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Sasakian geometry on lens space bundles over Riemann surfaces

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DISSERTATION

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Dedication

A la memoria de mi padre.

Para mi mami.

Para mi esposa,Junior y Meredith que les debo mas de lo que puedo expresar.

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Abstract

We study the Sasakian Geometry on Lens Space bundles over Riemann Surfaces, denoted $M^5_{l_1,l_2,\mathbf{w}}$. We describe the quotient $M^5_{l_1,l_2,\mathbf{w}}/S^1(v)$, where $S^1(v)$ is a circle action generated by the Reeb vector field ξ_v, as a complex fiber bundle over Σ_g whose fiber is a weighted projective space. We compute the cohomology of $M^5_{l_1,l_2,\mathbf{w}}$ and we see the depence of that cohomology on l_2. We show the existence of extremal Sasakian metrics on $M^5_{l_1,l_2,\mathbf{w}}$ with constant scalar curvature.

Contents

Chapter 1

Introduction

1.1 Introduction

In this dissertation we study Sasakian metrics of certain lens space bundles over a Riemann surface of genus $g \geq 1$. In the paper [BTF14a] Boyer and Tonnesen-Friedman studied S^3 bundles over a Riemann surface, they showed there that there are two homotopy types of those; the trivial bundle and the non trivial one. Furthermore they showed the existence of extremal Sasakian metrics on those manifolds. These manifolds can be constructed using the join, so these manifolds are written

$$M^5_{l_1,1,\mathbf{w}} = M^3_g *_{l_1,1} S^3_{\mathbf{w}}$$

where l_1, w_1, w_2 are certain integer parameters. In this dissertation we continue studying these manifolds written as

$$M^5_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}$$

with this notation Boyer and Tonnesen-Friedman did the case $l_2 = 1$. We do the case $l_2 > 1$. The problem of finding the "best" Kähler metrics in a given Kähler class on a Kähler manifold has a long and well developed history beginning with Calabi [Cal82]. He introduced the notion of an *extremal Kähler metric* as a critical point of the L^2-norm squared of the scalar curvature of a Kähler metric, and showed that extremal Kähler metrics are precisely those such that the $(1,0)$-gradient of the scalar curvature is a holomorphic vector field. Particular cases are Kähler-Einstein metrics and more generally those of *constant scalar curvature (CSC).*

On the other hand in the case of the odd dimensional version, namely Sasakian geometry, aside from the study of Sasaki-Einstein metrics, the study of extremal Sasaki metrics is much more recent [BGS08]. It is well known that a Sasaki metric has constant scalar curvature if and only if the transverse Kähler metric has constant scalar curvature. So for quasi-regular Sasaki metrics the existence of CSC Sasaki metrics coincides with the existence of CSC Kähler metrics on the quotient cyclic orbifolds $\mathcal{Z}$.

The main results of this dissertation are:

Proposition 1.1. *Let* $\mathbf{v} = (v_1, v_2)$ *with* $v_1, v_2 \in \mathbb{Z}^+$ *and* $\gcd(v_1, v_2) = 1$*, and let* $\xi_{\mathbf{v}}$ *be* *a Reeb vector field in the Sasaki cone $\mathfrak{t}_2^+(\mathbf{w})$. Then the quotient of*

$$M^5_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}$$

with

$$S^1 \to M^3_g \to \Sigma_g$$

by the circle action $S^1(\mathbf{v})$ generated by $\xi_{\mathbf{v}}$ is a complex fiber bundle over Σ_g whose fiber is the complex orbifold $\mathbb{CP}(\mathbf{v})/\mathbb{Z}_{\frac{l_2}{s}}$. Where

$$s = \gcd(|w_2v_1 - w_1v_2|, l_2).$$

Proposition 1.2. *The orbifold pseudo-Hirzebruch surface*

$$B_{l_1,l_2,\mathbf{v},\mathbf{w}} = M^5_{l_1,l_2,\mathbf{w}}/S^1(v)$$

can be realized as the orbifold log pair $(S_n, \Delta_{\mathbf{v}})$ where S_n is a pseudo-Hirzebruch surface of degree $n = \frac{l_1(w_1v_2 - w_2v_1)}{s}$. Where

$$s = \gcd(|w_2v_1 - w_1v_2|, l_2)$$

and the branch divisor $\Delta_{\mathbf{v}}$ is given by

$$\Delta_{\mathbf{v}} = (1 - \frac{1}{v_1})E_n + (1 - \frac{1}{v_2})E'_n \tag{1.1}$$

where E'_n is the infinity section and E_n is the zero section.

We also have:

Proposition 1.3. *The cohomology groups of $M^5_{l_1,l_2,\mathbf{w}}$ are given by*

$$H^r(M^5_{l_1,l_2,\mathbf{w}},\mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } r = 0,5 \\ \mathbb{Z}^{2g} & \textit{for } r = 1 \\ \mathbb{Z} & \textit{for } r = 2 \\ \mathbb{Z}\oplus\mathbb{Z}^{2g}_{l_2} & \textit{for } r = 3 \\ \mathbb{Z}^{2g} & \textit{for } r = 4 \end{cases}$$

Moreover the cohomology ring is given by

$$\mathbb{Z}[\alpha_i,\beta_i,x,y]/(J,x^2,xy=\gamma_g,l_2\alpha_i x,l_2\beta_i x)$$

where J is an ideal described by: For the canonical homology basis of Σ_g

$$H_1(\Sigma_g) = \{a_1,a_2,\cdots,a_g,b_1,b_2,\cdots,b_g\}$$

and

$$H^1(\Sigma_g) = \{\alpha_1,\alpha_2,\cdots\alpha_g,\beta_1,\beta_2,\cdots,\beta_g\}$$

the dual basis; then

$$\alpha_j\cup\beta_k = -\delta_{jk}x$$

for $j,k = 1,2,\cdots,g$ where deg x=2,deg y=3 and γ_g is the orientation class of $M^5_{l_1,l_2,\mathbf{w}}$.

For the homotopy groups we have:

Proposition 1.4.

$$\pi_1(M^5_{l_1,l_2,\mathbf{w}}) = \pi_1(M^3_g)/(l_2\mathbb{Z})$$

$$\pi_2(M^5_{l_1,l_2,\mathbf{w}}) = 0$$

$$\pi_i(M^5_{l_1,l_2,\mathbf{w}}) = \pi_i(S^3)$$

for $i > 2$.

For the existence of extremal Sasakian metrics we have :

Theorem 1.1. *For any choice of genus* $g = 1, 2, ..., 19$ *the regular ray in the Sasaki cone* $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ *admits an extremal representative with non-constant scalar curvature.*

For any choice of genus $g = 20, 21, ...$ *there exists a* $K_g \in \mathbb{Z}^+$ *such that if* $l_1|\mathbf{w}| \geq K_g$*, then the regular ray in the Sasaki cone* $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ *admits an extremal representative with non-constant scalar curvature.*

For any choice of genus $g = 20, 21, ...$ *there exist at least one choice of* (l_1, w_1, w_2) *such that the regular ray in the Sasaki cone* $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ *admits no extremal representative, despite the fact that the quasi-regular Sasaki structure*

$$\mathcal{S}_{l_1,l_2,\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{l_1,l_2,\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$$

is extremal.

We also have:

Theorem 1.2. *Let*

$$M^5_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}.$$

Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ *with gcd*$(w_1, w_2) = 1$ *and* $w_1 > w_2$ *there exists a* ξ_v *in the Sasaki cone on* $M^5_{l_1,l_2,\mathbf{w}}$ *such that the corresponding ray of Sasakian structures*

$$\mathcal{S}_a = (a^{-1}\xi_v, a\eta_v, \Phi, g_a)$$

has constant scalar curvature.

Most of these metrics are irregular.

Chapter 2

Basic Concepts

2.1 Spectral Sequences

2.1.1 Exact Couples

For the fundamentals of spectral sequences we refer to [BT82].

An Exact Couple is an exact sequence of Abelian groups of the form $A \xrightarrow{i} A \xrightarrow{j} B \xrightarrow{k} A$ where i,j, k are group homomorphisms.Define $d : B \to B$ by $d = j \circ k$.Then $d^2 = j(k \circ j)k = 0$ so the homology group $H(B) = Kern\ d/Im\ d$ is defined. Out of a given exact couple we can construct a new exact couple, called the derived couple: $A' \xrightarrow{i\prime} A' \xrightarrow{j\prime} B' \xrightarrow{k\prime} A'$

by making the following definitions:

- $A' = i(A)$, $B' = H(B)$.
- $i'i(a) = i(i(a))$.
- If $a' = ia \in A'$,$a \in A$ then $j'a' = [ja]$. And $j'$is well defined.
- k' is induced from k.Let $[b] \in H(B)$ then $jkb = 0$ so $kb = ia$ for some $a \in A$. Define $k'[b] = kb \in i(A)$. With these definitions (1) is an exact couple.

2.2 The Spectral Sequence of a Filtered Complex

Let K be a differential complex with differential operator D, i. e. K is an Abelian group and $D : K \to K$ a group homomorphism such that $D^2 = 0$. A subcomplex $K'$of K is a subgroup such that $DK' \subset K'$. A sequence of subcomplexes $K = K_0 \supset K_1 \supset K_2 \supset \cdots$ is called a filtration on K. This makes K into a filtered complex with associated graded complex: $GK = \bigoplus K_p/K_{p+1}$ Let $A = \bigoplus K_p$. A is a differential complex with operator D.Define $i : K_{p+1} \to K_p$ the inclusion and B the quotient $0 \to A \to A \to B \to 0$ then B is the associated graded complex GK of K.In the previous short exact sequence each group is a complex with operator induced from D. In the graded case we get from this short exact sequence a long exact sequence of cohomology: $\cdots \to H^k(A) \to H^k(A) \to H^k(B) \to H^{k+1}(A) \to \cdots$ which we may write as

$$H(A) \xrightarrow{i_1} H(A) \xrightarrow{j_1} H(B) \xrightarrow{k_1} H(A)$$

this sequence we write as

$$A_1 \xrightarrow{i} A_1 \xrightarrow{j_1} B_1 \xrightarrow{k_1} A_1$$

where the map i need no longer be an inclusion.Since this diagram is an exact couple, it gives rise to a sequence of exact couples

$$A_r \xrightarrow{i} A_r \xrightarrow{j_r} B_r \xrightarrow{k_r} A_r$$

each being the derived couple of its predecessor. The sequence of subcomplexes $K \supset K_1 \supset K_2 \supset \cdots$ induces a sequence in cohomology $\cdots H(K) \leftarrow H(K) \leftarrow H(K_1) \leftarrow H(K_2) \cdots$

Let F_p be the image of F_p in $H(K)$. Then there is a sequence of inclusions

$$H(K) = F_0 \supset F_1 \supset F_2 \supset \cdots$$

making $H(K)$ into a filtered complex this; this filtration is called the induced filtration on $H(K)$. A filtration K_p on the filtered complex K is said to have length l if $K_l \neq 0$ and $K_p = 0$ for $p > l$.Whenever the filtration on K has finite length then A_r and B_r are eventually stationary and the stationary value B_∞ is the associated graded complex $\bigoplus F_p/F_{p+1}$ of the filtered complex $H(K)$ with filtration given by (3). It is customary to write E_r for B_r. Hence $E_1 = H(B)$ with differential $d_1 = j_1k_1$. $E_2 = H(E_1)$ with differential $d_2 = j_2k_2$ $E_3 = H(E_2)$ etc. A sequence of differential groups $\{E_r, d_r\}$ in which each E_r is the homology of its predecessor is called a **spectral sequence**. If E_r eventually becomes stationary, we denote the stationary value by E_∞. If E_∞ is equal to the associated graded group of some filtered group H, then we say that the spectral sequence converges to H.

The proof of the following theorem is in [Spa66].

Theorem 2.1. *[Spa66] Given a fibration*

$$p : E \to B$$

with fiber F there is a spectral sequence E_r converging to $H^*(E)$ with

$$E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q)$$

with a good cover $\mathcal{U}$ of B, where $\mathcal{H}^q$ is the locally constant presheaf $\mathcal{H}^q(U) = H^q(\pi^{-1}(U))$ on $\mathcal{U}$. If B is simply connected and $H^q(F)$ is finite dimensional then

$$E_2^{p,q} = H^p(M) \otimes H^q(F)$$

2.3 Orbifolds

For the fundamentals of Orbifolds we refer to [BG08].

Definition 2.1. *Let X be a paracompact Hausdorff space. An orbifold chart or local uniformizing system on X is a triple $(\tilde{U}, \Gamma, \phi)$, where $\tilde{U}$ is connected, open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$, Γ is a finite group acting effectively as of $\tilde{U}$ and $\phi : \tilde{U} \to U$ is a continuous map onto an open set $U \subset X$ such that $\phi \circ \gamma = \phi$ for all $\gamma \in \Gamma$ and the induced natural map of*

$$\tilde{U}/\Gamma \longrightarrow U$$

is a homeomorphism. An embedding between two such charts $(\tilde{U}, \Gamma, \phi)$ and $(\tilde{U}\prime, \Gamma\prime, \phi\prime)$

is a smooth (or holomorphic) embedding

$$\lambda : \tilde{U} \longrightarrow \tilde{U}\prime$$

such that $\phi\prime \circ \lambda = \phi$. *An orbifold atlas on* X *is a collection* $\mathcal{U} = (\tilde{U}_i, \Gamma_i, \phi_i)$ *of orbifolds charts such that*

- $X = \bigcup \phi(\tilde{U}_i)$

- *Given two charts* $(\tilde{U}_i, \Gamma_i, \phi_i)$ *and* $(\tilde{U}_j, \Gamma_j, \phi_j)$ *with* $U_i = \phi(\tilde{U}_i)$ *and* $U_j = \phi(\tilde{U}_j)$ *and a point* $x \in U_i \bigcap U_j$ *there exist an open neighborhood* U_k *of* x *and a chart* $(\tilde{U}_k, \Gamma_k, \phi_k)$ *such that there are injections*

  $$\lambda_{ik} : (\tilde{U}_k, \Gamma_k, \phi_k) \longrightarrow (\tilde{U}_i, \Gamma_i, \phi_i)$$

  and

  $$\lambda_{jk} : (\tilde{U}_k, \Gamma_k, \phi_k) \longrightarrow (\tilde{U}_j, \Gamma_j, \phi_j).$$

An atlas $\mathcal{U}$ *is said to be a refinement of an atlas* $\mathcal{V}$ *if there is an injection of every chart of* $\mathcal{U}$ *into every chart of* $\mathcal{V}$. *Two orbifolds atlases are said to be equivalent if they have a common refinement. A smooth orbifold is a paracompact Hausdorff space* X *with an equivalence class of orbifold atlases. We denote the orbifold by* $\mathcal{X} = (X, U)$. *If every finite group* Γ *consists of orientation preserving diffeomorphisms and there is an atlas such that all the injections are orientation preserving the orbifold is orientable.*

2.4 The Join construction

For the Join Construction we refer to [BG08].

We denote by $\mathcal{SO}$ the set of compact quasiregular Sasakian orbifolds, by $\mathcal{SM}$ the subset of $\mathcal{SO}$ that are smooth manifolds and by $\mathcal{R} \subset \mathcal{SM}$ the subset of compact regular Sasakian manifolds. For each pair of positive integers (k_1, k_2) we define a graded multiplication

$$*_{k_1,k_2} : \mathcal{SO}_{2n_1+1} \times \mathcal{SO}_{2n_2+1} \to \mathcal{SO}_{2(n_1+n_2)+1}$$

by: Let M_1 and $M_2 \in \mathcal{SO}$ of dimension $2n_1 + 1$ and $2n_2 + 1$ respectively. Since each orbifold (M_i, S_i) has a quasiregular Sasakian structure S_i; its Reeb vector field generates a locally free circle action and the quotient space by this action has a natural orbifold structure $\mathcal{Z}_i$. So there is a locally free action of the torus T^2 on $M_1 \times M_2$ and the quotient orbifold is the product of the orbifolds $\mathcal{Z}_i$. The Sasakian structure on M_i determines a Kähler structure ω_i on the orbifold $\mathcal{Z}_i$ but in order to obtain an integral orbifold cohomology class $[\omega_i] \in H^2_{orb}(\mathcal{Z}_i, \mathbb{Z})$ we need to assure that the period of a generic orbit is 1.By a result of Wadsley the period function on a quasiregular Sasakian orbifold is lower semicontinuos and constant on the dense open set of regular orbits;because on Sasakian orbifolds all Reeb orbits are geodesics. By a transverse homothety we can normalize the period function to be 1; on the dense open set of regular orbits.In this case the Kähler forms ω_i define integer orbifold cohomology classes $[\omega_i] \in H^2_{orb}(\mathcal{Z}_i, \mathbb{Z})$. Each pair of positive integers k_1, k_2 give a Kähler form $k_1\omega_1 + k_2\omega_2$ on the product, and $[k_1\omega_1 + k_2\omega_2] \in H^2_{orb}(\mathcal{Z}_1 \times \mathcal{Z}_2, \mathbb{Z})$ and so defines a V-bundle over the orbifold $\mathcal{Z}_1 \times \mathcal{Z}_2$ whose total space is an orbifold that we denote by $M_1 *_{k_1,k_2} M_2$. This is called the (k_1, k_2) join of (M_1, S_1) and (M_2, S_2).

Chapter 3

Kähler manifolds

3.1 Kähler Manifolds

Let M be a real manifold of dimension $2n$. An almost complex structure J is a smooth section of $\mathrm{End}(TM)$, such that $J^2 = -\mathbb{I}$. We can extend J to act on $TM \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathbb{C}$-linearity. Then J induces a splitting $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}(M) \oplus T^{0,1}(M)$ where $T^{1,0}(M), T^{0,1}(M)$ are eigenspaces with eigenvalues $\pm i$ respectively.

An almost complex structure is said to be integrable if M admits an atlas of complex charts with holomorphic transition functions such that J corresponds to the induced complex multiplication on $TM \otimes_{\mathbb{R}} \mathbb{C}$.

Let (M, J) be an almost complex manifold and let g be a Riemannian metric on M such that

$$g(JX, JY) = g(X, Y), X, Y \in \Gamma(TM)$$

then g is called an Hermitian metric and (M, J, g) is called an almost Hermitian manifold. Given g a Hermitian metric we define its fundamental 2-form ω_g of g by

$$\omega_g(X, Y) = g(X, JY), X, Y \in \Gamma(TM)$$

and (J, g, ω_g) is called an almost Hermitian structure on M.

An almost Hermitian structure with integrable J is called Hermitian. A hermitian manifold (M, J, g, ω_g) is said to be Kähler if ω_g is a closed 2-form. g is called a Kähler metric, ω_g its Kähler form and (J, g, ω_g) a Kähler structure on M.

3.2 Kähler Orbifolds

It is straightforward to generalize the Geometry of Kähler manifolds as seen above to Kähler Orbifolds. If $\mathcal{X} = (X, \mathcal{U})$ is a complex orbifold with an Hermitian metric g and corresponding 2 form ω_g defined by

$$\omega_g(X, Y) = g(X, JY)$$

Then $\mathcal{X}$ is a Kähler orbifold if ω_g is a closed 2 form. Here g,J and ω_g are sequences of Γ_i invariant Hermitian metrics, almost complex structures and 2 forms respectively, on each local uniformizing neighborhood that are compatible with the injections.

Definition 3.1. *The branch divisor Δ of an orbifold $\mathcal{X} = (X, \mathcal{U})$ is a $\mathbb{Q}$ divisor on X of the form*

$$\Delta = \sum(1 - \frac{1}{m_\alpha})D_\alpha,$$

where the sum is taken over all Weil divisors D_α that lie in the orbifold singular locus $\sum^{orb}(\mathcal{X})$ and m_α is the gcd of the orders of the local uniformizing groups taken over all points of D_α and is called the ramification index of D_α.

Definition 3.2. *A canonical divisor K_X is any divisor on X such that its restriction to X_{reg} is associated to the canonical bundle $\Lambda^n_{X_{reg}}$.*

Definition 3.3. *A Baily divisor on an orbifold $\mathcal{X}$ is a Cartier divisor $D_{\tilde{U}}$ on each local uniformizing system $(\tilde{U}, \Gamma, \phi)$ that satisfies:*
i) for each $x \in X$ and $\gamma \in \Gamma$, and $f \in D_{\gamma x}$ implies $f \circ \gamma \in D_x$.
ii) if

$$\lambda : (\tilde{U}, \Gamma, \phi) \to (\tilde{U}^{'}, \Gamma^{'}, \phi^{'})$$

is an injection and $f \in D^{'}_{\lambda(x)}$ then $f \circ \lambda \in D_x$, where D is the divisor sheaf on $\tilde{U}$.

Definition 3.4. *The orbifold canonical divisor $K^{orb}_{\mathcal{X}}$ is any Baily divisor associated to the canonical orbibundle $K_{\mathcal{X}}$.*

We have

Theorem 3.1. *[BG08] The orbifold canonical divisor $K^{orb}_{\mathcal{X}}$ and canonical divisor K_X are related by*

$$K^{orb}_{\mathcal{X}} = \phi^* K_X + \sum (1 - \frac{1}{m_\alpha}) \phi^* D_\alpha$$

where

$$\phi_i : \tilde{U}_i \to U_i$$

are the local branch covers.

We have also

Theorem 3.2. *[BG08] For $G = U(n)$. The orbifold Chern classes $c_i^{orb}(E)$ are defined for any V-bundle E on the orbifold $\mathcal{X}$ with group G. Moreover, $p^* c_i^{orb}(E)$ are integral classes in $H^*_{orb}(\mathcal{X}, \mathbb{Z})$, here $p : B\mathcal{X} \to X$, is the Haefliger classifying space [Hae84].*

Chapter 4

Contact Geometry

Definition 4.1. *A manifold M^{2n+1} is a (strict)* **contact manifold** *if there exists a 1-form η, called a* **contact 1-form***, on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A **contact structure** *on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.*

A contact structure gives rise to a codimension one subbundle of TM, denoted $\mathcal{D}$ by $\mathcal{D} = \ker\eta$, $\mathcal{D}$ is called the *contact subbundle.*

If the vector bundle $\mathcal{D}$ is oriented we say that the contact manifold M is **co-oriented**.

Lemma 4.1. *[BG08] On a strict contact manifold (M, η) there is a unique vector* *field ξ, called the* **Reeb vector field**, *satisfying*

$$\xi\rfloor\eta = 1, \qquad \xi\rfloor d\eta = 0.$$

The Reeb vector field uniquely determines a 1-dimensional foliation $\mathcal{F}_\xi$ on (M,η) called the *characteristic foliation*. Let L_ξ denote the trivial line bundle consisting of tangent vectors that are tangent to the leaves of $\mathcal{F}_\xi$. This splits the tangent bundle as $TM = \mathcal{D} \bigoplus L_\xi$.

Definition 4.2. *The foliation $\mathcal{F}_\xi$ of a strict contact structure is said to be quasi-regular if there is a positive integer k such that each point has a foliated coordinate chart $(\mathcal{U}, x)$ such that each leaf of $\mathcal{F}_\xi$ passes through $\mathcal{U}$ at most k times. If $k = 1$ the foliation is called regular. If $\mathcal{F}_\xi$ is not quasi-regular it is said to be irregular. We say also that the Reeb vector field ξ is regular (quasi-regular, irregular).*

We have:

Theorem 4.1. *[BW58, BG00] Let $(\mathcal{M}, \eta)$ be a regular compact strict manifold. Then $\mathcal{M}$ is the total space of a principal circle bundle $\pi : \mathcal{M} \longrightarrow Z, \quad Z = \mathcal{M}/\mathcal{F}_\xi$, and Z is a compact symplectic manifold with symplectic form Ω, $[\Omega] \in H^2(Z, \mathbb{Z})$ and η is a connection form on the bundle with curvature $d\eta = \pi^*\Omega$.*

We are going to use this theorem in the orbifold version the proof of which is in [BG00].

Definition 4.3. *A (strict)* **almost contact structure** *on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$, ξ is a vector field, η is* *a 1-form satisfying*

$$\eta(\xi) = 1 \quad \textit{and} \quad \Phi \circ \Phi = -\mathbb{I} + \xi \otimes \eta,$$

. A smooth manifold with such a structure is called an **almost contact manifold***.*

Definition 4.4. *Let M be a (strict) almost contact manifold. A Riemannian metric g on M is said to be* **compatible** *with the almost contact structure if for any vector fields X, Y on M we have:*

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y).$$

An almost contact structure with a compatible metric is called an **almost contact metric structure***.*

Lemma 4.2. *[BG08] Let (M, ξ, η, Φ) be an almost contact manifold. Then the following are equivalent:*

(i) There exists a compatible Riemannian metric on M such that the orbits of ξ are geodesics.

(ii) $\pounds_\xi \eta = 0$.

(iii) $\xi \rfloor d\eta = 0$.

Definition 4.5. *An almost complex structure J in $\mathcal{D} = ker\ \eta$, η a 1-form, is said to be compatible with the symplectic form $d\eta$ if $d\eta(JX, JY) = d\eta(X, Y)$ for all vector fields X, Y, and $d\eta(JX, X) > 0$ for all $X \neq 0$ hold. We can extend the endomorphism J to an endomorphism Φ on all of TM by setting $\Phi\xi = 0$. So we have the conditions*

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y) \textit{ for all } X, Y, \qquad d\eta(\Phi X, X) > 0 \textit{ for all } X \neq 0, \qquad (4.1)$$

and (ξ, η, Φ) defines an almost contact structure on M.

Definition 4.6. *Let (M, η) be a contact manifold. Then an almost contact structure (ξ, η', Φ) is said to be* **compatible** *with the contact structure if $\eta' = \eta$, ξ is its Reeb vector field and the endomorphism Φ satisfies the conditions 4.1. We denote by $\mathcal{AC}(\eta)$ the set of compatible almost contact structures on (M, η).*

Every compatible almost contact structure is determined uniquely by the endomorphism Φ which in turn is determined uniquely by the almost complex structure on $\mathcal{D}$. So we can give the set $\mathcal{AC}(\eta)$ the subspace topology of the space of sections $\Gamma(\mathrm{End}\, TM)$ with the C^∞ compact open topology. We have

Theorem 4.2. *[BG08] Let (M, η) be a strict contact manifold. Then there is a one to one correspondence between compatible almost complex structures J on the symplectic vector bundle $(\mathcal{D}, d\eta)$ and compatible almost contact structures (ξ, η, Φ). Moreover, the space $\mathcal{AC}(\eta)$ is contractible.*

Definition 4.7. *A (strict) contact manifold (M, η) with a compatible strict almost contact metric structure (ξ, η, Φ, g) such that $g(X, \Phi Y) = d\eta(X, Y)$ is called a* **contact metric structure**.

Definition 4.8. *A contact metric structure (ξ, η, Φ, g) is called* **K-contact** *if ξ is a Killing vector field of g, i.e. if $\pounds_\xi g = 0$. A manifold with such a structure is called a* **K-contact manifold**.

Theorem 4.3. *[BG08] On a complete contact metric manifold (M, η, g), the following conditions are equivalent:*

1. *The characteristic foliation $\mathcal{F}_\xi$ is a Riemannian foliation.*

2. *g is bundle-like.*

3. *The Reeb flow is an isometry.*

4. *The Reeb flow is a $\mathcal{CR}$ transformation.*

5. *The Reeb flow leaves the $(1,1)$ tensor field Φ invariant.*

6. *The contact metric structure (M, η, g) is K-contact.*

Let M be a smooth manifold, and consider the *cone on M* as $C(M) = M \times \mathbb{R}^+$. We identify M with $M \times \{1\}$.

In this case the natural Riemannian structure on $C(M)$ is not the product metric, but the so-called *warped product $M \times_{r^2} \mathbb{R}^+$*, where r denotes the coordinate on $\mathbb{R}^+$.

Definition 4.9. *For any Riemannian metric g_M on M, the warped product metric on $C(M) = \mathbb{R}^+ \times M$ is the Riemannian metric defined by*

$$g = dr^2 + \phi^2(r) g_M, \quad r \in \mathbb{R}^+$$

and $\phi(r)$ is a smooth function. If $\phi(r) = r$ then $(C(M), g)$ is called the metric cone onM.

There is a one-to-one correspondence between Riemannian metrics on M and cone metrics on $C(M)$. The cone metric admits a group of homothety transformations defined by $(x, r) \mapsto (x, \lambda r)$. The infinitesimal generator of the homothety group is the Liouville vector field defined by

$$\Psi = r \frac{\partial}{\partial r}.$$

The cone metric is homogeneous of degree 2 and satisfies

$$\pounds_\Psi g = 2g.$$

Given an almost contact structure (ξ, η, Φ) on M we define a section I of $End(TC(M))$ by

$$IY \;=\; \Phi Y + \eta(Y)\Psi$$

and

$$I\Psi \;=\; -\,\xi.$$

Then I defines an almost complex structure on $C(M)$ that is homogeneous of degree 0 in r. I is invariant under the flow of Ψ and $\pounds_\Psi I = 0$.

Conversely, we can begin with an almost complex structure I on $C(M)$ such that

1. I is invariant under Ψ,.
2. $I\Psi$ is tangent to M.

From this one sees that $\xi = I\Psi$ defines a nowhere vanishing vector field on M. Letting L_ξ denote the trivial line bundle generated by ξ, we have an exact sequence

$$0 \longrightarrow L_\xi \longrightarrow TM \longrightarrow Q \longrightarrow 0,$$

and there is a one-to-one correspondence between the splittings of this exact sequence and 1-forms η on M that satisfy $\eta(\xi) = 1$. This correspondence is given by $\eta \mapsto \ker \eta$.

We can define an endomorphism Φ of TM by

$$\Phi = \begin{cases} I, & \text{on } \ker \eta \\ 0 & \text{on } L_\xi. \end{cases} \tag{4.2}$$

then $\Phi^2 = -\mathbb{I} + \xi \otimes \eta$, so (ξ, η, Φ) defines an almost contact structure on M.

Definition 4.10. *A* **symplectic cone** *is a cone $C(M) = M \times \mathbb{R}^+$ with a symplectic structure ω which has a one parameter group ρ_t of homotheties whose infinitesimal generator is a vector field on $\mathbb{R}^+$.*

We have:

Proposition 4.1. *[BG08] Let η be a 1-form on the manifold M. Then η defines a strict contact structure on M if and only if the 2-form $d(r^2\eta)$ defines a symplectic structure on $C(M) = M \times \mathbb{R}^+$.*

and

Proposition 4.2. *[BG08] There is a one-to-one correspondence between the contact metric structures (ξ, η, Φ, g) on M and almost Kähler structures $(dr^2 + r^2 g, d(r^2\eta), I)$.*

and also

Proposition 4.3. *[BG08] A contact metric structure (ξ, η, Φ, g) is K-contact if and only if $\Psi - i\xi$ is pseudo-holomorphic with respect to the almost Kähler structure $(dr^2 + r^2 g, d(r^2\eta), I)$ on $C(M)$.*

Definition 4.11. *An almost contact structure (ξ, η, Φ) is said to be* **normal** *if the corresponding almost complex structure I on $C(M)$ is integrable.*

So we have

Proposition 4.4. *[BG08] There is a one-to-one correspondence between the normal almost contact structures (ξ, η, Φ) on M and almost complex structures I on $C(M)$ that are integrable. Furthermore, if g is a Riemannian metric on M that is compatible with the almost contact structure, then the cone metric $h = dr^2 + r^2 g$ is Hermitian with respect to the complex structure I.*

Definition 4.12. *A normal contact metric structure $S = (\xi, \eta, \Phi, g)$ on M is called a* **Sasakian** *structure. A pair (M, S) is called a* **Sasakian manifold**.

Definition 4.13. *A contact metric manifold (M, ξ, η, Φ, g) is* **Sasakian** *if its metric cone $(C(M), dr^2 + r^2 g, d(r^2\eta), I)$ is Kähler.*

Chapter 5

K Contact and Sasakian Structures

Definition 5.1. *The order of a quasi-regular K-contact structure S is the least common multiple of the orders of the leaf holonomy groups of the characteristic foliation, we denote the order of S by $\nu(S)$. A K-contact manifold with $\nu < \infty$ is said to be of finite order.*

We have

Proposition 5.1. *[BG08] Let (ξ, η) be a quasi-regular strict contact structure on M such that the leaves of the characteristic foliation $\mathcal{F}_\xi$ are compact. Then the Reeb flow generates a locally free circle action on M. Furthermore, there exists a Riemannian metric g such that the induced structure (ξ, η, g) is K-contact.*

We have:

Theorem 5.1. *[BG08] Let (M, ξ, η, Φ, g) be a quasi-regular K-contact manifold with compact leaves. Then*

1. *The space of leaves* $M/\mathcal{F}_\xi$ *is an almost Kähler orbifold such that the canonical projection* $\pi : M \longrightarrow M/\mathcal{F}_\xi$ *is an orbifold Riemannian submersion.*

2. M *is the total space of a principal* S^1*-orbibundle over* $M/\mathcal{F}_\xi$ *with connection 1-form* η *whose curvature* $d\eta = \pi^*\omega$*,* ω *a symplectic form on* $M/\mathcal{F}_\xi$*.*

3. ω *defines a nontrivial integral orbifold cohomology class*

$$[p^*\omega] \in H^2_{orb}(M/\mathcal{F}_\xi, \mathbb{Z})$$

   where p *is the natural projection.*

4. *The leaves of* $\mathcal{F}_\xi$ *are all geodesics.*

5. *If the characteristic foliation* $\mathcal{F}_\xi$ *is regular, then the circle action is free and* M *is the total space of a principal* S^1*-bundle over an almost Kähler, hence, symplectic manifold defining an integral class* $[\omega] \in H^2(M/\mathcal{F}_\xi, \mathbb{Z})$*.*

6. (M, ξ, η, Φ, g) *is Sasakian if and only if* $(M/\mathcal{F}_\xi, \omega)$ *is Kähler.*

We have also:

Theorem 5.2. *[BG08] Let* $(\mathcal{Z}, \omega, J)$ *be an almost Kähler orbifold with* $[p^*\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$*, and let* M *denote the total space of the circle V-bundle defined by the class* $[\omega]$*. Then the orbifold* M *admits a K-contact structure* (ξ, η, Φ, g) *such that* $d\eta = \pi^*\omega$*, where* $\pi : M \longrightarrow \mathcal{Z}$ *is the natural orbifold projection map. Furthermore, if all the local uniformizing groups of* $\mathcal{Z}$ *inject into the structure group* S^1*, then* M *is a smooth K-contact manifold.*

We have:

Proposition 5.2. *[BG08] Let (M, ξ, η, Φ, g) be a compact K-contact manifold. Then the leaf closures of the Reeb flow are the orbits of a torus $\mathfrak{T}$ lying in the center of the Lie group $\mathfrak{A}(M, \xi, \eta, \Phi, g)$ of automorphisms of (ξ, η, Φ, g) and the Reeb flow is conjugate to a linear flow on $\mathfrak{T}$.*

Basic Cohomology

Let $\mathcal{F}$ be a foliation of a smooth manifold M. A differential r-form ω on M is said to be *basic* if for all vector fields V on M that are tangent to the leaves of $\mathcal{F}$ the following conditions hold:

$$V \rfloor \omega = 0, \qquad \pounds_V \omega = 0. \tag{5.1}$$

Let $\mathit{\Omega}^r_B(\mathcal{F})$ denote sheaf of germs of basic r-forms on M, and $\Omega^r_B(\mathcal{F})$ the set of its global sections. We have $\Omega_B(\mathcal{F}) = \oplus_r \Omega^r_B(\mathcal{F})$ is closed under addition and exterior multiplication and so is a subalgebra of the algebra of differential forms on M. And we have

$$\pounds_V d\omega = d\pounds_V \omega = 0, \qquad \text{and} \qquad V \rfloor d\omega = \pounds_V \omega - d(V \rfloor \omega) = 0. \tag{5.2}$$

that is, exterior derivation takes basic forms to basic forms. So the subalgebra $\Omega_B(\mathcal{F})$ forms a subcomplex of the de Rham complex and its cohomology ring $H^*_B(\mathcal{F}) = H^*_B(\mathcal{F}, d_B)$ is called the *basic cohomology ring* of $\mathcal{F}$. Here d_B denotes the restriction of d to $\Omega_B(\mathcal{F})$.

The closures of the leaves of the characteristic foliation $\mathcal{F}_\xi$ is a torus

$$\mathfrak{T} \subset \mathfrak{A}(\xi, \eta, \Phi, g) \subset \mathfrak{I}(M, g).$$

Let $\Omega(M)^{\mathfrak{T}}$ denote the subalgebra of $\Omega(M)$ consisting of $\mathfrak{T}$-invariant forms. Since the orbit of ξ is dense in $\mathfrak{T}$ any basic form is $\mathfrak{T}$-invariant, so $\Omega_B(\mathcal{F}) \subset \Omega(M)^{\mathfrak{T}}$ and

Proposition 5.3. *[BG08] There is an exact sequence of complexes*

$$0 \longrightarrow \Omega^*_B(\mathcal{F}_\xi) \longrightarrow \Omega^*(M)^{\mathfrak{T}} \xrightarrow{\xi \rfloor} \Omega^{*-1}_B(\mathcal{F}_\xi) \longrightarrow 0.$$

and we have a long exact cohomology sequence

$$\cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \xrightarrow{\iota_*} H^p(M, \mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots \tag{5.3}$$

where $\delta[\alpha]_B = [d\eta]_B \cup [\alpha]_B$, and j_p is the composition of $\xi\rfloor$ with the isomorphism $H^r(\Omega(M)^{\mathfrak{T}}) \approx H^r(M, \mathbb{R})$. And we have

Proposition 5.4. *[BG08] Let (M, ξ, η, Φ, g) be a quasi-regular K-contact manifold with $\pi : M \longrightarrow M/\mathcal{F}_\xi$ an orbifold Riemannian submersion. Then π induces a ring isomorphism*

$$\pi^* : H^*(M/\mathcal{F}_\xi, \mathbb{R}) \xrightarrow{\approx} H^*_B(\mathcal{F}_\xi).$$

We will use the following theorem in the next chapter.

Theorem 5.3. *[BG08] Let (M^{2n+1}, S) be a compact Sasakian manifold. Then the pth Betti number $b_p(M)$ is even for p odd with $1 \leq p \leq n$, and for p even with $n < p \leq 2n$.*

Chapter 6

Sasakian Geometry on Lens space bundles over Riemann Surfaces

6.1 Complex Structures on $\Sigma_g \times S^2$ and $\Sigma_g \tilde{\times} S^2$

By a *ruled surface* we mean what is called a geometrically ruled surface that is a $\mathbb{CP}^1$ bundle over Σ_g. It is well known (cf. [MS98], pg. 203) that there are two diffeomorphism types of ruled surfaces of a fixed genus g. These are the trivial bundle $\Sigma_g \times S^2$ and non-trivial S^2-bundles over Σ_g denoted by $\Sigma_g \tilde{\times} S^2$, and they are distinguished by their second Stiefel-Whitney class w_2.

It is well known [Ati55, Ati57, BPVdV84] that all complex structures on ruled surfaces arise by considering them as projectivizations of rank two holomorphic vector bundles E over a Riemann surface Σ_g of genus g. Thus if (M, J) is a ruled surface of genus g we can write it as $(M, J) = \mathbb{P}(E) \to \Sigma_g$, where Σ_g is equipped with a complex structure J_τ where $\tau \in \mathcal{M}_g$, the moduli space of complex structures on Σ_g.

We will divide the vector bundles E into two types, the indecomposable bundles and the ones that can be written as the sum of line bundles. In the latter case we can by tensoring with a line bundle bring E to the form $\mathcal{O} \oplus L$ where $\mathcal{O}$ denotes the trivial line bundle and L is a line bundle of degree $n \in \mathbb{Z}$. Let us subdivide our bundles as:

1. E is indecomposable
   (a) E is stable
   (b) E is non-stable
2. $E = \mathcal{O} \oplus L$, where L is a degree 0 holomorphic line bundle on Σ_g and $\mathcal{O}$ denotes the trivial (holomorphic) line bundle on Σ_g.
3. $E = \mathcal{O} \oplus L$, where L is a holomorphic line bundle on Σ_g of non-zero degree n.

We will denote the complex manifolds occurring in case (2) and (3) by S_n, where $n = \deg L$, and call them *pseudo-Hirzebruch surfaces.*

Assume now that (M, J) is a pseudo-Hirzebruch surface S_n with $n \geq 0$. Let E_n denote the zero section, of $M \to \Sigma_g$. Then $E_n \cdot E_n = n$ (where $n = 0$ in case (2)). If F denotes a fiber of the ruling $M \to \Sigma_g$, then $F \cdot F = 0$, while $F \cdot E_n = 1$. Any real cohomology class in the two dimensional space $H^2(M, \mathbb{R})$ may be written as a linear combination of (the Poincare duals of) E_n and F,

$$m_1 PD(E_n) + m_2 PD(F)\,.$$

In particular, the Kähler cone $\mathcal{K}$ corresponds to $m_1 > 0, m_2 > 0$ (see [Fuj92] or Lemma 1 in [TF98]).

Consider the cohomology class $\alpha_{k_1,k_2} \in H^2(M, \mathbb{R})$ given by

$$\alpha_{k_1,k_2} = k_1 h + k_2 PD(F), \tag{6.1}$$

where $h = PD(E_0) = PD(E_n) - mPD(F)$ when $n = 2m$ is even and $h = PD(E_1) = PD(E_n) - mPD(F)$ when $n = 2m + 1$ is odd. Then it follows from the above that α_{k_1,k_2} is a Kähler class if and only if $k_1 > 0$ and $k_2/k_1 > m$.

The cohomology class in (6.1) is of course defined even if J belongs to case (1) and we have the following general Lemma which is essentially the rank$E = 2$ case of Proposition 1 in [Fuj92].

Lemma 6.1. *[Fuj92]*

- *For any (M, J) of case (1)(a) with $\deg(E)$ even, α_{k_1,k_2} is a Kähler class if and only if $k_1, k_2 > 0$.*

- *For any (M, J) of case (1)(a) with $\deg(E)$ odd, α_{k_1,k_2} is a Kähler class if and only if $k_1 > 0$ and $k_2 + \frac{k_1}{2} > 0$.*

- *For any (M, J) of case (1)(b), let $\tilde{m}$ equal the (non-negative and well-defined) number $\max(E) - \deg(E)/2$, where $\max(E)$ denotes the maximal degree of a sub line bundle of E.*

  - *If $\deg(E)$ is even, α_{k_1,k_2} is a Kähler class if and only if $k_1 > 0$ and $\frac{k_2}{k_1} > \tilde{m}$.*

  - *If* $\deg(E)$ *is odd,* α_{k_1,k_2} *is a Kähler class if and only if* $k_1 > 0$ *and* $\frac{k_2}{k_1} + \frac{1}{2} > \tilde{m}$.

- *For any* (M, J) *of case (2) with* $\deg(E)$ *even,* α_{k_1,k_2} *is a Kähler class if and only if* $k_1, k_2 > 0$.

- *For any* (M, J) *of case (3) above with* $n = 2m$ *or* $n = 2m + 1$ *and* $m \in \mathbb{Z}^+$, α_{k_1,k_2} *is a Kähler class if and only if* $k_1 > 0$ *and* $\frac{k_2}{k_1} > m$.

6.1.1 Extremal Kähler Metrics

Let (M, J) be a compact complex manifold admitting at least one Kähler metric. For a particular Kähler class α, let α^+ denote the set of all Kähler forms in α.

Calabi [Cal82] suggested that one should look for extrema of the following functional Φ on α^+:

$$\Phi : \alpha^+ \to \mathbb{R}$$

$$\Phi(\omega) = \int_M s^2 d\mu,$$

where s is the scalar curvature and $d\mu$ is the volume form of the Kähler metric corresponding to the Kähler form ω. Thus Φ is the square of the L^2-norm of the scalar curvature.

Proposition 6.1. *[Cal82] The Kähler form* $\omega \in \alpha^+$ *is an extremal point of* Φ *if and only if the gradient vector field* $grad\, s$ *is a holomorphic real vector field, that is* $\pounds_{grad\, s} J = 0$. *When this happens the metric* g *corresponding to* ω *is called an* extremal Kähler metric.

Notice that if $\pounds_{grad\,s} J = 0$, then $J grad\, s$ is a Hamiltonian Killing vector field inducing Hamiltonian isometries.

6.1.2 Circle Bundles over Riemann Surfaces

Let M^3_g denote the total space of an S^1 bundle over a Riemann surface Σ_g of genus $g \geq 1$ and for simplicity we assume that this bundle arises from a generator in $H^2(\Sigma_g, \mathbb{Z})$.

There are many inequivalent Sasakian structures on M^3_g with constant scalar curvature. These correspond to the inequivalent Kähler structures on the base Σ_g arising from the moduli space $\mathcal{M}_g$ of complex structures on Σ_g. When writing M^3_g we often assume that a transverse complex structure has been chosen without specifying which one. Thus, we write the Sasakian structure with constant scalar curvature on M^3_g as $\mathcal{S}_1 = (\xi_1, \eta_1, \Phi_1, g_1)$ and call it the *standard Sasakian structure*. However, when we do wish to specify the complex structure on Σ_g we shall denote it by $\tau \in \mathfrak{M}_g$ and denote the induced endomorphism on the circle bundle by Φ_τ.

We denote the fundamental group of M^3_g by $\Gamma_3(g)$. Then from the long exact homotopy sequence of the bundle $S^1 \longrightarrow M^3_g \longrightarrow \Sigma_g$ and the fact that $\pi_2(\Sigma_g) = 0$ we have

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_3(g) \longrightarrow \Gamma_2(g) \longrightarrow 1 \tag{6.2}$$

where $\Gamma_2(g)$ is the fundamental group of Σ_g. So $\Gamma_3(g)$ is an extension of $\Gamma_2(g)$ by $\mathbb{Z}$. Furthermore, it does not split [Sco83].

6.1.3 The Join Construction

Recall the definition of the lens spaces. Let $\omega = e^{\frac{2\pi\sqrt{-1}}{l_2}}$ and consider the action from $S^3 \mapsto S^3$ defined by $(u, v) \mapsto (\omega^{w_1}u, \omega^{w_2}v)$. The quotient space of S^3 by this action is called the lens space and we are going to denote it $L(l_2; w_1, w_2)$.

We describe the join of M_g^3 with the sphere $S^3_{\mathbf{w}}$. Recall the weighted sphere as presented in Example 7.1.12 of [BG08]. Let η_0 denote the standard contact form on S^3. It is the restriction to S^3 of the 1-form $\sum_{i=1}^{2}(y_i dx_i - x_i dy_i)$ in $\mathbb{R}^4$. Let $\mathbf{w} = (w_1, w_2)$ be a weight vector with $w_i \in \mathbb{Z}^+$. Then the weighted contact form is defined by

$$\eta_{\mathbf{w}} = \frac{\eta_0}{\eta_0(\xi_{\mathbf{w}})} \tag{6.3}$$

with Reeb vector field $\xi_{\mathbf{w}} = \sum_{i=1}^{2} w_i H_i$ where H_i is the vector field on S^3 induced by $y_i\partial_{x_i} - x_i\partial_{y_i}$ on $\mathbb{R}^4$.

We denote this weighted sphere by $S^3_{\mathbf{w}}$ and consider the manifold $M_g^3 \times S^3_{\mathbf{w}}$ with contact forms $\eta_1, \eta_{\mathbf{w}}$ on each factor, respectively. There is a 3-dimensional torus T^3 acting on $M_g^3 \times S^3_{\mathbf{w}}$ generated by the Lie algebra $\mathfrak{t}_3$ of vector fields ξ_1, H_1, H_2 that leaves both 1-forms $\eta_1, \eta_{\mathbf{w}}$ invariant. Now the join construction [BGO07, BG08] provides us with a new contact manifold by quotienting $M_g^3 \times S^3_{\mathbf{w}}$ with an appropriate circle subgroup of T^3. Let $(x, u) \in M_g^3$ with $x \in \Sigma_g$ and u in the fiber, and $(z_1, z_2) \in S^3_{\mathbf{w}}$. Consider the circle action on $M_g^3 \times S^3_{\mathbf{w}}$ given by

$$(x, u; z_1, z_2) \mapsto (x, e^{il_2\theta}u; e^{-iw_1\theta}z_1, e^{-iw_2\theta}z_2) \tag{6.4}$$

where the action $u \mapsto e^{il_2\theta}u$ is that generated by $l_2\xi_1$. We also assume, without loss of generality, that $\gcd(l_2, w_1, w_2) = 1$. The action (6.4) is generated by the vector field $l_2\xi_1 - \xi_{\mathbf{w}}$. It has period $1/l_2$ on the M^3_g part, and if $l_1 = \gcd(w_1, w_2)$ it will have period $-1/l_1$ on the $S^3_{\mathbf{w}}$ part. With this in mind, when considering quotients we shall always take the pair (w_1, w_2) to be relatively prime positive integers in which case the infinitesimal generator of the action is given by the vector field $l_2\xi_1 - l_1\xi_{\mathbf{w}}$. In order to construct the appropriate contact structure with 1-form $l_1\eta_1 + l_2\eta_{\mathbf{w}}$, we renormalize the vector field and consider

$$L_{\mathbf{w}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_{\mathbf{w}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}(w_1H_1 + w_2H_2). \tag{6.5}$$

This generates a free circle action on $M^3_g \times S^3_{\mathbf{w}}$ which we denote by $S^1(l_1, l_2, \mathbf{w})$.

Definition 6.1. *The quotient space of $M^3_g \times S^3_{\mathbf{w}}$ by the action $S^1(l_1, l_2, \mathbf{w})$ is called the (l_1, l_2)-join of M^3_g and $S^3_{\mathbf{w}}$, and is denoted by $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$.*

$M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$ will be a smooth manifold if $\gcd(l_2, \upsilon_2 l_1) = 1$ where $\upsilon_2 = w_1w_2$. Moreover, since the 1-form $l_1\eta_1 + l_2\eta_{\mathbf{w}}$ on $M^3_g \times S^3_{\mathbf{w}}$ is invariant under T^3, we get a contact form on $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$, denoted by $\eta_{l_1,l_2,\mathbf{w}}$, which is invariant under the factor group $T^2(l_1, l_2, \mathbf{w}) = T^3/S^1(l_1, l_2, \mathbf{w})$. The corresponding contact structure is $\mathcal{D}_{l_1,l_2,\mathbf{w}}$=ker $\eta_{l_1,l_2,\mathbf{w}}$, and the Reeb vector field $R_{l_1,l_2,\mathbf{w}}$ of $\eta_{l_1,l_2,\mathbf{w}}$ is the restriction to $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$ of the vector field

$$\tilde{R}_{l_1,l_2,\mathbf{w}} = \frac{1}{2l_1}\xi_1 + \frac{1}{2l_2}\xi_{\mathbf{w}} \tag{6.6}$$

on $M^3_g \times S^3_{\mathbf{w}}$.

The quotient of $M^3_g \times S^3_{\mathbf{w}}$ by the 2-torus generated by $L_{\mathbf{w}}$ and $R_{\mathbf{w}}$ splits giving the complex orbifold $\Sigma_g \times \mathbb{CP}(\mathbf{w})$ with the product complex structure and symplectic form $\omega = l_1\omega_g + l_2\omega_{\mathbf{w}}$ where $\omega_g, \omega_{\mathbf{w}}$ are the standard symplectic form on Σ_g and $\mathbb{CP}(\mathbf{w})$, respectively. Then $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$ is the total space of the S^1 orbibundle

$$\pi : M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}} \longrightarrow \Sigma_g \times \mathbb{CP}(\mathbf{w})$$

which satisfies $\pi^*\omega = d\eta_{l_1,l_2,\mathbf{w}}$. This is the orbifold Boothby-Wang construction, the orbifold $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$ not only inherits a quasi-regular contact structure, but also a natural Sasakian structure $\mathcal{S}_{\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{l_1,l_2,\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$ from the product Kähler structure on the base. In particular, the underlying CR structure which is inherited from the product complex structure on $\Sigma_g \times \mathbb{CP}(\mathbf{w})$ is $(\mathcal{D}_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ where $J_{\mathbf{w}} = \Phi_{\mathbf{w}}|_{\mathcal{D}_{l_1,l_2,\mathbf{w}}}$.

It seems a complicated problem to determine the homotopy type of

$$M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$$

when $l_2 > 1$. In [BTF14a] it is done the case when $l_2 = 1$. There is shown that $M^3_g \star_{l_1,1} S^3_{\mathbf{w}}$ are S^3 bundles over Σ_g and that there are two diffeomorphism types; the trivial bundle and the nontrivial one. For the case $l_2 > 1$, we compute the cohomology groups and cohomology ring of $M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$.

It follows from Proposition 7.6.7 of [BG08] that

$$M^3_g \star_{l_1,l_2} S^3_{\mathbf{w}}$$

is a lens space bundle over Σ_g.

6.1.4 The Sasaki Cone and Deformed Sasakian Structures

Recall the (unreduced) Sasaki cone [BGS08]. Let $\mathcal{S}_0 = (\xi_0, \eta_0, \Phi_0, g_0)$ be a Sasakian structure and let $\mathfrak{Aut}(\mathcal{S}_0)$ its group of automorphisms. We denote the Lie algebra of infinitesimal automorphisms of $\mathcal{S}_0$ by $\mathfrak{aut}(\mathcal{S}_0)$. We let $\mathfrak{t}_k \subset \mathfrak{aut}(\mathcal{S}_0)$ denote the Lie algebra of a maximal torus in $\mathfrak{Aut}(\mathcal{S}_0)$, which is unique up to conjugacy. It has rank k. The unreduced Sasaki cone is given by

$$\mathfrak{t}_k^+ = \{\xi \in \mathfrak{t}_k \mid \eta_0(\xi) > 0\}.$$

Here we consider the Sasaki cone of our Sasakian structure

$$\mathcal{S}_{l_1,l_2,\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{l_1,l_2,\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$$

Recall from Section 6.1.3 that on $M_g^3 \times S_{\mathbf{w}}^3$ we have the Lie algebra $\mathfrak{t}_3$ generated by ξ_1, H_1, H_2. Let $\mathfrak{t}_1(\mathbf{w})$ be the Lie algebra generated by the vector field $L_{\mathbf{w}} \in \mathfrak{t}_3$ of Equation (6.5). There is an exact sequence of Abelian Lie algebras

$$0 \longrightarrow \mathfrak{t}_1(\mathbf{w}) \longrightarrow \mathfrak{t}_3 \longrightarrow \mathfrak{t}_2(\mathbf{w}) \longrightarrow 0,$$

and we view the quotient algebra $\mathfrak{t}_2(\mathbf{w}) = \mathfrak{t}_3/\mathfrak{t}_1(\mathbf{w})$ as a Lie algebra on $M^5_{l_1,l_2,\mathbf{w}}$. Then the unreduced Sasaki cone $\mathfrak{t}_2^+(\mathbf{w})$ of $M^5_{l_1,l_2,\mathbf{w}}$ is defined by

$$\mathfrak{t}_2^+(\mathbf{w}) = \{\xi \in \mathfrak{t}_2(\mathbf{w}) \mid \eta_{l_1,l_2,\mathbf{w}}(\xi) > 0\}. \tag{6.7}$$

We can take $\xi_{\mathbf{w}}, H_1$ as a basis for $\mathfrak{t}_2(\mathbf{w})$. Then for $R \in \mathfrak{t}_2^+(\mathbf{w})$ writing $R = a\xi_{\mathbf{w}} + bH_1$ we must have $a > 0$ and $aw_1 + b > 0$. We can also write

$$R = a\xi_{\mathbf{w}} + bH_1 = (aw_1 + b)H_1 + aw_2H_2 = v_1H_1 + v_2H_2 = \xi_{\mathbf{v}}$$

which identifies the Sasaki cone of $M^5_{l_1,l_2,\mathbf{w}}$ with the Sasaki cone of $S^3_{\mathbf{w}}$. All Sasakian structures $\mathcal{S}_{\mathbf{v}} = (\xi_{\mathbf{v}}, \eta_{l_1,l_2,\mathbf{v}}, \Phi_{\mathbf{v}}, g_{\mathbf{v}})$ in the Sasaki cone $\mathfrak{t}_2^+(\mathbf{w})$ have the same underlying CR structure, namely $(\mathcal{D}_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$. We have

$$\eta_{l_1,l_2,\mathbf{v}} = \frac{\eta_{l_1,l_2,\mathbf{w}}}{\eta_{l_1,l_2,\mathbf{w}}(\xi_{\mathbf{v}})}, \qquad \Phi_{\mathbf{v}}|_{\mathcal{D}_{l_1,l_2,\mathbf{w}}} = \Phi_{\mathbf{w}}|_{\mathcal{D}_{l_1,l_2,\mathbf{w}}} = J_{\mathbf{w}}. \tag{6.8}$$

It is the reduced Sasaki cone $\kappa(\mathcal{D}, J)$ that can be thought of as the moduli space of Sasakian structures associated to an underlying CR structure. It is simply the quotient of the unreduced Sasaki cone by the Weyl group of the CR automorphism group. This action amounts to ordering either the w_is or the v_i. The proof of the following lemma is in [BTF14a]

Lemma 6.2. *Let M be a quasiregular Sasakian manifold with Sasakian structure*

$$\mathcal{S} = (\xi, \eta, \Phi, g)$$

and let $\pi : M \longrightarrow \mathcal{Z}$ be the orbifold Boothby-Wang map to the Kähler orbifold $\mathcal{Z}$ with

Kähler form ω. Let $\check{X}$ be a vector field on $\mathcal{Z}$ leaving both the Kähler form ω and the complex structure J invariant. Then $\check{X}$ lifts to an infinitesimal automorphism X of the Sasakian structure $\mathcal{S}$ that is unique modulo the ideal $\mathcal{I}_\xi$ generated by ξ if and only if it is Hamiltonian. Furthermore, if $\check{X}$ is Hamiltonian with Hamiltonian function H satisfying $\check{X} \rfloor \omega = -dH$, then X can be represented by $\check{X}^h + \pi^* H\xi$ where $\check{X}^h$ denotes the horizontal lift of $\check{X}$.

We first consider the smooth join, that is, $\mathbf{w} = (1,1)$.

Lemma 6.3. *Consider the Sasakian structure*

$$\mathcal{S}_{l_1,l_2,(1,1)} = (\xi_{l_1,l_2,(1,1)}, \eta_{l_1,l_2,(1,1)}, \Phi_\tau, g)$$

on the 5-manifold $M^5_{l_1,l_2,(1,1)}$ with

$$\Phi_\tau|_{\mathcal{D}_{l_1,l_2,(1,1)}} = J \in \mathfrak{J}$$

Let X denote the infinitesimal generator of the induced Hamiltonian circle action on $M^5_{l_1,l_2,(1,1)}$. Meaning that X is the fundamental vector field of the action so X is Hamiltonian. Then we have that the Sasaki cone has dimension two.

Proof. The proof of this lemma is similar to the corresponding lemma in [BTF14a]. For $i = 1,2$ we let (η_i, ξ_i) denote the contact 1-form and its Reeb vector field on M^3_g and S^3, respectively. We use the lemma above to lift the Hamiltonian circle action: $\tilde{\mathcal{A}}(\lambda)(w,[u,v]) = (w,[u,\lambda v])$ where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This action is clearly holomorphic.

The Reeb vector field together with the lift $X = \hat{X}^h + \eta(X)\xi$ span the Lie algebra $\mathfrak{t}_2$ of a maximal torus $\mathfrak{T}^2 \in \mathfrak{Con}(M^5_{l_1,l_2}, \eta_{l_1,l_2,\mathbf{w}})$. So the Sasaki cone has dimension two. □

We now want to describe the Kähler orbifold associated to the Sasakian structure $\mathcal{S}_{\mathbf{v}}$ when this structure is quasi-regular. For this purpose we can assume that v_1 and v_2 are relatively prime positive integers.

Proposition 6.2. *Let* $\mathbf{v} = (v_1, v_2)$ *with* $v_1, v_2 \in \mathbb{Z}^+$ *and* $\gcd(v_1, v_2) = 1$, *and let* $\xi_{\mathbf{v}}$ *be a Reeb vector field in the Sasaki cone* $\mathfrak{t}_2^+(\mathbf{w})$. *Then the quotient of* $M^5_{l_1,l_2,\mathbf{w}}$ *by the circle action* $S^1(\mathbf{v})$ *generated by* $\xi_{\mathbf{v}}$ *is a complex fiber bundle over* Σ_g *whose fiber is the complex orbifold* $\mathbb{CP}(\mathbf{v})/\mathbb{Z}_{\frac{l_2}{s}}$. *Where*

$$s = \gcd(|w_2 v_1 - w_1 v_2|, l_2).$$

Proof. The computations here are similar to the corresponding ones in [BTF13]. We know that the quotient $M^5_{l_1,l_2,\mathbf{w}}/S^1(\mathbf{v})$ is a projective algebraic orbifold with an induced orbifold Kähler structure. We denote this Kähler orbifold by $B_{\mathbf{v},\mathbf{w}}$, and consider the 2-dimensional subalgebra $\mathfrak{t}_2(\mathbf{v}, \mathbf{w})$ of $\mathfrak{t}_3$ generated by the vector fields $L_{\mathbf{w}}$ and $\xi_{\mathbf{v}}$ on $M^3_g \times S^3_{\mathbf{w}}$. The S^1 action on the lens space is done as follows:
First

$$L(l_2; l_1 w_1, l_1 w_2) = S^3_{\mathbf{w}}/\mathbb{Z}_{l_2}$$

then the $S^1 = S^1_\theta/\mathbb{Z}_{l_2}$ action given by

$$(x, u; z_1, z_2) \mapsto (x, e^{i\theta}; [e^{-i\frac{l_1 w_1 \theta}{l_2}} z_1, e^{-i\frac{l_1 w_2 \theta}{l_2}} z_2])$$

where the brackets denote the equivalence class defined below.

Then the $T^2 = S^1_\phi \times (S^1_\theta/\mathbb{Z}_{l_2})$ action on $M^3_g \times L(l_2; l_1w_1, l_1w_2)$ is given by

$$(x, u; [z_1, z_2]) \mapsto (x, e^{i\theta}u; [e^{i(v_1\phi - \frac{l_1w_1\theta}{l_2})}z_1, e^{i(v_2\phi - \frac{l_1w_2\theta}{l_2})}z_2]), \tag{6.9}$$

where $(x, u) \in M^3_g$ with u in the fiber of the bundle $\rho : M^3_g \longrightarrow \Sigma_g$, and $[z_1, z_2] \in L(l_2; l_1w_1, l_1w_2)$. The brackets in the action (6.9) denote the equivalence class defined by $(z_1', z_2') \sim (z_1, z_2)$ if $(z_1', z_2') = (\lambda^{l_1w_1}z_1, \lambda^{l_1w_2}z_2)$ for $\lambda^{l_2} = 1$. By taking the quotient first by the circle action generated by $L_{\mathbf{w}}$ gives the commutative diagram

$$\begin{tikzcd} M^3_g \times L(l_2; l_1w_1, l_1w_2) \arrow[d, "\pi_B"] \arrow[dr] & \\ B_{\mathbf{v},\mathbf{w}} & M^5_{l_1,l_2,\mathbf{w}} \arrow[l] \end{tikzcd} \tag{6.10}$$

where π_B is the quotient projection by the 2-torus generated by $\mathfrak{t}_2(\mathbf{v}, \mathbf{w})$, the southeast arrow is the quotient projection by the circle action generated by $L_{\mathbf{w}}$, and the southwest arrow is the quotient projection generated by $S^1(\mathbf{v})$. A point of $B_{\mathbf{v},\mathbf{w}}$ is given by the equivalence class $[x, u; z_1, z_2]$ defined by the T^2 action (6.9). We claim that $B_{\mathbf{v},\mathbf{w}}$ is a bundle over Σ_g with fiber $\mathbb{CP}(\mathbf{v})/\mathbb{Z}_{\frac{l_2}{s}}$. To see this consider the projection $\pi : M^3_g \times L(l_2; l_1w_1, l_1w_2) \longrightarrow \Sigma_g$ defined by $\pi = \rho \circ \pi_1$ where $\pi_1 : M^3_g \times L(l_2; l_1w_1, l_1w_2) \longrightarrow M^3_g$ is projection onto the first factor. We have

$$\pi(x, e^{i\theta}u; [e^{i(v_1\phi - \frac{lw_1\theta}{l_2})}z_1, e^{i(v_2\phi - \frac{lw_2\theta}{l_2})}z_2]) = \pi(x, u; [z_1, z_2]) = x,$$

so the torus acts in the fibers of π. This gives a map $\tau : B_{\mathbf{v},\mathbf{w}} \longrightarrow \Sigma_g$ defined by $\tau([x,u;z_1,z_2]) = x$, so π factors through $B_{\mathbf{v},\mathbf{w}}$ giving the commutative diagram

$$\begin{tikzcd}
M_g^3 \times L(l_2; l_1w_1, l_1w_2) \arrow[dd, "\pi"] \arrow[dr, "\pi_B"] & \\
& B_{\mathbf{v},\mathbf{w}} \arrow[dl, "\tau"] \\
\Sigma_g &
\end{tikzcd} \tag{6.11}$$

To see the fibers of τ let us do the analysis of the T^2 action given by Equation (6.9).We look for fixed points under a subgroup of the circle S^1_ϕ. So, we set

$$(e^{iv_1\phi}z_1, e^{iv_2\phi}z_2) = (e^{-2\pi\frac{l_1w_1}{l_2}ri}z_1, e^{-2\pi\frac{l_1w_2}{l_2}ri}z_2)$$

for some $r = 0, \dots l_2 - 1$. If $z_1z_2 \neq 0$ we have

$$v_1\phi = 2\pi(-\frac{l_1w_1}{l_2}r + k_1), \qquad v_2\phi = 2\pi(-\frac{l_1w_2}{l_2}r + k_2) \tag{6.12}$$

for some integers k_1, k_2. We get

$$r = \frac{l_2}{l_1}\frac{k_2v_1 - k_1v_2}{w_2v_1 - w_1v_2} \tag{6.13}$$

which must be a nonnegative integer less than l_2. We can solve equation (6.12) for ϕ, we get

$$\phi = 2\pi\frac{k_1w_2 - k_2w_1}{w_2v_1 - w_1v_2}. \tag{6.14}$$

Now we write (6.13) as

$$r = \frac{l_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)} \frac{k_2v_1 - k_1v_2}{l_1\frac{w_2v_1 - w_1v_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)}} \tag{6.15}$$

Since v_1 and v_2 are relatively prime, we can choose k_1 and k_2 so that

$$\frac{k_2v_1 - k_1v_2}{l_1\frac{w_2v_1 - w_1v_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)}} = 1$$

This gives

$$r = \frac{l_2}{\gcd(|w_2v_1 - w_1v_2|, l_2)} \tag{6.16}$$

Next suppose that $z_2 = 0$. Then we have $e^{iv_1\phi} = e^{-2\pi\frac{l_1w_1}{l_2}ri}$ for some $r = 1, \dots l_2$. We get

$$\phi = 2\pi(-\frac{l_1w_1r}{v_1l_2} + \frac{k}{v_1}). \tag{6.17}$$

A similar calculation at $z_1 = 0$ gives

$$\phi = 2\pi(-\frac{l_1w_2r'}{v_2l_2} + \frac{k'}{v_2}). \tag{6.18}$$

We want to find out when regularity can happen. For this we need the minimal angle at the two endpoints to be equal. We get

$$-\frac{l_1w_2r'}{v_2l_2} + \frac{k'}{v_2} = -\frac{l_1w_1r}{v_1l_2} + \frac{k}{v_1}$$

for some choice of integers k, k' and nonnegative integers $r, r' < \frac{l_2}{l_1}$. We have

$$\frac{-l_1 w_2 r' + k' l_2}{v_2} = \frac{-l_1 w_1 r + k l_2}{v_1}. \tag{6.19}$$

We assume that $\mathbf{w} \neq (1,1)$. We want to find the periods of the orbits of the flow of the Reeb vector field defined by the weight vector $\mathbf{v} = (v_1, v_2)$. In particular, we want to know when there is a regular Reeb vector field in the $\mathbf{w}$-Sasaki cone.

Let us find the minimal angle, hence the generic period of the Reeb orbits, on the dense open subset Z defined by $z_1 z_2 \neq 0$. Set

$$s = \gcd(|w_2 v_1 - w_1 v_2|, l_2)$$

Lemma 6.4. *The minimal angle on Z is $\frac{2\pi}{s}$. Thus, $S^1_\phi/\mathbb{Z}_s$ acts freely on the dense open subset Z.*

Proof. We choose k_1, k_2 in Equation (6.15) so that

$$\frac{k_2 v_1 - k_1 v_2}{l_1 \frac{w_2 v_1 - w_1 v_2}{\gcd(|w_2 v_1 - w_1 v_2|, l_2)}} = 1.$$

We get

$$l_1 \frac{w_2 v_1 - w_1 v_2}{s} = k_2 v_1 - k_1 v_2.$$

Rewriting this becomes

$$(s k_2 - l_1 w_2) v_1 = (s k_1 - l_1 w_1) v_2.$$

Since v_1 and v_2 are relatively prime this equation implies $sk_i = l_1 w_i + m v_i$ for $i = 1, 2$ and some integer m. Putting this into Equation (6.14) we get $\phi = \frac{2\pi m}{s}$, so the minimal angle is $\frac{2\pi}{s}$. □

We have for the endpoints defined by $z_2 = 0$ and $z_1 = 0$.

Proposition 6.3. *The following hold:*

1. *The period on Z, namely $\frac{2\pi}{s}$, is an integral multiple of the periods at the endpoints. Hence, $S^1_\phi/\mathbb{Z}_s$ acts effectively on $M_{l_1,l_2,\mathbf{w}}$.*

2. *The period at the endpoint $z_j = 0$ is $2\pi\frac{1}{v_i l_2}$ where $i \equiv j + 1 \mod 2$. So the end points have equal periods if and only if $\mathbf{v} = (1, 1)$.*

3. *The Sasaki cone contains a regular Reeb vector field if and only if $\mathbf{v} = (1, 1)$ and l_2 divides $w_1 - w_2$.*

Proof. A Reeb vector field will be regular if and only if the period of its orbit is the same at all points. We know that it is $\frac{2\pi}{s}$ on Z. We need to determine the minimal angle at the endpoints. From Equation (6.17) the angle at $z_2 = 0$ is

$$\phi = 2\pi\left(\frac{-l_1 w_1 r + k l_2}{v_1 l_2}\right).$$

Now

$$\gcd(l_2, l_1 w_1) = 1$$

so we can choose k and r such that $-l_1 w_1 r + k l_2 = 1$. This gives period $2\pi\frac{1}{v_1 l_2}$. Similarly, at $z_1 = 0$ we have the period $2\pi\frac{1}{v_2 l_2}$. So the period is the same at the endpoints if and only if $v_1 = v_2$ which is equivalent to $\mathbf{v} = (1,1)$ since v_1 and v_2 are relatively prime which proves (2).

Moreover, the period is the same at all points if and only if

$$\mathbf{v} = (1,1), \qquad l_2 = s = \gcd(|w_2v_1 - w_1v_2|, l_2). \tag{6.20}$$

But the last equation holds if and only if l_2 divides $w_1 - w_2$ proving (3).

(1) follows from the fact that for each $i = 1, 2$, $\frac{v_i}{l_2}$ is an integral multiple of $\gcd(|w_2v_1 - w_1v_2|, l_2) = s$. □

We have the action of the 2-torus $S^1_\phi/\mathbb{Z}_s \times (S^1_\theta/\mathbb{Z}_{l_2})$ on $M^3_g \times L(l_2; l_1w_1, l_1w_2)$ given by expression (6.9) whose quotient space is $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$. It follows from the action (6.9) that $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a bundle over Σ_g with fiber a weighted projective space of complex dimension one. By (1) of Proposition 6.3 the generic period is an integral multiple, say m_i, of the period at the divisor D_i. Thus, for $i = 1, 2$ we have

$$m_i = v_i l_2 s = v_i m. \tag{6.21}$$

Observe that from its definition $m = \frac{l_2}{s}$, so m_i is a positive integer. It is the ramification index of the branch divisor D_i. We think of D_1 as the zero section and D_2 as the infinity section of the bundle $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$. Thus, $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a fiber bundle over Σ_g with fiber

$$\mathbb{CP}^1[v_1, v_2]/\mathbb{Z}_m$$

The complex structure of $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is the projection of the transverse complex structure on $M_{l_1,l_2,\mathbf{w}}$ which in turn is the lift of the product complex structure on $\Sigma_g \times$

$\mathbb{CP}^1[\mathbf{w}]$. □

We call the Kähler orbifold $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ an *orbifold pseudo-Hirzebruch surface.*

6.1.5 Regular Sasakian Structures

It follows from Proposition 6.2 that each Sasaki cone $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ contains a unique ray of regular Sasakian structures determined by setting $\mathbf{v} = (1,1)$. We have the Reeb vector field $R = H_1 + H_2$ and $B_{1,\mathbf{w}}$ is a pseudo-Hirzebruch surface with trivial orbifold structure. By the Leray-Hirsch Theorem the cohomology groups are obtained from the tensor product of the cohomology groups of the base and the fiber (see Section 1.3 of [ACGTF08]). Thus, the first Chern class satisfies

$$c_1(B_{1,\mathbf{w}}) = (2PD(E_n) + (2-2g-n)PD(F) \tag{6.22}$$

where the divisors E_n and F satisfy $E_n \cdot E_n = n, E_n \cdot F = 1$ and $F \cdot F = 0$. Since the second Stiefel-Whitney class is the mod 2 reduction of c_1, we see that $B_{1,\mathbf{w}}$ is diffeomorphic to $\Sigma_g \times S^2$ when n is even and diffeomorphic to $\Sigma_g \tilde{\times} S^2$ when n is odd.

When $n = 2m$ is even, we have $PD(F) = [\omega_g]$ and $PD(E_{2m}) = m[\omega_g] + [\omega_0]$ where the class $[\omega_g]([\omega_0])$ represents the area form of Σ_g (the fiber $\mathbb{CP}^1$), respectively. If $\pi : M^5_{g,l,\mathbf{w}} \longrightarrow B_{1,\mathbf{w}}$ denotes the S^1 bundle map, we have

$$\pi^* c_1(B_{1,\mathbf{w}}) = c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}). \tag{6.23}$$

Writing the symplectic class on $B_{1,\mathbf{w}}$ as

$$[\omega] = k_1[\omega_0] + k_2[\omega_g] = k_1 PD(E_{2m}) + (k_2 - mk_1)PD(F) \tag{6.24}$$

for some relatively prime positive integers k_1, k_2.

When $n = 2m + 1$ is odd, we have $PD(F) = [\omega_g]$ and $PD(E_{2m+1}) = PD(E_1) + mPD(F)$, so

$$[\omega] = k_1 h + k_2[\omega_g] = k_1 PD(E_{2m+1}) + (k_2 - mk_1)PD(F) \tag{6.25}$$

where $h = PD(E_1)$.

We have:

Lemma 6.5. *The following relation holds:*

$$n = l_1|\mathbf{w}| - 2l_1 w_2 = l_1(w_1 - w_2).$$

Proof. Let L_n denote a line bundle on Σ_g of degree n. Then after defining $\chi = \phi - \frac{l_1 w_1 \theta}{l_2}$ the T^2 action (6.9) with $\mathbf{v} = (1,1)$ becomes

$$(x, u; z_1, z_2) \mapsto (x, e^{i\theta}u; e^{i\chi}z_1, e^{i(\chi + \frac{(l_1|\mathbf{w}| - 2l_1 w_2)}{l_2}\theta)}z_2), \tag{6.26}$$

So we can identify $B_{1,\mathbf{w}}$ with $\mathbb{P}(E)$ where $E = \mathcal{O} \oplus L_n$ where n is given by the equation of the lemma. $\square$

6.1.6 Quasi-regular Sasakian Structures

We now consider the general case $\mathbf{v} = (v_1, v_2)$ where $v_1, v_2 \in \mathbb{Z}^+$ and we assume that $\gcd(v_1, v_2) = 1$. The base space $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is now an orbifold pseudo-Hirzebruch surface as discussed in Section 6.1.4. As a complex manifold $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ is a smooth pseudo-Hirzebruch surface S_n for some $n \in \mathbb{Z}$, but there are branch divisors making the orbifold structure essential. We want to prove:

Proposition 6.4. *The orbifold pseudo-Hirzebruch surface $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ can be realized as the orbifold log pair $(S_n, \Delta_\mathbf{v})$ where S_n is a pseudo-Hirzebruch surface of degree*

$$n = \frac{l_1(w_1v_2 - w_2v_1)}{s}.$$

Where

$$s = \gcd(|w_2v_1 - w_1v_2|, l_2).$$

and the branch divisor $\Delta_\mathbf{v}$ is given by Equation (6.30).

Proof. Here the computations are similar to the corresponding ones in [BTF13].
It is convenient to represent $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ as a log pair $(B_{1,\mathbf{w}'}, \Delta)$ for some weight vector $\mathbf{w}'$ and some branch divisor Δ. To do this we consider the map

$$\tilde{h}_\mathbf{v} : M_g^3 \times L(l_2; l_1w_1, l_1w_2) \times \mathbb{R} \longrightarrow M_g^3 \times L(l_2; l_1w_1, l_1w_2) \times \mathbb{R}$$

defined by

$$\tilde{h}_\mathbf{v}(x, u; [z_1, z_2]) = (x, u; [z_1^{v_2}, z_2^{v_1}]). \tag{6.27}$$

It is a v_1v_2-fold covering map. Consider the $S^1 \times \mathbb{C}^*$ action $\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda, \tau)$ on $M_g^3 \times$

$L(l_2; l_1w_1, l_1w_2) \times \mathbb{R}$ defined by

$$\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda,\tau)(x,u;[z_1,z_2]) = (x,\lambda u;[\tau^{v_1}\lambda^{\frac{-l_1w_1}{l_2}}z_1, \tau^{v_2}\lambda^{\frac{-l_1w_2}{l_2}}z_2]), \tag{6.28}$$

where $\lambda, \tau \in \mathbb{C}^*$ with $|\lambda| = 1$. Almost by definition we have

$$B_{l_1,l_2,\mathbf{v},\mathbf{w}} = \big(M_g^3 \times L(l_2; l_1w_1, l_1w_2) \times \mathbb{R}\big)/\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda,\tau).$$

A computation gives a commutative diagram:

$$\begin{array}{ccc} M_g^3 \times L(l_2; l_1w_1, l_1w_2) \times \mathbb{R} & \xrightarrow{\mathcal{A}_{\mathbf{v},l,\mathbf{w}}(\lambda,\tau)} & M_g^3 \times L(l_2; l_1w_1, l_1w_2) \times \mathbb{R} \\ \Big\downarrow{\scriptstyle \tilde{h}_{\mathbf{v}}} & & \Big\downarrow{\scriptstyle \tilde{h}_{\mathbf{v}}} \\ M_g^3 \times L(l_2; l_1w_1', l_1w_2') \times \mathbb{R} & \xrightarrow{\mathcal{A}_{1,l,\mathbf{w}'}(\lambda,\tau^{v_1v_2})} & M_g^3 \times L(l_2; l_1w_1', l_1w_2') \times \mathbb{R}, \end{array} \tag{6.29}$$

where $\mathbf{w}' = (v_2w_1, v_1w_2)$. Now $\tilde{h}_{\mathbf{v}}$ induces a fiber preserving biholomorphism

$$h_{\mathbf{v}} : B_{l_1,l_2,\mathbf{v},\mathbf{w}} \longrightarrow B_{1,\mathbf{w}'}$$

given by

$$h(x,[z_1,z_2]) = (x,[z_1^{v_2}, z_2^{v_1}]).$$

As ruled surfaces $B_{1,\mathbf{w}'} = S_n$ where

$$n = \frac{l_1(w_1v_2 - w_2v_1)}{s}.$$

and

$$s = \gcd(|w_2v_1 - w_1v_2|, l_2).$$

We can thus write $B_{l_1,l_2,\mathbf{v},\mathbf{w}}$ as the log pair $(S_n, \Delta_{\mathbf{v}})$ where $\Delta_{\mathbf{v}}$ is the branch divisor

$$\Delta_{\mathbf{v}} = (1 - \frac{1}{v_1})E_n + (1 - \frac{1}{v_2})E_n' \tag{6.30}$$

where E_n' is the infinity section which satisfies $E_n' \cdot E_n' = -n.$.

The T^2 action

$$\mathcal{A}_{1,\mathbf{l},\mathbf{w}'} : M \times L(l_2; l_1 w_1', l_1 w_2') \times \mathbb{R} \longrightarrow M \times L(l_2; l_1 w_1', l_1 w_2') \times \mathbb{R}$$

is given by

$$(x, u; [z_1, z_2]) \mapsto (x, e^{i\theta}u; [e^{i(\phi - \frac{l_1 w_1'}{l_2}\theta)} z_1, e^{i(\phi - \frac{l_1 w_2'}{l_2}\theta)} z_2]), \tag{6.31}$$

Defining $\chi = \phi - \frac{l_1 w_1'}{l_2}\theta$ gives

$$(x, u; [z_1, z_2]) \mapsto (x, e^{i\theta}u; [e^{i\chi} z_1, e^{i(\chi + \frac{l_1}{l_2}(w_1' - w_2')\theta)} z_2]). \tag{6.32}$$

as shown above this action is generally not free, but has branch divisors at the zero $(z_2 = 0)$ and infinity $(z_1 = 0)$ sections with ramification indices both equal to m.

Equation (6.32) tells us that the T^2-quotient space $B_{1,\mathbf{w}'}$ is the projectivization of the holomorphic rank two vector bundle $E = \mathcal{O} \oplus L_n$ over Σ_g where $\mathcal{O}$ denotes the trivial line bundle and L_n is a line bundle of 'degree'

$$n = \frac{l_1}{s}(w_1 v_2 - w_2 v_1)$$

with

$$s = \gcd(|w_1 v_2 - w_2 v_1|, l_2).$$

So $S_n = \mathbb{P}(\mathcal{O} \oplus L_n)$ is a smooth projective algebraic variety.

$\square$

For the first Chern class of the contact bundle we have:

Lemma 6.6. *Let $\mathcal{D}_{l_1,l_2,\mathbf{w}}$ be the contact structure on $M^5_{l_1,l_2,\mathbf{w}}$. Then*

$$c_1(\mathcal{D}_{l_1,l_2,\mathbf{w}}) = ((2-2g)l_2 - l_1|\mathbf{w}|)\gamma \tag{6.33}$$

where γ is a generator of

$$H^2(M^5_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}$$

and $|\mathbf{w}| = w_1 + w_2$.

For the second Stiefel-Whitney class,$w_2(M^5_{l_1,l_2,\mathbf{w}})$ we have

$$w_2(M^5_{l_1,l_2,\mathbf{w}}) = l_1|\mathbf{w}|\gamma(mod 2)$$

Proof. The orbifold canonical divisor of $\Sigma_g \times \mathbb{CP}(\mathbf{w})$ is

$$\begin{aligned} K^{orb} &= K_{\Sigma_g \times \mathbb{CP}^1} + (1 - \frac{1}{w_1})E_0 + (1 - \frac{1}{w_2})E_0 \\ &= -(2-2g)F - 2E_0 + (1 - \frac{1}{w_1})E_0 - (1 + \frac{1}{w_2})E_0 \\ &= -(2-2g)F - \frac{|\mathbf{w}|}{w_1 w_2}E_0. \end{aligned} \tag{6.34}$$

While the orbifold first Chern class c_1^{orb} of $-K^{orb}$ is a rational class in $H^2(\Sigma_g \times \mathbb{CP}(\mathbf{w}), \mathbb{Q})$, it defines an integral class in the orbifold cohomology $H^2_{orb}(\Sigma_g \times \mathbb{CP}(\mathbf{w}), \mathbb{Z})$ defined as the cohomology of the classifying space of the orbifold (see Section 4.3 of

[BG08]). Namely, the orbifold first Chern class c_1^{orb} of $\Sigma_g \times \mathbb{CP}(\mathbf{w})$ satisfies

$$p^* c_1^{orb}(\Sigma_g \times \mathbb{CP}(\mathbf{w})) = 2(1-g)\alpha + |\mathbf{w}|\beta \tag{6.35}$$

where p is the classifying map of the orbifold $\Sigma_g \times \mathbb{CP}(\mathbf{w})$ and α is a generator in $H^2(\Sigma_g, \mathbb{Z})$ and β is a generator in $H^2_{orb}(\mathbb{CP}(\mathbf{w}), \mathbb{Z})$. It follows from the definition of the $(l_1, 1_2)$-join that α pulls back to l_2 times a generator and β pulls back to l_1 times a generator. Thus, since $\pi^*\omega = d\eta_{l,\mathbf{w}}$ we have $l_1\pi^*\alpha + \pi^* l_2\beta = 0$. So we can take $\pi^*\alpha = l_2\gamma$ and $\pi^*\beta = -l_1\gamma$. This gives Equation (6.33) and proves the result. □

Concerning the Kähler class we are going to use the diagram:

$$\begin{array}{ccccc} & & M_g^3 \times S^3_{\mathbf{w}} & & \\ & & \downarrow \pi_L & & \\ & & M_{l_1,l_2,\mathbf{w}} & & \\ & \swarrow \pi_{\mathbf{w}} & & \searrow \pi_{\mathbf{v}} & \\ \Sigma_g \times \mathbb{CP}^1[\mathbf{w}] & & & & (S_n, \Delta) \\ & p_{\mathbf{w}} \searrow & & \swarrow p_{\mathbf{v}} & \\ & & \Sigma_g & & \end{array} \tag{6.36}$$

where $p_{\mathbf{w}}, p_{\mathbf{v}}$ are the obvious projections.

For the following lemma I follow the corresponding lemma in [BTF13].

Lemma 6.7. *The induced Kähler class on $B_{l_1,l_2,\mathbf{w}} = (S_n, \Delta)$ takes the form*

$$k_1 p_v^*[\omega_g] + k_2 PD(D_1)$$

Where D_1 is the zero section.

Proof. From the commutative diagram (6.36) on degree 2 cohomology

$$ker\pi_B^* = (\pi_v \circ \pi_L)^*$$

has dimension 2. Let us see that $p_v^*[\omega_g]$ and $PD(D_1)$ span $ker\pi_B^*$. From the definition of the join $p_v^*[\omega_g]$ is in $ker\pi_B^*$. Furthermore

$$(p_v \circ \pi_v \circ \pi_L)^* : H^2(\Sigma_g;\mathbb{Z}) \to H^2(M_g^3 \times L(l_2; l_1w_1, l_1w_2);\mathbb{Z})$$

has dimension one so it must be spanned by $[\omega_g]$. Since $p_v^*[\omega_g]$ is in $ker\pi_B^*$ and $(p_v \circ \pi_v)^*[\omega_g] = l_2\gamma$ we have that $\pi_L^*\gamma = 0$. So we get that $PD(D_1)$ is in the kernel of π_B^* and since it is independent of $p_v^*[\omega_g]$ we must have that $p_v^*[\omega_g]$ and $PD(D_1)$ span $ker\pi_B^*$. From the observation that the induced Kähler class on

$$B_{l_1,l_2,\mathbf{w}} = (S_n, \Delta)$$

is in $ker\pi_B^*$ the lemma follows. □

Writing the induced Kähler class $[\omega_B]$ on (S_n, Δ) as

$$[\omega_B] = k_1 p_v^*[\omega_g] + k_2 PD(D_1)$$

We have

Lemma 6.8. *The following is satisfied:*

1. $k_2 = l_2$
2. $k_1 = m_1 l_1 w_2$

Proof. We know that $\pi_v^*[\omega_g]$ is a trivial class in $M_{l_1,l_2,\mathbf{w}}$ and $(p_v \circ \pi_v)^*[\omega_g] = l_2\gamma$ and $\pi_v^* PD(D_1) = -m_1 l_1 w_2 \gamma$ we see that

$$k_1 l_2 - k_2 m_1 l_1 w_2 = 0$$

and since $gcd(k_1, k_2) = m = l_2/s$ we have that $k_2 = l_2$ and $k_1 = m_1 l_1 w_2$. □

6.1.7 Extremal Sasakian Structures

Given a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on a compact manifold M^{2n+1} we deform the contact 1-form by $\eta \mapsto \eta(t) = \eta + t\zeta$ where ζ is a basic 1-form with respect to the characteristic foliation $\mathcal{F}_\xi$ defined by the Reeb vector field ξ. Here t lies in a suitable interval containing 0 and such that $\eta(t) \wedge d\eta(t) \neq 0$. This gives rise to a family of Sasakian structures $\mathcal{S}(t) = (\xi, \eta(t), \Phi(t), g(t))$ that we denote by $\mathcal{S}(\xi, \bar{J})$ where $\bar{J}$ is the induced complex structure on the normal bundle $\nu(\mathcal{F}_\xi) = TM/L_\xi$ to the Reeb foliation $\mathcal{F}_\xi$ which satisfy the initial condition $\mathcal{S}(0) = \mathcal{S}$. On the space $\mathcal{S}(\xi, \bar{J})$ we consider the "energy functional" $E : \mathcal{S}(\xi, \bar{J}) \longrightarrow \mathbb{R}$ defined by

$$E(g) = \int_M s_g^2 d\mu_g, \tag{6.37}$$

i.e. the L^2-norm of the scalar curvature s_g of the Sasaki metric g. Critical points g of this functional are called *extremal Sasakian metrics*. Similar to the Kählerian case, the Euler-Lagrange equations for this functional says [BGS08] that g is critical if and only if the gradient vector field $J\mathrm{grad}_g s_g$ is transversely holomorphic, so, in particular, Sasakian metrics with constant scalar curvature are extremal. Since the scalar curvature s_g is related to the transverse scalar curvature s_g^T of the transverse Kähler metric by $s_g = s_g^T - 2n$, a Sasaki metric is extremal if and only if its transverse Kähler metric is extremal. Hence, in the regular (quasi-regular) case, an extremal Kähler metric lifts to an extremal Sasaki metric, and conversely an extremal Sasaki metric projects to an extremal Kähler metric.

6.1.8 Admissible metrics

This is the admissible construction for the smooth case, for the quasiregular case see [BTF14b].

Assume (M, J) equals the total space of $\mathbb{P}(\mathcal{O} \oplus L_n) \to \Sigma_g$, where $L_n \to \Sigma_g$ is a holomorphic line bundle of degree $n > 0$.

Let us consider a ruled manifold of the form $S_n = \mathbb{P}(\mathcal{O} \oplus L_n) \to \Sigma_g$, where L_n is a holomorphic line bundle of degree n, where $n \in \mathbb{Z}^+$ on Σ and $\mathcal{O}$ is the trivial holomorphic line bundle. Let g_{Σ_g} be the Kähler metric on Σ_g of constant scalar curvature $2s_{\Sigma_g}$, with Kähler form ω_{Σ_g}, such that $c_1(L_n) = [\frac{\omega_{\Sigma_g}}{2\pi}]$. That is, S_n is the $\mathbb{CP}^1$ bundle over Σ_g associated to a principal S^1 bundle over Σ_g with curvature ω_g. Let $\mathcal{K}_{\Sigma_g}$ denote the canonical bundle of Σ_g. Since $c_1(\mathcal{K}_{\Sigma_g}^{-1}) = [\rho_{\Sigma_g}/2\pi]$, where ρ_{Σ_g} denotes the Ricci form, we have the relation $s_{\Sigma_g} = 2(1-g)/n$. For each smooth function $\Theta(\mathfrak{z}), \mathfrak{z} \in [-1,1]$ satisfying

$$\begin{aligned} (i)\ \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii)\ \Theta(\pm 1) = 0, \\ (iii)\ \Theta'(-1) = 2, \quad \Theta'(1) = -2 \end{aligned} \tag{6.38}$$

and each $0 < r < 1$ we obtain admissible Kähler metrics;

$$g = \frac{1 + r\mathfrak{z}}{r} g_{\Sigma_g} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \tag{6.39}$$

with Kähler form

$$\omega = \frac{1 + r\mathfrak{z}}{r} \omega_{\Sigma_g} + d\mathfrak{z} \wedge \theta\,. \tag{6.40}$$

as in [ACGTF08]

Notice that $\mathfrak{z} : S_n \to [-1,1]$ is a moment map of ω and the circle action $\tilde{\mathcal{A}}_n(\lambda)$ (generated by the vector field $K = Jgrad\mathfrak{z}$). Further θ is a 1-form such that $\theta(K) = 1$ and $d\theta = \pi^*\omega_{\Sigma_g}$.

As usual $E_n = \mathbb{P}(\mathcal{O} \oplus 0)$ denotes the zero section and F denotes the fiber of the bundle $S_n \to \Sigma_g$. Then $E_n^2 = n$. Note that in the admissible set-up $E_n = \mathfrak{z}^{-1}(1)$.

The Kähler class of this metric satisfies

$$\mathrm{PD}([\omega]) = 4\pi E_n + \frac{2\pi(1-r)n}{r} F.$$

Writing $F(\mathfrak{z}) = \Theta(\mathfrak{z})(1 + r\mathfrak{z})$, we see from Proposition 1 in [ACGTF08] that the corresponding metric is extremal exactly when $F(\mathfrak{z})$ is a polynomial of degree at most 4 and $F''(-1/r) = 2rs_{\Sigma_g}$. This, as well as the endpoint conditions of (6.38), is satisfied precisely when $F(\mathfrak{z})$ is given by

$$F(\mathfrak{z}) = \frac{(1-\mathfrak{z}^2)h(\mathfrak{z})}{4(3-r^2)}, \tag{6.41}$$

where

$$\begin{aligned} h(\mathfrak{z}) &= (12 - 8r^2 + 2r^3 s_{\Sigma_g}) \\ &+ 4r(3-r^2)\mathfrak{z} \\ &+ 2r^2(2 - r s_{\Sigma_g})\mathfrak{z}^2, \end{aligned}$$

and $-1 < \mathfrak{z} < 1$.

When $s_{\Sigma_g} \geq 0$, i.e., $g \leq 1$, we can then check that $h(\mathfrak{z}) > 0$ for $-1 < \mathfrak{z} < 1$ hence $\Theta(\mathfrak{z})$, as defined via $F(\mathfrak{z})$ above, satisfies all the conditions of (6.38). Thus in this case, for all $r \in (0,1)$ we have an extremal Kähler metric. However, for $g \geq 2$, i.e., $s_{\Sigma_g} < 0$ the positivity of $h(\mathfrak{z})$, hence $\Theta(\mathfrak{z})$ for $-1 < \mathfrak{z} < 1$ holds only when $0 < r < 1$ is sufficiently small.

The following theorem is a special case of theorem 5.1 in [BTF14a]

Theorem 6.1. *For any choice of genus $g = 1, 2, \ldots, 19$ the regular ray in the Sasaki cone $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ admits an extremal representative with non-constant scalar curvature.*

For any choice of genus $g = 20, 21, \ldots$ there exists a $K_g \in \mathbb{Z}^+$ such that if $l_1|\mathbf{w}| \geq K_g$, then the regular ray in the Sasaki cone $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ admits an extremal representative with non-constant scalar curvature.

For any choice of genus $g = 20, 21, \ldots$ there exist at least one choice of (l_1, w_1, w_2)

such that the regular ray in the Sasaki cone $\kappa(M^5_{l_1,l_2,\mathbf{w}}, J_{\mathbf{w}})$ admits no extremal representative, despite the fact that the quasi-regular Sasaki structure

$$\mathcal{S}_{l_1,l_2,\mathbf{w}} = (\xi_{\mathbf{w}}, \eta_{l_1,l_2,\mathbf{w}}, \Phi_{\mathbf{w}}, g_{\mathbf{w}})$$

is extremal.

The following theorem is a special case of theorem 1.1 in [BTF14b]

Theorem 6.2. *Let*

$$M^5_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}.$$

Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with gcd$(w_1, w_2) = 1$ and $w_1 > w_2$ there exists a ξ_v in the Sasaki cone on $M^5_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures

$$\mathcal{S}_a = (a^{-1}\xi_v, a\eta_v, \Phi, g_a)$$

has constant scalar curvature.

Most of these metrics are irregular.

6.2 The Cohomology of the Join

In this section we want to compute the cohomology groups of the join

$$M^5_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}.$$

The fibration π_L in Diagram (6.11) together with the torus bundle with total space $M^3_g \times S^3_{\mathbf{w}}$ gives the commutative diagram of fibrations

$$\begin{array}{ccccc} M^3_g \times S^3_{\mathbf{w}} & \longrightarrow & M_{l_1,l_2,\mathbf{w}} & \longrightarrow & \mathsf{B}S^1 \\ \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{\psi} \\ M^3_g \times S^3_{\mathbf{w}} & \longrightarrow & \Sigma_g \times \mathsf{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}] & \longrightarrow & \mathsf{B}S^1 \times \mathsf{B}S^1 \end{array} \tag{6.42}$$

where $\mathsf{B}G$ is the classifying space of a group G or Haefliger's classifying space [Hae84] of an orbifold if G is an orbifold. Note that the lower fibration is a product of fibrations. In particular, the fibration

$$S^3_{\mathbf{w}} \longrightarrow \mathsf{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}] \longrightarrow \mathsf{B}S^1 \tag{6.43}$$

is rationally equivalent to the Hopf fibration, so over $\mathbb{Q}$ the only non-vanishing differentials in its Leray-Serre spectral sequence are $d_4(\beta) = s^2$ where β is the orientation class of S^3 and s is a basis in $H^2(\mathsf{B}S^1, \mathbb{Q}) \approx \mathbb{Q}$ and those induced from d_4 by naturality. However, we want the cohomology over $\mathbb{Z}$. The proof of the following lemma is in [BTF13].

Lemma 6.9. *For w_1 and w_2 relatively prime positive integers we have*

$$H^r_{orb}(\mathbb{C}\mathbb{P}^1[\mathbf{w}], \mathbb{Z}) = H^r(\mathsf{B}\mathbb{C}\mathbb{P}^1[\mathbf{w}], \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 2, \\ \mathbb{Z}_{w_1 w_2} & \text{for } r > 2 \text{ even}, \\ 0 & \text{for } r \text{ odd}. \end{cases}$$

We can see that Lemma 6.9 implies

Lemma 6.10. *The only non-vanishing differentials in the Leray-Serre spectral sequence of the fibration (6.43) are those induced naturally by* $d_4(\alpha) = w_1 w_2 s^2$ *for* $s \in H^2(\mathsf{B}S^1, \mathbb{Z}) \approx \mathbb{Z}[s]$ *and* α *the orientation class of* S^3.

Now the map ψ of Diagram (6.42) is that induced by the inclusion $e^{i\theta} \mapsto (e^{il_2\theta}, e^{-il_1\theta})$. So noting

$$H^*(\mathsf{B}S^1 \times \mathsf{B}S^1, \mathbb{Z}) = \mathbb{Z}[s_1, s_2]$$

we see that $\psi^* s_1 = l_2 s$ and $\psi^* s_2 = -l_1 s$. This together with Lemma 6.10 gives $d_4(\alpha) = w_1 w_2 l_1^2 s^2$ in the Leray-Serre spectral sequence of the top fibration in Diagram (6.42). Here is the main result of this section

Proposition 6.5. *The cohomology groups of* $M^5_{l_1,l_2,\mathbf{w}}$ *are given by*

$$H^r(M^5_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{for } r = 0, 5 \\ \mathbb{Z}^{2g} & \textit{for } r = 1 \\ \mathbb{Z} & \textit{for } r = 2 \\ \mathbb{Z} \oplus \mathbb{Z}^{2g}_{l_2} & \textit{for } r = 3 \\ \mathbb{Z}^{2g} & \textit{for } r = 4 \end{cases}$$

Moreover the cohomology ring of $M^5_{l_1,l_2,\mathbf{w}}$ *is given by*

$$\mathbb{Z}[\alpha_i, \beta_i, x, y, \gamma]/(J, x^2, xy = \gamma, l_2\alpha_i \cup x, l_2\beta_i \cup x)$$

where J is an ideal described by: For the canonical homology basis of Σ_g

$$H_1(\Sigma_g) = \{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$$

and

$$H^1(\Sigma_g) = \{\alpha_1, \alpha_2, \cdots \alpha_g, \beta_1, \beta_2, \cdots, \beta_g\}$$

the dual basis; then

$$\alpha_j \cup \beta_k = -\delta_{jk} x$$

for $j, k = 1, 2, \cdots, g$ *where deg x=2,deg y=3 and* γ *is the orientation class in*

$$M^5_{l_1,l_2,\mathbf{w}}$$

Proof. From the Leray-Serre spectral sequence of the bundle

$$S^1 \to M^3_g \to \Sigma_g$$

but in the form:

$$M^3_g \to \Sigma_g \to BS^1$$

we get the differentials:

$$d_2(\alpha_i) = 0, i = 1, \cdots, 2g$$

where α_i are classes in dimension 1.

$$d_2(\beta_i) = \alpha_i \otimes s, i = 1, \cdots, 2g$$

where β_i are classes in dimension 2 and

$$H^2(\mathsf{B}S^1, \mathbb{Z}) \approx \mathbb{Z}[s]$$

$$d_4(\gamma_g) = s^2$$

where γ_g, is a class in dimension 3.

From here we can compute the cohomology of M_g^3 namely :

$$H^r(M_g^3, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 3, \\ \mathbb{Z}^{2g} & \text{for } r = 1, 2 \end{cases}$$

From the Leray-Serre spectral sequence of the fibration

$$M_g^3 \times S^3_{\mathbf{w}} \to M^5_{l_1, l_2, \mathbf{w}} \to BS^1$$

We get the differentials

$$d_2(\alpha_i) = 0, i = 1, \cdots, 2g$$

where α_i are classes in dimension 1.

$$d_2(\beta_i) = \alpha_i \otimes l_2 s,$$

$i = 1, \cdots, 2g$ where β_i are classes in dimension 2.

$$d_2(\gamma_0) = 0$$

$$d_2(\gamma_g) = 0$$

where γ_g,γ_0 are classes in dimension 3.

$$d_4(\gamma_0) = w_1 w_2 l_1^2 s^2,$$

$$d_4(\gamma_g) = l_2^2 s^2$$

since

$$d_4(w_1 w_2 l_1^2 \gamma_g - l_2^2 \gamma_0) = 0$$

and since $\gcd(l_2, w_1 w_2 l_1) = 1$ there are a and b such that

$$a l_2^2 + b w_1 w_2 l_1^2 = 1$$

and

$$d_4(a\gamma_g + b\gamma_0) = (a l_2^2 s^2 + b w_1 w_2 l_1^2 s^2) = s^2$$

and

$$d_4(\alpha_i \otimes \gamma_0) = \alpha_i \otimes d_4(\gamma_0) = \alpha_i \otimes w_1 w_2 l_1^2 s^2,$$

where $s^2 \in \mathbb{Z}_{l_2}$ that is in dimension four there is no torsion.

In dimension 1 the cohomology is generated by $\{\alpha_i\}$, $i = 1 \cdots 2g$ from here follows that

$$H^1(M^5_{l_1,l_2,\mathbf{w}};\mathbb{Z}) = \mathbb{Z}^{2g}$$

In dimension 2 the class s generates a $\mathbb{Z}$ and there is no torsion so

$$H^2(M^5_{l_1,l_2,\mathbf{w}};\mathbb{Z}) = \mathbb{Z}$$

in dimension 3 by Poincaré duality the free part of the cohomology in dimension 3 is equal to the free part of the cohomology in dimension 2 which is $\mathbb{Z}$ and the torsion is generated by $d_2(\beta_i) = \alpha_i \otimes \bar{s}$, for $\bar{s} \in \mathbb{Z}_{l_2}$ for $i = 1 \cdots, 2g$ so

$$H^3(M^5_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}^{2g}_{l_2}$$

In dimension 4 by Poincaré duality the free part of the cohomology in dimension 4 is equal to the cohomology in dimension 1 which is $\mathbb{Z}^{2g}$ and there is no torsion so

$$H^4(M^5_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}^{2g}$$

from here follows that the cohomology of $M^5_{l_1,l_2,\mathbf{w}}$ is

$$H^r(M^5_{l_1,l_2,\mathbf{w}}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } r = 0, 5 \\ \mathbb{Z}^{2g} & \text{for } r = 1 \\ \mathbb{Z} & \text{for } r = 2 \\ \mathbb{Z} \oplus \mathbb{Z}^{2g}_{l_2} & \text{for } r = 3 \\ \mathbb{Z}^{2g} & \text{for } r = 4 \end{cases}$$

For the proof of the cohomology ring notice that the ideal J is the cohomology ring of the Riemann surface Σ_g and the relations defining J pass in the spectral sequence from Σ_g to $M^5_{l_1,l_2,\mathbf{w}}$ and x is a generator of $H^2(M^5_{l_1,l_2,\mathbf{w}},\mathbb{Z}) = \mathbb{Z}$ so $x^2 = 0$. That $xy = \gamma$ is clear because xy has degree 5 and there is only one class in degree 5 namely γ. Notice that $\alpha_i \cup l_2 x = 0$ and $\beta_i \cup l_2 x = 0$ because they have degree 3 and the torsion part of the cohomology in dimension 3 is generated by $\alpha_i \otimes l_2 s$ and $\beta_i \otimes l_2 s$ and in the limit of the spectral sequence this becomes $\alpha_i \cup l_2 x$ and $\beta_i \cup l_2 x$. $\square$

We also have:

Proposition 6.6.

$$\pi_1(M^5_{l_1,l_2,\mathbf{w}}) = \pi_1(M^3_g)/(l_2\mathbb{Z})$$

$$\pi_2(M^5_{l_1,l_2,\mathbf{w}}) = 0$$

$$\pi_i(M^5_{l_1,l_2,\mathbf{w}}) = \pi_i(S^3)$$

for $i > 2$

Proof. First to prove $\pi_2(M^5_{l_1,l_2,\mathbf{w}}) = 0$ we represent $M^5_{l_1,l_2,\mathbf{w}}$ as a $L(l_2; w_1, w_2)$ bundle over Σ_g that is

$$L(l_2; w_1, w_2) \longrightarrow M^5_{l_1,l_2,\mathbf{w}} \longrightarrow \Sigma_g$$

by taking the homotopy exact sequence we get that

$$\cdots \longrightarrow \pi_2(L(l_2; w_1, w_2)) \longrightarrow \pi_2(M^5_{l_1,l_2,\mathbf{w}}) \longrightarrow \pi_2(\Sigma_g) \cdots$$

and from the well known facts $\pi_2(L(l_2;w_1,w_2))=0$ and $\pi_2(\Sigma_g)=0$ we get

$$\pi_2(M^5_{l_1,l_2,\mathbf{w}})=0$$

To compute the $\pi_1(M^5_{l_1,l_2,\mathbf{w}})$ we represent $M^5_{l_1,l_2,\mathbf{w}}$ as

$$S^1 \longrightarrow M^3_g \times S^3_{\mathbf{w}} \longrightarrow M^5_{l_1,l_2,\mathbf{w}}$$

by taking the homotopy exact sequence we get that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\delta} \pi_1(M^3_g \times S^3_{\mathbf{w}}) \longrightarrow \pi_1(M^5_{l_1,l_2,\mathbf{w}}) \longrightarrow 1$$

and observe that the map δ is multiplication by l_2 so we have

$$\pi_1(M^5_{l_1,l_2,\mathbf{w}})=\pi_1(M^3_g)/(l_2\mathbb{Z})$$

Now we want to calculate the homotopy groups of $M^5_{l_1,l_2,\mathbf{w}}$ for this from

$$M^3_g \to \Sigma_g \to BS^1$$

by taking the exact homotopy sequence we have

$$\cdots \to \pi_{i+1}(BS^1) \to \pi_i(M^3_g) \to \pi_i(\Sigma_g) \to \pi_i(BS^1) \to \cdots$$

and by using $\pi_{i+1}(BS^1)=\pi_i(S^1)$ and $\pi_i(S^1)=0$ and $\pi_i(\Sigma_g)=0$ for $i>1$ we have

$$0 \to \pi_i(M^3_g) \to 0$$

for $i > 1$ and so we get

$$\pi_i(M_g^3) = 0$$

for $i > 1$ so we see that

$$M_g^3 = K(\Gamma_g^3, 1)$$

is an Eilenberg Mc Lane space. Now from

$$S^1 \to M_g^3 \times S^3 \to M^5_{l_1,l_2,\mathbf{w}}$$

by taking the exact homotopy sequence we have

$$\cdots \to \pi_i(S^1) \to \pi_i(M_g^3 \times S^3) \to \pi_i(M^5_{l_1,l_2,\mathbf{w}}) \to \pi_{i-1}(S^1) \to \cdots$$

for $i > 2$ we have

$$0 \to \pi_i(S^3) \to \pi_i(M^5_{l_1,l_2,\mathbf{w}}) \to 0$$

so we see that

$$\pi_i(M^5_{l_1,l_2,\mathbf{w}}) = \pi_i(S^3)$$

for $i > 2$. Notice that

$$\pi_i(S^3)$$

for $i > 2$ are known for some i[Tod62] □

6.3 Minimal Models

In this section we want to compute the Minimal Model in the sense of Sullivan [Sul77] of

$$M = M_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}.$$

Definition 6.2. *A differential graded algebra is a graded algebra*

$$\mathcal{A} = \oplus_{k\geq 0}\mathcal{A}^k$$

with a differential $d : \mathcal{A} \to \mathcal{A}$ *of degree* $+1$, *such that*

- $\mathcal{A}$ *is graded commutative i.e.*

  $$x \cdot y = (-1)^k y \cdot x,$$

  $x \in \mathcal{A}^k$,$y \in \mathcal{A}^l$.

- d *is a derivation, i.e.*

  $$d(x \cdot y) = dx \cdot y + (-1)^k x \cdot dy,$$

  for $x \in \mathcal{A}^k$.

- $$d^2 = 0.$$

The cohomology $H^*(\mathcal{A})$ is an algebra, we shall always assume that it is finite dimensional in each degree. $\mathcal{A}$ is connected if $H^0(\mathcal{A})$ is the ground field and $\mathcal{A}$ is one connected if $H^1(\mathcal{A}) = 0$. A map between two differential algebras $\mathcal{A}$ and $\mathcal{B}$ is an algebra homomorphism

$$f : \mathcal{A} \to \mathcal{B}$$

preserving the grading and d, such a map induces an algebra map

$$f^* : H^*(\mathcal{A}) \to H^*(\mathcal{B})$$

. If $\mathcal{A}^0$ is the ground field, we define the augmentation ideal

$$A(\mathcal{A}) = \oplus_{k>0}\mathcal{A}^k$$

and the graded space of indecomposables

$$I(\mathcal{A}) = A(\mathcal{A})/(A(\mathcal{A}) \cdot A(\mathcal{A}))$$

In such algebra the derivation d is decomposable if for each $x \in \mathcal{A}$ $dx \in \mathcal{A} \cdot \mathcal{A}$. An elementary extension of $(\mathcal{A}, d_{\mathcal{A}})$ is any algebra $\mathcal{B}$ of the form

$$(\mathcal{B} = \mathcal{A} \otimes \Lambda(V_k), d_{\mathcal{B}})$$

where $d_{\mathcal{B}}$ restricted to $\mathcal{A}$ is $d_{\mathcal{A}}$, $d_{\mathcal{B}}(V_k) \subset \mathcal{A}$ and V_k is a finite dimensional vector space. $\mathcal{M}$ is a minimal differential algebra if $\mathcal{M}$ (if $\mathcal{M}_0$ is the ground field)can be written as

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$$

$$\mathcal{M} = \bigcup_{i \geq 0} \mathcal{M}_i$$

with $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ an elementary extension, d decomposable and $\mathcal{M}$ free as an algebra. Such a collection of subalgebras is a series for $\mathcal{M}$. Let $\mathcal{M}^{(i)}$ denote the subalgebra of $\mathcal{M}$ generated by elements of degree $\leq i$. d being decomposable implies that these are subalgebras of $\mathcal{M}$. If $\mathcal{M}$ is a one connected minimal algebra then $\mathcal{M}^1 = 0$ and

$$\mathcal{M}^{(2)} \subset \mathcal{M}^{(3)} \subset \cdots$$

is a series for $\mathcal{M}$. Thus a one connected minimal algebra has a canonical series. For any minimal differential algebra $\mathcal{M}$, $\mathcal{M}^{(1)}$ also has a canonical series

$$0 \subset \mathcal{M}_1^{(1)} \subset \mathcal{M}_2^{(1)} \subset \cdots$$

$$\mathcal{M}^{(1)} = \bigcup_{i \geq 0} \mathcal{M}_i^{(1)}$$

where $\mathcal{M}_i^{(1)}$ = algebra generated by $x \in \mathcal{M}^{(1)}$ such that $dx \in \mathcal{M}_{i-1}^{(1)}$.

Now if $\mathcal{A}$ is a differential algebra then $\rho : \mathcal{M}_{\mathcal{A}} \to \mathcal{A}$ is a k-stage minimal model for $\mathcal{A}$ if

- $\mathcal{M}_{\mathcal{A}}$ is a minimal algebra generated in dimensions $\leq k$.
- ρ induces an isomorphism on cohomology in dimensions $\leq k$ and an injection in dimension $k+1$. When $k = \infty$ $\mathcal{M}_{\mathcal{A}}$ is a minimal model for $\mathcal{A}$.

We have the following

Theorem 6.3. *[DGMS75, GM78]*

- *Every one connected differential algebra has a minimal model unique up to iso-*

morphism.

- *Every connected differential algebra has a 1- stage minimal model unique up to isomorphism.*

Define the de Rham homotopy groups of a one connected differential algebra $\mathcal{A}$, $\pi_*(\mathcal{A})$ (with Whitehead product), to be those of any minimal model for $\mathcal{A}$. Define the de Rham fundamental group of a differential algebra to be the de Rham fundamental group of any minimal model for the algebra. Let $\mathcal{M}$ be a minimal differential algebra and $H^*(\mathcal{M})$ the cohomology viewed as a differential algebra with $d = 0$. $\mathcal{M}$ is formal if there is a map of differential algebras

$$\psi : \mathcal{M} \to (H^*(\mathcal{M}), d = 0)$$

inducing an isomorphism on cohomology. The homotopy type of a differential algebra $\mathcal{A}$ is a formal consequence of its cohomology if its minimal model is formal.

We have the following

Theorem 6.4. *[DGMS75, GM78] Let $\mathcal{M}$ be a differential graded algebra. Let V_i be a vector space containing elements of degree i only and $C_i \subset V_i$, containing only closed elements.Let N_i be a vector space. $\mathcal{M}$ is formal iff*

$$V_i = C_i \oplus N_i,$$

such that any closed form in the ideal $I(\oplus N_i)$ is exact.

Aaron Tievsky proved in his thesis [Tie08] that

If M is a Sasakian manifold the Minimal model of M is given by

$$H_{\mathcal{B}}(\mathcal{F}_\xi) \otimes \Lambda(y)$$

Where

$$H_{\mathcal{B}}(\mathcal{F}_\xi)$$

is the basic cohomology of M and $\mathcal{F}_\xi$ is the characteristic foliation of M and

$$dy = [d\eta]_B$$

is the Kähler class. Moreover deg y= 1. We want to compute the Minimal Model of

$$M = M_{l_1,l_2,\mathbf{w}} = M^3_g *_{l_1,l_2} S^3_{\mathbf{w}}.$$

Here is the main result of this section.

Proposition 6.7. *The Minimal Model of $M^5_{l_1,l_2\mathbf{w}}$ is given by*

$$\mathbb{R}[\alpha_i, \beta_i]/J \otimes \Lambda(\beta) \otimes \Lambda(y)$$

where β is a 2 class and J is an ideal described by: For the canonical homology basis of Σ_g,

$$H_1(\Sigma_g) = \{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$$

and

$$H^1(\Sigma_g) = \{\alpha_1, \alpha_2, \cdots \alpha_g, \beta_1, \beta_2, \cdots, \beta_g\}$$

the dual basis; then

$$\alpha_j \cup \beta_k = -\delta_{jk}\eta_{\Sigma_g}$$

for $j, k = 1, 2, \cdots, g$ *where* η_{Σ_g} *is the fundamental class of* Σ_g*.*

Proof. If $\pi : M^5_{l_1,l_2\mathbf{w}} \longrightarrow B_{1,\mathbf{w}}$ denotes the S^1 bundle map, we have by using the result of Tievsky that the Minimal Model of $M^5_{l_1,l_2\mathbf{w}}$ is given by

$$H_{\mathcal{B}}(\mathcal{F}_\xi) \otimes \Lambda(y)$$

But by proposition 7.2.2 in [BG08] which says that

$$H_{DR}(B_{1,\mathbf{w}}; \mathbb{R}) \approx H_{\mathcal{B}}(\mathcal{F}_\xi)$$

as rings.

So we have that the Minimal Model of $M^5_{l_1,l_2\mathbf{w}}$ is given by

$$H_{DR}(B_{1,\mathbf{w}}; \mathbb{R}) \otimes \Lambda(y)$$

but we know that

$$B_{1,\mathbf{w}} = \Sigma_g \times \mathbb{CP}[w]$$

and the cohomology ring of $\Sigma_g \times \mathbb{CP}[w]$ is

$$\mathbb{R}[\alpha_i, \beta_i]/J \otimes \Lambda(\beta)$$

with the ideal J given in the proposition, which is the cohomology ring of Σ. So the minimal model of $M^5_{l_1,l_2\mathbf{w}}$ is given by

$$\mathbb{R}[\alpha_i, \beta_i]/J \otimes \Lambda(\beta) \otimes \Lambda(y)$$

with the J given in the proposition. $\square$

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