)}{h} \\ \ddot{y} &= \lim_{h\to 0}\frac{\dot{y}(t) - \dot{y}(t-h)}{h} \\ &= \lim_{h\to 0}\frac{y(t) - 2y(t-h) + y(t-2h)}{h^2} \end{aligned}$$

etc

Modify the controller by replacing the derivatives by their finite differences to get

[2] Such pole-zero cancellations for time-delay systems were pointed out to the first author by Mark Spong.

$$u \triangleq - k^T \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix}^{-1} \left(\begin{bmatrix} y(t) \\ \dfrac{y(t)-y(t-h)}{h} \\ \dfrac{y(t)-2y(t-h)+y(t-2h)}{h^2} \\ \vdots \\ \cdots \end{bmatrix} - \begin{bmatrix} 0 \\ cbu(t) \\ cAbu(t) + cb\dfrac{u(t)-u(t-h)}{h} \\ \vdots \\ \cdots \end{bmatrix} \right)$$

The closed-loop transfer function has the form

$$G(s, e^{-hs})$$

where $G(s,1) = \lim_{s\to 0} G(s, e^{-hs})$ is stable by construction. Consequently, as h is increased from 0, there is a first $h^* > 0$ such that $G(s, e^{-hs})$ has poles on the $j\omega$ axis. So, for any $0 < h < h^*$, the system is stabilizable by output feedback. In practice, one would hope to find an h large enough such that the finite differences do not amplify the noise and also small enough so that the relative stability of the system is acceptable (i.e., the poles are far enough in the lhp.).

2.1 Stabilization by Delayed Output Feedback

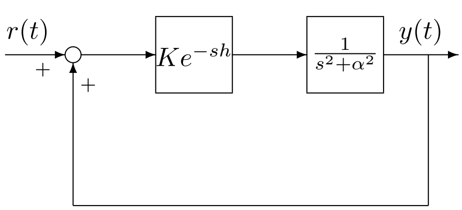

Fig. 1. Feedback stabilization of oscillatory systems.

Figure 1 shows a feedback system (from Petterson and Werner (1991)Niculescu and Abdallah (1999) Nicolescu (2001)) whose open-loop transfer function is $G_0(s) = 1/(s^2+\alpha^2)$. It follows from theorem 2 that this system is stabilizable by output feedback. However, to compare with the results in Petterson and Werner (1991)Niculescu and Abdallah (1999), the compensator is chosen to be the positive feedback $G_c(s) = Ke^{-hs}$, $K > 0$ resulting in the closed-loop transfer function

$$G_{CL}(s) = \frac{Ke^{-hs}}{s^2+\alpha^2-Ke^{-hs}}.$$

The interest here is to study the stability of this system, that is, the roots of the closed-loop characteristic polynomial $a(s, e^{-hs}) \triangleq s^2+\alpha^2 - Ke^{-hs}$.

Case 1. $0 < K < \alpha^2$.

This system is clearly unstable for $h = 0$ having poles on the $j\omega$ axis at $\pm j\sqrt{\alpha^2-K}$. The polynomials $a(s,z), \tilde{a}(s,z)$ are given by

$$a(s,z) \triangleq s^2+\alpha^2-Kz$$
$$\tilde{a}(s,z) \triangleq z\left(s^2+\alpha^2-K/z\right) = zs^2+z\alpha^2-K.$$

Eliminating s^2 gives $z^2K-K=0$ or $z=\pm 1$. The four common zeros are therefore

$$\{(s_k,z_k) \mid a(s_k,z_k) = \tilde{a}(s_k,z_k) = 0\} = \left\{(\pm j\sqrt{(\alpha^2-K)}, 1), (\pm j\sqrt{(\alpha^2+K)}, -1)\right\}.$$

Solving for the delay values corresponding to these common zeros results in

$$1 = e^{-hj\sqrt{(\alpha^2-K)}} \implies h = h_{1m} = \frac{m2\pi}{\sqrt{(\alpha^2-K)}},\ m = 0,1,2,...$$
$$-1 = e^{-hj\sqrt{(\alpha^2+K)}} \implies h = h_{2n} = \frac{(2n+1)\pi}{\sqrt{(\alpha^2+K)}},\ n = 0,1,2,...$$

The set of delays $\{h_{1m}, h_{2n} \mid m = 0,1,2,...\ n = 0,1,2,...\}$ can be arranged in increasing order. The above theorem implies that for *all* delay values between any two adjacent delays in this set, the system is either stable or unstable. Of course, for $h \in \{h_{1m}, h_{2n} \mid m = 0,1,2,...\ n = 0,1,2,...\}$, the system is unstable having poles on the $j\omega$ axis. According to Theorem **??**, to determine whether a particular interval corresponds to the system being stable requires finding a specific h in the interval for which the system is stable for this delay value. To do so, consider the first two values of h, that is, $m = 0$ and $n = 0$ resulting in $h = 0$ and $h = \pi/\sqrt{(\alpha^2+K)}$. For h small, $a(s,e^{-hs}) \triangleq s^2+\alpha^2-Ke^{-hs} \approx s^2+\alpha^2-K(1-hs) = s^2+Khs+\alpha^2-K$. As $\alpha^2 < K$ in this case, it follows that $a(s,e^{-hs})$ is stable for $h > 0$ small enough. Consequently, by the above theorem, the system is stable for all h satisfying $0 < h < h_{20} = \pi/\sqrt{(\alpha^2+K)}$. For $h < h_{20}$ the system is stable while for $h = h_{20}$ there are two poles $\left(s_0 = \pm j\sqrt{(\alpha^2+K)}\right)$ on the $j\omega$ axis. Consequently, for $h = h_{20}$ there can be *no* poles in the *open* right-half plane. Using this information, one can conclude that next interval, i.e., $h_{20} < h < \min\left\{\frac{2\pi}{\sqrt{(\alpha^2-K)}}, \frac{3\pi}{\sqrt{(\alpha^2+K)}}\right\}$[3] is

[3] A straightforward calculation shows

$$\min\left\{\frac{2\pi}{\sqrt{(\alpha^2-K)}}, \frac{3\pi}{\sqrt{(\alpha^2+K)}}\right\} = \frac{2\pi}{\sqrt{(\alpha^2-K)}} \text{ for } 0 < K < \frac{5}{13}\alpha^2 = \frac{3\pi}{\sqrt{(\alpha^2+K)}} \text{ for } \frac{5}{13}\alpha^2 < K < \alpha^2$$

unstable. To do so, $a(s, e^{-hs})$ is first expanded around the delay value h_{20} to get

$$a(s, e^{-hs}) \triangleq s^2 + \alpha^2 - Ke^{-(h_{20}+\Delta h)s} \approx s^2 + \alpha^2 - Ke^{-h_{20}s}(1 - (\Delta h)s)$$

where $\Delta h \triangleq h - h_{20}$ is small. Next s is expanded about the corresponding s_0 on the $j\omega$ axis, that is, $s_0 = \pm j\sqrt{(\alpha^2 + K)}$. Carrying out this Taylors series expansion and dropping terms of second order or higher in $\Delta h, \Delta s$ gives

$$\begin{aligned} & a(s_0 + \Delta s, e^{-(h_{20}+\Delta h)(s_0+\Delta s)}) \\ & \triangleq (s_0 + \Delta s)^2 + \alpha^2 - Ke^{-(h_{20}+\Delta h)(s_0+\Delta s)} \\ & \approx \left(2s_0 + Kh_0 e^{-h_{20}s_0}\right)\Delta s + K(\Delta h)e^{-h_{20}s_0}s_0 \\ & + s_0^2 + \alpha^2 - Ke^{-h_{20}s_0} \end{aligned}$$

Using the fact that $s_0^2 = -\left(\alpha^2 + K\right)$ and $e^{-h_{20}s_0} = -1$, this reduces to

$$\begin{aligned} & a(s_0 + \Delta s, e^{-(h_{20}+\Delta h)(s_0+\Delta s)}) \\ & \approx (2s_0 - Kh_{20})\Delta s - K(\Delta h)s_0 \end{aligned} \tag{6}$$

and is valid to first order in $\Delta h, \Delta s$. Setting equation (6) to zero and solving for Δs, results in

$$\begin{aligned} \Delta s &\approx \frac{K(\Delta h)s_0}{(2s_0 - Kh_{20})} = \frac{K(\Delta h)s_0}{2s_0 - Kh_0}\frac{2\bar{s}_0 - Kh_{20}}{2\bar{s}_0 - Kh_{20}} \\ &= \frac{K(\Delta h)}{4|s_0|^2 + (Kh_{20})^2}\left(2|s_0|^2 - Kh_{20}s_0\right) \\ \mathrm{Re}(\Delta s) &\approx \frac{2|s_0|^2 K(\Delta h)}{4|s_0|^2 + (Kh_{20})^2} > 0 \text{ for } \Delta h > 0 \end{aligned}$$

As pointed out above, for this particular system, there are *no* poles in the *open* right-half plane for $h = h_{20}$. As h is increased slightly from h_{20}, the analysis shows that the poles at $s_0 = \pm j\sqrt{(\alpha^2 + K)}$ migrate to the right-half plane since $\mathrm{Re}(\Delta s) > 0$. As the system is unstable for $h = h_{20} + \Delta h$, with $\Delta h > 0$ and small, theorem **??** implies the system is unstable for $h_{20} < h < \min\left\{\frac{2\pi}{\sqrt{(\alpha^2-K)}}, \frac{3\pi}{\sqrt{(\alpha^2+K)}}\right\}$. These results are consistent with those given in Petterson and Werner (1991)Niculescu and Abdallah (1999).

Case 2. $K > \alpha^2$

This system is clearly unstable for $h = 0$ having poles at $\sqrt{K - \alpha^2}, -\sqrt{K - \alpha^2}$. The polynomials $a(s, z), \tilde{a}(s, z)$ are given by

$$\begin{aligned} a(s, z) &\triangleq s^2 + \alpha^2 - Kz \\ \tilde{a}(s, z) &\triangleq z\left(s^2 + \alpha^2 - K/z\right) = zs^2 + z\alpha^2 - K. \end{aligned}$$

Eliminating s^2 gives $z^2K - K = 0$ or $z = \pm 1$. The four common zeros are therefore

$$\begin{aligned} & \{(s_k, z_k) \mid a(s_k, z_k) = \tilde{a}(s_k, z_k) = 0\} \\ & = \left\{(\pm\sqrt{(K - \alpha^2)}, 1), (\pm j\sqrt{(\alpha^2 + K)}, -1)\right\}. \end{aligned}$$

According to Definition 2, only the common zeros (s_k, z_k) for which $\mathrm{Re}(s_i) = 0, s_i \neq 0$, and $|z_i| = 1$ are considered. Solving for the delay values corresponding to these particular common zeros results in

$$\begin{aligned} & -1 = e^{-hj\sqrt{(\alpha^2+K)}} \\ & \quad \Longrightarrow h = h_{2n} = \frac{(2n+1)\pi}{\sqrt{(\alpha^2 + K)}}, \; n = 0, 1, 2, ... \end{aligned}$$

By the continuity property given in lemma 2, it follows that $\sigma(h) > 0$ for h small enough and, consequently, the system must be unstable for $0 < h < \pi/\sqrt{(\alpha^2 + K)}$. In the next interval, $\pi/\sqrt{(\alpha^2 + K)} < h < 3\pi/\sqrt{(\alpha^2 + K)}$ nothing can be concluded based on theorem **??**. Although the system has poles on the $j\omega$ axis for $h_{20} = \pi/\sqrt{(\alpha^2 + K)}$, a perturbation analysis cannot be used to test stability there. This is because it cannot be assumed that the system has no other poles in the open right-half plane since the system is unstable for $h < h_{20}$.

Example Petterson and Werner (1991)Niculescu and Abdallah (1999) Here the previous system is reconsidered from the point of view of theorem 2. The open-loop system is still $G_0(s) = 1/(s^2 + \alpha^2)$ with a statespace realization given by

$$\begin{aligned} \frac{dx}{dt} &= \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= c \begin{bmatrix} 1 & 0 \end{bmatrix} x. \end{aligned}$$

Following theorem 2, the control is chosen as

$$u \triangleq -\begin{bmatrix} k_1 & k_0 \end{bmatrix} \begin{bmatrix} c \\ cA \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \frac{y(t) - y(t-h)}{h} \end{bmatrix} + r$$

resulting in the closed-loop transfer function

$$\begin{aligned} \frac{Y(s, e^{-hs})}{R(s, e^{-hs})} &= \frac{1}{s^2 + \omega^2 + k_0 + \frac{k_1}{h}(1 - e^{-sh})} \\ &= \frac{1}{a(s, e^{-hs})} \end{aligned}$$

Choosing $k_0 = a_0 - \omega^2, k_1 = a_1$ results in

$$a(s, 1) = \lim_{h \to 0} a(s, e^{-hs}) = s^2 + a_1 s + a_0$$

which is stable for $a_0 > 0, a_1 > 0$. Now rewrite $a(s, e^{-sh}) = s^2 + \alpha^2 - Ke^{-sh}$ where $\alpha^2 = \omega^2 + k_0 + k_1/h = a_0 + k_1/h, K = k_1/h$. (Note that $K < \alpha^2$). The polynomials $a(s, z), \tilde{a}(s, z)$ are given by

$$\begin{aligned} a(s, z) &\triangleq s^2 + \alpha^2 - Kz \\ \tilde{a}(s, z) &\triangleq z\left(s^2 + \alpha^2 - K/z\right) = zs^2 + z\alpha^2 - K. \end{aligned}$$

The system is stable for $h = 0$ and, as h is increased, there is a first value h^* for which $a(s, e^{-h^*s})$ has roots on the imaginary axis. This means that when $h = h^*$ the polynomials $a(s, z)$ and $\tilde{a}(s, z)$ must have a common zero (s^*, z^*) satisfying $\mathrm{Re}(s^*) = 0, |z^*| = 1$. To find the common zeros, s^2 is eliminated giving $z^2K -$

$K = 0$ or $z = \pm 1$. The four common zeros are therefore

$$\begin{aligned}&\{(s_k, z_k) \mid a(s_k, z_k) = \tilde{a}(s_k, z_k) = 0\}\\ &= \left\{(\pm j\sqrt{(\alpha^2 - K)}, 1), (\pm j\sqrt{(\alpha^2 + K)}, -1)\right\}\\ &= \left\{(\pm j\sqrt{a_0}, 1), (\pm j\sqrt{a_0 + 2a_1/h}, -1)\right\}.\end{aligned}$$

Solving for the delay values corresponding to these common zeros results in

$$\begin{aligned}1 &= e^{-hj\sqrt{(\alpha^2-K)}}\\ &\Longrightarrow h = h_{1m} = \frac{m2\pi}{\sqrt{(\alpha^2 - K)}},\ m = 0, 1, 2, ...\\ -1 &= e^{-hj\sqrt{(\alpha^2+K)}}\\ &\Longrightarrow h = h_{2n} = \frac{(2n + 1)\pi}{\sqrt{(\alpha^2 + K)}},\ n = 0, 1, 2, ...\end{aligned}$$

The delay $h_{10} = 0$ is eliminated since it has already been shown the system is stable for $h = 0$. Consequently, the first value of h corresponding to roots on the imaginary axis is $h_{20} = \pi/\sqrt{a_0 + 2k_1/h}$. The system is stable for

$$0 \le h < \frac{\pi}{\sqrt{a_0 + 2a_1/h}}$$

or

$$0 \le h < \frac{-a_1 + \sqrt{a_1^2 + \pi^2 a_0}}{a_0}$$

Setting $a_0 = r^2, a_1 = 2r$ places the closed-loop poles at $-r$ when $h = 0$. The above bounds on the delay become $0 \le h < \left(-2 + \sqrt{4 + \pi^2}\right)/r = 1.724/r$. Consequently, this indicates that the further these poles are placed in the lhp, the smaller the range of stability in the delay h.

3. ROBUST STABILITY OF THE SMITH PREDICTOR

The Smith Predictor control structure shown in Figure 2 provides a methodology to stabilize a time-delay system that has a pure process delay in the open-loop plant (Power and Simpson (1978), p.230).

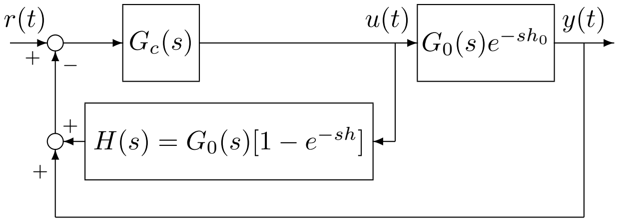

Fig. 2. The Smith-predictor feedback system.

Here the open-loop system is $G_0(s)e^{-sh_0}$ and application of the feedback $H(s) = G_0(s)[1-e^{-hs}]$ *with* $h = h_0$ results in the closed-loop system transfer function

$$\frac{G_c(s)G_0(s)}{1 + G_c(s)G_0(s)} e^{-sh_0}. \qquad (7)$$

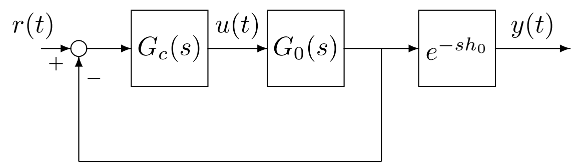

Fig. 3. The equivalent closed-loop system when $h = h_0$.

Thus, if the delay h_0 is known precisely, the feedback structure shown in Figure 3 shows that it is sufficient to stabilize the *delay-free* system $G_0(s)$ using the compensator $G_c(s)$. However, if the delay h is uncertain, then the perfect cancellation of the Smith predictor structure does not materialize. Instead, the closed-loop transfer function is given by

$$\frac{G_c(s)G_0(s)}{1 + G_c(s)G_0(s)\left[1 - (e^{-sh_0} - e^{-sh})\right]} e^{-sh_0}. \qquad (8)$$

With $G_c(s)G_0(s) = n(s)/d(s)$, the characteristic equation of (8) may be written as

$$a(s, e^{-h_0 s}, e^{-hs}) \triangleq a_0(s) + a_1(s)\left[e^{-sh_0} - e^{-sh}\right] \qquad (9)$$

where $a_0(s) = n(s)+d(s), a_1(s) = -n(s)$. Here the delays h_0, h are noncommensurate in general. By design, the compensator $G_c(s)$ is chosen so that (9) (or equivalently (7)) is stable for $h = h_0$, that is, $a_0(s)$ is stable. The interest here is to determine the range of values of h about h_0 for which (9) is stable. As h is increased (or decreased) in value from h_0, there is a first value h^* for which the roots of (9) are on the $j\omega$ axis, that is,

$$\begin{aligned}a(s, e^{-h_0 s}, e^{-h^* s}) &\triangleq a_0(s) + a_1(s)\left(e^{-sh_0} - e^{-sh^*}\right)\\ &= 0 \qquad (10)\end{aligned}$$

for some $s^* = j\omega^*$. To exploit this idea set $z_0 = e^{-sh_0}, z = e^{-sh}$ and define

$$\begin{aligned}a(s, z_0, z) &= a_0(s) + a_1(s)(z_0 - z)\\ \tilde{a}(s, z_0, z) &= z_0 z a(-s, 1/z_0, 1/z)\\ &= z_0 z\left[a_0(-s) + a_1(-s)(1/z_0 - 1/z)\right]\\ &= z_0 z a_0(-s) + a_1(-s)(z - z_0).\end{aligned}$$

Equation (10) implies that for $h = h^*$, the polynomials $\{a(s, z_0, z), \tilde{a}(s, z_0, z)\}$ must have a common zero (s^*, z_0^*, z^*) satisfying $\mathrm{Re}(s^*) = 0, |z_0^*| = 1, |z^*| = 1$.

Solving $a(s, z_0, z) = 0$ for z_0 gives

$$z_0 = z - \frac{a_0(s)}{a_1(s)} \qquad (11)$$

and substituting this into $\tilde{a}(s, z_0, z)$ results in

$$
\begin{aligned}
&\tilde{a}(s, z - \frac{a_0(s)}{a_1(s)}, z) \\
&= \frac{1}{a_1(s)}\left(z^2 a_0(-s)a_1(s) - a_0(s)a_0(-s)z + a_1(-s)a_0(s)\right) \\
&= \frac{a_0(s)a_0(-s)}{a_1(s)}\left(z^2\frac{a_1(s)}{a_0(s)} - z + \frac{a_1(-s)}{a_0(-s)}\right).
\end{aligned}
$$

As $a_0(s)$ is stable by design, $a_0(s)$ and $a_0(-s)$ are not zero on the $j\omega$ axis. Further, if $a_1(s)$ has a root on the $j\omega$ axis, then this root cannot correspond to a root of $a(s, z_0, z) = a_0(s) + a_1(s)(z_0 - z) = 0$ as this would obviously imply $a_0(s)$ has a root on $j\omega$ axis.

The problem now is to find the roots of

$$
z^2\frac{a_1(s)}{a_0(s)} - z + \frac{a_1(-s)}{a_0(-s)} = 0. \tag{12}
$$

for $\mathrm{Re}(s^*) = 0, |z^*| = 1$.

Solving (12) for z gives

$$
z = \frac{1 \pm \sqrt{1 - 4\frac{a_1(s)}{a_0(s)}\frac{a_1(-s)}{a_0(-s)}}}{2\frac{a_1(s)}{a_0(s)}} \tag{13}
$$

and substituting this into (11) gives

$$
z_0 = \frac{-1 \pm \sqrt{1 - 4\frac{a_1(s)}{a_0(s)}\frac{a_1(-s)}{a_0(-s)}}}{2\frac{a_1(s)}{a_0(s)}} \tag{14}
$$

Lemma For those values of ω for which $|a_1(j\omega)/a_0(j\omega)| < 1/2$, equation (12) with $z = e^{-hj\omega}$ cannot be zero for any h.

Proof With $z = e^{-hj\omega}$ and $\frac{a_1(j\omega)}{a_0(j\omega)} = \left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| e^{j\angle\frac{a_1(j\omega)}{a_0(j\omega)}}$, equation (12) can be rewritten as

$$
e^{-hj\omega}\frac{a_1(j\omega)}{a_0(j\omega)} - 1 + e^{hj\omega}\frac{a_1(-j\omega)}{a_0(-j\omega)} = 0
$$

or

$$
2\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right| \cos\left(h\omega + \angle\frac{a_1(j\omega)}{a_0(j\omega)}\right) = 1.
$$

Consequently, for those values of ω for which $|a_1(j\omega)/a_0(j\omega)| < 1/2$, there can be no roots of (12) on the $j\omega$ axis.

Corollary If for all ω, $|a_1(j\omega)/a_0(j\omega)| < 1/2$, then the system (9) is stable independent of the delay value h.

Proof Follows directly from lemma 3.

3.1 Stability test

This lemma shows that one need only consider (14)(19) for $|a_1(j\omega)/a_0(j\omega)| > 1/2$. Then, with h_0 the *fixed* nominal value of the delay and $z_0 = e^{-jh_0\omega}, s = j\omega$, one solves (14) for $\omega \in \{\omega : |a_1(j\omega)/a_0(j\omega)| > 1/2\}$. That is, solve

$$
e^{-jh_0\omega} = \frac{-1 \pm j\sqrt{4\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right|^2 - 1}}{2\frac{a_1(j\omega)}{a_0(j\omega)}}, \tag{15}
$$

for $\omega_i \in \{\omega : |a_1(j\omega)/a_0(j\omega)| > 1/2\}$. With $z_1 = e^{-jh\omega}, s = j\omega$ in (19), each of the solutions ω_i of (15) are then substituted into

$$
e^{-jh\omega} = \frac{1 \pm j\sqrt{4\left|\frac{a_1(j\omega)}{a_0(j\omega)}\right|^2 - 1}}{2\frac{a_1(j\omega)}{a_0(j\omega)}} \tag{16}
$$

and solved for the corresponding $h_{ik}, k = 1, 2, ...$ Then for each i, one selects the h_{ik_1} that is closest and less than h_0 and the h_{ik_2} that is closest and greater than h_0. Finally, $h_1^* = \max_i\{h_{ik_1}\}, h_2^* = \min_i\{h_{ik_2}\}$.

Example Let the system model be $G_0(s) = 1/s$ and the controller $G_c(s) = K > 0$ in Figure 2. The closed-loop characteristic polynomial is $a(s, e^{-h_0 s}, e^{-hs}) = s + K + K\left(e^{-h_0 s} - e^{-hs}\right)$. Here $a_0(s) = s + K, a_1(s) = K$ resulting in $a_1(j\omega)/a_0(j\omega) = K/(j\omega + K)$. Substituting into equation (15) results

$$
\begin{aligned}
e^{-jh_0\omega} &= \frac{-1 \pm j\sqrt{4\frac{K^2}{\omega^2+K^2} - 1}}{2K/(j\omega + K)} \\
&= -\frac{1}{2} \mp \frac{\omega}{2K}\sqrt{4\frac{K^2}{\omega^2 + K^2} - 1} \\
&+ j\left(-\frac{\omega}{2K} \pm \frac{1}{2}\sqrt{4\frac{K^2}{\omega^2 + K^2} - 1}\right) \\
&= e^{j\gamma(\omega)}
\end{aligned}
$$

With $h_0 = .25, K = .1$, this can be solved numerically by finding the roots of $r(\omega) \triangleq -h_0\omega - \gamma_i(\omega) = 0$ for $i = 1, 2$. It turns out that $r(\omega)$ has no roots so this system is stable for all $h \geq 0$.

Example Consider a variation of the previous example where $a(s, e^{-h_0 s}, e^{-hs}) = s + K - K\left(e^{-h_0 s} - e^{-hs}\right)$ with $K > 0$. Here $a_0(s) = s + K, a_1(s) = -K$ resulting in $a_1(j\omega)/a_0(j\omega) = -K/(j\omega + K)$. Substituting into equation (15) results in

$$
\begin{aligned}
e^{-jh_0\omega} &= \frac{-1 \pm j\sqrt{4\frac{K^2}{\omega^2+K^2} - 1}}{-2K/(j\omega + K)} \\
&= \frac{1}{2} \pm \frac{\omega}{2K}\sqrt{4\frac{K^2}{\omega^2 + K^2} - 1} \\
&+ j\left(\frac{\omega}{2K} \mp \frac{1}{2}\sqrt{4\frac{K^2}{\omega^2 + K^2} - 1}\right) \\
&= e^{j\gamma_i(\omega)}, i = 1, 2
\end{aligned}
$$

With $h_0 = .25, K = .1$, this can be solved numerically by finding the roots of $r(\omega) \triangleq -h_0\omega - \gamma_i(\omega) = 0$ for $i = 1, 2$. Doing so results in the single root $\omega = .0976$ giving $z_0 = e^{-jh_0\omega} |_{\omega=.0976}= 0.9997 - 0.024398j$. Substituting this into (16) (or using equation (11)) gives $z = -0.00029767 - 1.0003975789j$. Finally, h is obtained from solving

$$e^{-hj\omega} |_{\omega=.0976}= -0.00029767 - 1.0003975789j$$

giving $h = h_2^* = 16.1$ seconds. That is, for $0 \leq h < 16.1$ it follows

$$a(s, e^{-h_0 s}, e^{-hs}) \neq 0 \text{ for } \mathrm{Re}(s) \geqq 0$$

and

$$a(s, e^{-h_0 s}, e^{-h_2^* s}) = 0 \text{ for some } s = j\omega.$$

4. NUMERICAL EXAMPLE

In this section, an example is presented to illustrate the proposed technique. The double integrator (briefly discussed in the introduction, see also **?**) and some simple extensions of it are considered.

4.1 Double integrator

Let us now re-consider equation (**??**) written as

$$a(s,z) = s^2 + z[k_1 - k_2 e^{-sr_0}] \qquad (17)$$

where $z = e^{-sh}$. Next, form the auxiliary polynomial

$$\begin{aligned}\tilde{a}(s,z) &= a(-s, 1/z) \\ &= z(-s)^2 + [k_1 - k_2 e^{sr_0}] \qquad (18)\end{aligned}$$

Recalling the results in **?**, set $a(s,z) = 0$ and solve for

$$z = \frac{s^2}{k_2 e^{-sr_0} - k_1}. \qquad (19)$$

This is then substituted for z in (18) to obtain

$$\tilde{a}(s,z) = \frac{s^2(-s^2)}{k_2 e^{-sr_0} - k_1} + [k_1 - k_2 e^{sr_0}]$$

If we now let $s = j\omega$, and $\tilde{a}(j\omega, e^{-j\omega h}) = 0$, we obtain

$$\begin{aligned}(-j\omega)^2(j\omega)^2 &= (k_1 - k_2 e^{j\omega r_0})(k_1 - k_2 e^{-j\omega r_0}) \\ \omega^4 &= |k_1 - k_2 e^{-j\omega r_0}|^2. \qquad (20)\end{aligned}$$

The method here is that k_1, k_2, r_0 are chosen to satisfy have already been chosen according to (**??**) to make (17) stable when $h = 0$. Then, if the system becomes unstable as h is increased, then there must be a first value of h_0 for which (17) has a root on the $j\omega_0$ axis. Further, (18) must then be stable for $0 < h < h_0$. The point of the analysis leading to (20) is that any root on the $j\omega_0$ axis for $h = h_0$ must satisfy (20). In fact, we have the following **?**:

Proposition 1. The closed-loop system (**??**) is stable for all parameters k_1, k_2, r_0, and h in the following ranges

$$\begin{aligned} 0 &< k_1 \\ 0 &< k_2 < \frac{1+4n}{1+4n+8n^2} k_1 \\ \frac{2n\pi}{\sqrt{k_1 - k_2}} &< r_0 < \frac{(2n+1)\pi}{\sqrt{k_1+k_2}} \\ h &< h_0 \triangleq \frac{\tan^{-1}\left[\frac{-k_2\sin(\omega_0 r_0)}{-k_2\cos(\omega_0 r_0)+k_1}\right]}{\omega_0} \qquad (21)\end{aligned}$$

where ω_0 is the smallest positive root of the equation

$$\omega^4 - [k_1^2 + k_2^2 - 2k_1k_2\cos(\omega r_0)] = 0$$

Proof. Let ω_0 be the smallest positive root of

$$\omega^4 - [k_1^2 + k_2^2 - 2k_1k_2\cos(\omega r_0)] = 0.$$

This root is then substituted into equation (19)

$$e^{-j\omega_0 h} = \frac{\omega_0^2}{k_2 e^{-j\omega_0 r_0} - k_1}$$

and solved for h. This gives

$$\begin{aligned} e^{-j\omega_0 h} &= \frac{\omega_0^2}{k_2\cos(\omega_0 r_0) - k_1 - jk_2\sin(\omega_0 r_0)} \\ &= \frac{\omega_0^2}{\left| k_2\cos(\omega_0 r_0) - k_1 - j\sin(\omega_0 r_0)\right| e^{j\tan^{-1}\left(\frac{-k_2\sin(\omega_0 r_0)}{k_2\cos(\omega_0 r_0)-k_1}\right)}} \\ &= \frac{\omega_0^2}{\omega_0^2 e^{j\tan^{-1}\left(\frac{-\sin(\omega_0 r_0)}{k_2\cos(\omega_0 r_0)-k_1}\right)}} = e^{j\tan^{-1}\left(\frac{k_2\sin(\omega_0 r_0)}{k_1 - k_2\cos(\omega_0 r_0)}\right)} \\ \Longrightarrow h_0 &= \tan^{-1}\left(\frac{-k_2\sin(\omega_0 r_0)}{k_1 - k_2\cos(\omega_0 r_0)}\right)/\omega_0 \end{aligned}$$

where $k_1 - k_2\cos(\omega_0 r_0) > 0$ as $k_1 > k_2$.

■

Let us extend the analysis to the multiple-delay case for the following closed-loop system (see also **?**)

$$\ddot{y}(t) + \sum_{k=0}^{n} a_k y\left(t - \frac{k}{n}r - h\right) = 0, \qquad (22)$$

with h and r rationally independent, but we have some rational dependence in terms of r (some delays are natural multiples of $\frac{r}{n}$, etc.). It is quite evident that choosing $n = 1$, and $a_0 = k_1$, $a_1 = -k_2$, one recovers the previous example. One assumes that the system free of delays is an *oscillator*, that is

$$\sum_{k=0}^{n} a_k > 0, \qquad (23)$$

and that

$$\sum_{k=1}^{n} k a_k < 0. \quad (24)$$

Note that condition (24) corresponds to the asymptotic stability of the system (22) for *sufficiently small delays* $h = \varepsilon_1 > 0$ and $r = \varepsilon_2 > 0$. Next, we provide some stability results using the matrix pencils approach, in order to address the multiple-delay case. The matrix pencils Λ_1 and Λ_2 may be rewriten as

$$\Lambda_1(z) = z \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \\ 0 & \dots & 0 & a_n \end{bmatrix} + \\ + \begin{bmatrix} 0 & -1 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & \\ 0 & & & & & & & -1 \\ -a_n & -a_{n-1} & \dots & -a_1 & 0 & a_1 & \dots & a_{n-1} \end{bmatrix} \quad (25)$$

$$\Lambda_2(z) = z \begin{bmatrix} 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \\ 0 & \dots & 0 & a_n \end{bmatrix} + \begin{bmatrix} 0 & -1 & & 0 \\ & & \ddots & \\ 0 & 0 & & -1 \\ a_0 & a_1 & \dots & a_{n-1} \end{bmatrix}, \quad (26)$$

With the notations above and using the main result in the linear case, one has the following results (see also **?**)

Proposition 2. (First switch if $h = 0$). The system (22) satisfying the constraints (23)-(24) is asymptotically stable for all delay values r

$$0 < r < r_1(a_0, \dots, a_n),$$

where

$$r_1(a_0, \dots a_n) = \mathbf{inf}\{\gamma \quad (\gamma, \alpha) \in \Lambda_{0,+}\}. \quad (27)$$

Furthermore, if $r = 0$ or $r = r_1(a_0, \dots, a_n)$, the corresponding associated characteristic equation has at least one pair of roots on the imaginary axis.

Proposition 3. (Delay bounds). The system (22) satisfying the constraints (23)-(24) is asymptotically stable for all delays r and h

$$\begin{cases} 0 < r < r_1(a_0, \dots, a_n) \\ 0 < h < h(r) = \mathbf{min}_{\omega_s} \left\{ \dfrac{1}{\omega_s} \cdot \mathbf{arctan} \dfrac{-\sum_{k=1}^{n} a_k sin \frac{\omega_s k r}{n}}{\sum_{k=0}^{n} a_k cos \frac{\omega_s k r}{n}} \right\}, \end{cases} \quad (28)$$

where ω_s belongs to the set of positive solutions of the equation

$$\omega^4 = \sum_{k=0}^{n} a_k^2 + 2 \sum_{k=1}^{n} \sum_{h=0}^{k-1} a_k a_h cos\left(\frac{(k-h)\omega r}{n}\right). \quad (29)$$

Furthermore, assume that the chosen delay r and the solution $\tilde{\omega}_s$ defining the corresponding upper bound $h(r)$ in (28) satisfy the condition

$$\tilde{\omega}_s^3 > \frac{1}{2} \sum_{k=1}^{n} \sum_{h=0}^{k-1} \frac{a_k a_h (k-h) r}{n} sin\left(\frac{(k-h)\tilde{\omega}_s r}{n}\right). \quad (30)$$

Then

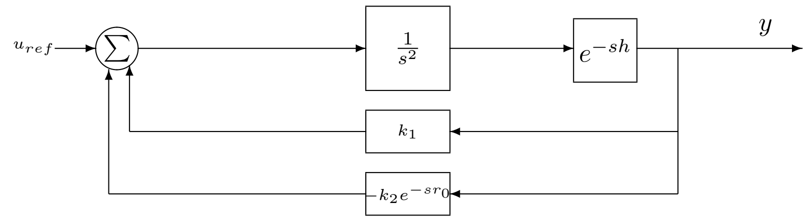

Fig. 4. The Closed-Loop System

i) if $h = h(r) + \varepsilon$, with $\varepsilon > 0$ sufficiently small, the system (22) is unstable.
ii) if the equation (29) has only one positive solution, then there does not exist any $h > h(r)$ such that the system (22) is asymptotically stable.

Note while these are stability analysis results, they may be used to provide stability designs by appropriate interpretation of h and r.

5. CONCLUSIONS

Given that a delay system is stable tor some delay $h_0 > 0$, an algorithm to compute how much the delay h can be varied about h_0 and still be stable. In a finite number of steps, the algorithm gives the stability interval $[h_1^*, h_2^*]$ such that $h_0 \in [h_1^*, h_2^*]$ and, for all h satisfying $h_1^* < h < h_2^*$, the system is stable. Further, the system was shown to be unstable if $h = h_1^*$ or $h = h_2^*$. Finally, it was shown that this test is especially useful to test the robust stability (with respect to delay values) of a Smith-Predictor based controller. Finally, a recent paper Stojić and L.S.Draganović (2001) also considers a complementary problem of the robustness of the Smith predictor due to perturbation $\Delta G_0(s)$ (the non delayed part of the plant) and using a finite power series expansion of the assumed known delay term $e^{-h_0 s}$.

REFERENCES

R. Bellman and K.L. Cooke. *Differential-Difference Equations.* New York: Academic, 1963.

J. Chiasson. A method for computing the interval of delay values for which a differential-delay system. *IEEE Transactions on Automatic Control*, 33(12):1176–1178, December 1988.

S.D.Brierley Chiasson, J.N. and E.B. Lee. A simplified derivation of the zeheb-walach 2-d stability test with applications to time-delay systems. *IEEE Transactions on Automatic Control*, 1985.

K.L. Cooke and J.M. Ferreira. Stability conditions for linear retarded functional differential equations. *Journal of Mathematical Analysis and Applications*, 96(4), October 1983.

J. Benitez-Read C.T. Abdallah, P. Dorato and R. Byrne. Delayed positive feedback can stabilize oscillatory systems. In *Proceedings of the American Control Conference*, pages 3106–3107, June 1993. San Francisco CA.

R. Datko. A procedure for determination of the exponential stability of certain differential-difference equations. *Quarterly Applied Mathematics*, October 1978.

S. A. van Gils-S. M. Verduyn Lunel Diekmann, O. and H.O. Walther. *Delay Equations*. Springer-Verlag, 1995.

J.K. Hale and S.M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer-Verlag, 1993.

E.I. Jury Hertz, D. and E. Zeheb. Simplified analytic stability test for systems with commensurate time delays. *IEE Proceedings*, 131, 1984a. part D.

E.I. Jury Hertz, D. and E. Zeheb. Stability independent and dependent of delay for delay differential systems. *J. Franklin Institute*, 1984b.

E.W. Kamen. Linear systems with commensurate time delays: Stability and stabilization independent of delay. *IEEE Transactions on Automatic Control*, pages 367–375, April 1982.

S.I. Nicolescu. *Delay effects on Stability*. Springer-Verlag, 2001. Berlin.

S.I. Niculescu and C. T. Abdallah. Delay effects on static ouput feedback stabilization. In *Proceedings of the IEEE CDC*, pages 53–54, 1999. Sydney, Australia.

R.D. Robinett Petterson, B.J. and J.C. Werner. Lag-stabilized force feedback damping. *SAMD91-0194, UC-406*, 1991.

H.M. Power and R.J. Simpson. *Introduction to Dynamics and Control*. McGraw-Hill, 1978.

M.S. Matjević Stojić, M.R. and L.S.Draganović. A robust smith predictor modified by internal models for integrating process with dead time. *IEEE Transactions on Automatic Control*, 46: 1293–1298, 2001.