10-8-2004

Almost Global Asymptotic Formation Stabilization Using Navigation Functions

Amit Kumar

Herbert G. Tanner

Department of Mechanical Engineering

The University of New Mexico

Almost Global Asymptotic Formation Stabilization Using Navigation Functions

Amit Kumar
Department of Mechanical Engineering
The University of New Mexico
Albuquerque, NM 87131
e-mail: akumar@unm.edu

Herbert G. Tanner
Department of Mechanical Engineering
The University of New Mexico
Albuquerque, NM 87131
e-mail: tanner@unm.edu

UNM Technical Report: ME-TR-04-001

Report Date: October 8, 2004

Abstract

We present a navigation function through which a group of mobile agents can be coordinated to achieve a particular formation, both in terms of shape and orientation, while avoiding collisions between themselves and with obstacles in the environment. Convergence is global and complete, subject to the constraints of the navigation function methodology. Algebraic graph theoretic properties associated with the interconnection graph are shown to affect the shape of the navigation function. The approach is centralized but the potential function is constructed in a way that facilitates complete decentralization. The strategy presented will also serve as a point of reference and comparison in quantifying the cost of decentralization in terms of performance.

Keywords

Formations, navigation functions, formation function, cooperative control, graph theory

1 Introduction

In the recent years the problem of collision free navigation of multiple agents to achieve a desired formation has attracted considerable attention. The basic motivation arises from the fact that multi agent navigation, forms an integral part of the systems which require coordination to achieve a certain task. There is a lot of work in the area of formation control for multi-agent systems [7, 14, 10, 5, 18], with applications in the field of networked UAVs, aircraft formations, satellite clusters, etc. Occasionally, formation control is linked to motion planning, obstacle avoidance and navigation.

There are primarily two ways to address the problem of motion planning for a team of mobile agents. The problem has been addressed in the past with centralized navigation scheme by a number of groups [1, 2, 8]. In the centralized architecture there is single control law and the collision free trajectories are constructed in the composite configuration space. A similar idea has been exploited in [4], where the formation objectives and constraints have been encoded in a *Formation function*. In paper [8], a centralized cooperative control strategy using an artificial potential function is presented for a planar world.

The centralized approach involves computational complexity and is based on the premises that agent state information can always be communicated infinitely fast, but it guarantees completeness of solution. In order to allow the control architecture to scale nicely with the size of the group, decentralized solutions are alternatively sought. These typically involve the combined effect of individual, agent-based local potential fields [3, 6, 13, 17] Obstacle avoidance in a moving formation using potential field based [15, 13, 3], and reactive or optimal control approaches [9] have also been addressed.

The problem with current decentralized motion planning and formation control schemes stems from the inability to predict and control the critical points of the combined, resultant potential field. This has always been the case with conventional potential field strategies. In a seminal paper [12], Rimon and Koditschek introduced the navigation function methodology and offered the first formal solution to the problem of local minima in potential field motion planning. Navigation functions are smooth real valued maps realized through suitably chosen scalar valued cost functions. Integrating the negated gradient vector field of the cost function automatically gives rise to trajectories that guarantee collision free motion and convergence to the destination from almost all initial conditions. A set of measure zero including a number of critical points has to be excluded, but it has been shown [11] that this is the best that can be done in such situations. The only decentralized approach which can guarantee convergence is the one described in [3], however, decentralization is limited in the sense that each agent essentially carries a copy of the centralized scheme, requiring full knowledge of the system and environment state.

In this paper we construct a formation constraint function which is also a navigation function, while taking special care to facilitate *complete subsequent decentralization*. The use of this navigation function guarantees that no agent will collide with environment obstacles or other agents and that the desired formation (both in terms of shape as well as orientation) will be achieved asymptotically. Having a provably correct centralized coordination scheme will not only lead us to decentralized solutions, but also provide a point of reference and comparison in order to quantify the cost of decentralization in terms of performance. An interesting feature in our analysis is that the topology of the interconnection graph –specifically its algebraic properties, as expressed by the Laplacian and Incidence matrices– finds its way into the tuning parameters of the potential field.

The rest of the paper is organised as follows. In Section 2 we present the problem statement and a brief review of the formation graph theory. In Section 3 we define the formation navigation function and discuss the construction of the goal function and the obstacle function. Section 4 we show that the potential function we have presented is indeed a navigation function. Section 5 we present the simulation results for triangular formation case. Finally in Section 6, we conclude the paper listing the issues for further work.

2 Problem Formulation

We consider a homogeneous group of mobile agents, each with dynamics given by

$$\dot{q}_i = u_i, \quad q_i \in \mathbb{R}^n,\ i = 1,\dots,N \tag{1}$$

where q_i and u_i are the state and control input of agent i, respectively. In the remaining, q and u will denote the stack vectors of q_i and u_i. The agents are treated as autonomous point-robots.

The objective here is to construct a potential field that will enable the Nagents to stabilize with respect to their groupmates in configurations that make a particular formation, while avoiding collisions between themselves and with obstacles in the environment. The desired formation is specified in terms of a labeled directed graph.

Definition 2.1 (Formation graph). *The formation graph, $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{L}\}$, is a directed labeled graph consisting of:*

- *a set of vertices (nodes), $\mathcal{V} = \{v_1,\dots,v_N\}$, indexed by the mobile agents in the group,*
- *a set of edges, $\mathcal{E} = \{(v_i,v_j) \in \mathcal{V}\times\mathcal{V}\}$, containing ordered pairs of nodes that represent inter-agent position constraints, and*
- *a set of labels, $\mathcal{L} = \{\gamma_{ij} \mid \gamma_{ij} = \left\|q_i - q_j - c_{ij}\right\|^2,\ c_{ij} \in \mathbb{R}^n,\ (v_i,v_j) \in \mathcal{E}\}$, indexed by the edges in $\mathcal{E}$.*

We will use forms from algebraic graph theory to rewrite our γ_{ij} in terms of a brief review that follows

An orientation in a graph is the assignment of a direction to each edge, so that each edge $e_{ij} = (v_i,v_j)$ is an arc from vertex i to vertex j. We denote by $\mathcal{G}^\sigma$, the graph $\mathcal{G}$ with σ orientation. The incidence matrix of a directed graph $\mathcal{G} = \{\mathcal{V},\mathcal{E},\mathcal{L}\}$, is a is a $\mathcal{V}\times\mathcal{E}$ matrix $B(\mathcal{G}^\sigma) = (b_{ij})$ such that, $b_{ij} = -1$ if edge j leaves vertex j, $b_{ij} = 1$ if edge j enters vertex j and zero otherwise.

The Laplacian of $\mathcal{G}$ is the symmetric matrix defined as $L(\mathcal{G}) = B(\mathcal{G}^\sigma)B(\mathcal{G}^\sigma)^T$, and it is independent of the orientation σ. The Laplacian matrix captures many topological properties of the graph. L is a positive semi-definite matrix and for a connected graph L has a single zero eigenvalue.

Throughout the paper, the use of the Euclidian norm is implied, i.e. $\|\cdot\| \equiv \|\cdot\|_2$. Specifying the formation constraints as edge labels in the form

$$\left\|q_i - q_j - c_{ij}\right\|^2 = 0, \quad \forall (v_i,v_j) \in \mathcal{E},$$

not only specifies inter-agent distances, but also their *relative orientation*. The workspace $W = \{q \mid \|q\| \leq R\} \subset \mathbb{R}^n$, common for all agents, is assumed to be populated by a set of disjoint configurations $p_j,\ j = 1,\dots,s$ that represent point-obstacles. The assumption that both the robots as well as the obstacles are represented by points is not as restrictive as it may seem, since it has been shown [16] that a large class of shapes can be mapped to single points through a series of transformations; this "point-world" topology can be regarded as a degenerate case of the "sphere-world" topology of Rimon and Koditchek [12].

The potential function, $\varphi(q)$, has to be constructed so that setting the agent control inputs equal to its (scaled) negated gradient:

$$u = -K\nabla\varphi(q),$$

gives rise to a closed loop system in which trajectories from *all initial conditions* (except for a set of measure zero) are collision free and converge to configurations corresponding to the desired formation.

3 Formation Navigation Function

Requiring (almost) global convergence to a desired equilibrium, suggests navigation functions as a natural design choice. In its original form [12], a navigation function $\varphi(q)$ is defined on a compact connected analytic manifold with boundary, $F \subset \mathbb{R}^n$, in the interior of which there is a destination point q_d. The navigation function is a map $\varphi : F \to [0,1]$ if it is:

1. smooth on F (at least a $C^{(2)}$ function);
2. polar on F, (q_d is a unique minimum);
3. admissible on F, (uniformly maximal on ∂F);
4. a Morse function, (its critical points are nondegenerate).

Our navigation function does not conform strictly to the above definition, since we have relaxed the admissibility requirement. This relaxation does not affect the convergence properties but rather produces a potential field with nonuniform norm on ∂F. The proposed function has the following form:

$$\varphi(q) \triangleq \frac{\gamma_d(q)}{e^{\beta(q)^{1/k}}}, \tag{2}$$

where

- $\gamma_d(q) : F \to \mathbb{R}_+$ is a positive semi-definite scalar function, assuming the value of zero only when $q = q_d$.
- $\beta(q) : F \to [0,1]$ function that vanishes only when agents are in contact with the obstacles or with one another.
- k is a (positive) tuning parameter.

In the remaining of this section we will describe how one can construct a function in the form of (2), in a way that facilitates *decentralization*. In the approach that we follow, components of γ_d and β that involve the coordinates of agent i, should not be affected by "events" that happen outside of its local neighborhood.

3.1 γ_d: the Goal Function

Function γ_d encodes the control objective: converging to the destination configuration q_d. This configuration corresponds to a desired formation, as defined by all the labels of the formation graph being set to zero. With that in mind, a reasonable choice for γ_d is

$$\gamma_d(q) \triangleq \sum_{i=1}^{|\mathcal{E}|} \gamma_{ij}(q_i, q_j). \tag{3}$$

The reasoning behind this definition is that a sum of terms involving neighboring agents in the numerator of (2) will be easier to decompose into agent-specific individual potentials.

3.2 β: the Obstacle Function

The definition of β is partially inspired by our previous work. In the original work of Rimon and Koditschek [12], β is made up as a product of several "obstacle functions", $\beta(q) = \prod \beta_j$, where each β_j is vanishing whenever the system came into contact with one of the obstacles in the environment. In this case, collisions can occur not only between an agent and an obstacle, but between agents as well. For all possible combinations of an obstacle at p_t and agent $i = 1, \ldots, N$, we define a function

$$\beta_{it} \triangleq \left(1 - \lambda \frac{(\|q_i - p_t\|^2 - d^2)^2}{(\|q_i - p_t\|^2 - d^2)^2 + 1}\right)^{\frac{\operatorname{sign}(\|q_i - p_t\| - d) + 1}{2}}, \tag{4}$$

where $i = 1, \ldots, N$, $t = 1, \ldots, s$ and d is a parameter chosen so that β_{it} is smooth:

$$\lambda = \frac{1 + d^4}{d^4}. \tag{5}$$

The workspace boundary is seen as an obstacle, (labeled 0) and is being modelled in a similar way:

$$\beta_{i0} \triangleq \left(1 - \lambda \frac{[R^2 - \|q_i\|^2 - d^2]^2}{[R^2 - \|q_i\|^2 - d^2] + 1}\right)^{\frac{\mathrm{sign}(\|q_i\| - R + d) + 1}{2}},$$

Similarly, for every pair $i, j \in \{1, \ldots, N\}$ we can have:

$$b_{ij} \triangleq \left(1 - \lambda \frac{(\left\|q_i - q_j\right\|^2 - d^2)^2}{(\left\|q_i - q_j\right\|^2 - d^2)^2 + 1}\right)^{\frac{\mathrm{sign}(\left\|q_i - q_j\right\| - d) + 1}{2}}. \tag{6}$$

These are functions that vary in $[0, 1]$ and attain their maximum value whenever the distances $\left\|q_i - q_j\right\|$ or

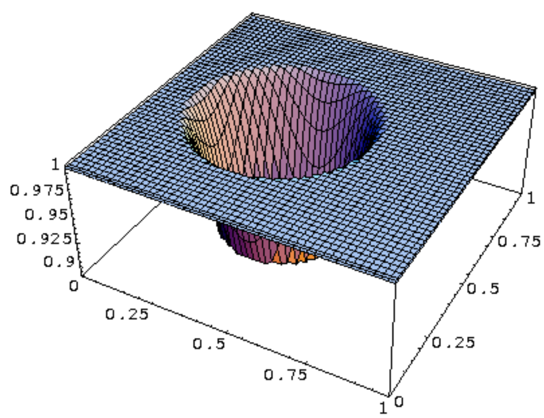

Figure 1: Obstacle functions are constant outside a certain "sensing" region.

$\left\|q_i - p_j\right\|$ are larger than d. In this way, the effect of the presence of obstacles and other agents on the motion of an agent will remain "local", within a region of radius d (Figure 1).

Then, the net effect of all obstacles and neighboring distances is captured in $\beta(q)$:

$$\beta(q) \triangleq \prod_{i,k} \beta_{ik} \prod_{i,j} b_{ij}, \quad i, j \in \{1, \ldots, N\},\ k = 0, \ldots, s.$$

4 Proof of Correctness

In this section we will formally show that the function (2), constructed in Section 3 is indeed a navigation function –with the exception of the admissibility property. Let $F = W \setminus \{p_1, \ldots, p_s\}$ denote the space remaining after removing all the obstacle points. We define $B_i(\varepsilon) = \{q \mid 0 < \beta_i < \varepsilon, \varepsilon > 0\}$. In the following, F is partitioned into five subsets:

1. the destination set, $F_d \triangleq \{q \mid \gamma_d(q) = 0\}$

2. the free space boundary, $\partial F \triangleq \beta^{-1}(0)$;
3. the set "near the obstacles", $F_0(\varepsilon) \triangleq \bigcup_{i=0}^{s+|\mathcal{E}|} B_i(\varepsilon) \setminus F_d$;
4. the set "near the workspace boundary", $F_1(\varepsilon) \triangleq B_0(\varepsilon) \setminus (F_d \cup F_0(\varepsilon))$;
5. the set "away from the obstacle", $F_2(\varepsilon) \triangleq F \setminus (F_d \cup \partial F \cup F_0(\varepsilon) \cup F_1(\varepsilon))$.

We follow the steps in [11] to establish the convergence properties of the potential field. These are formally stated in Propositions 4.2, 4.3, 4.4, 4.5, and 4.6. Since the desired formation can be achieved everywhere in the free space, Proposition 4.2 cannot formally ensure that the goal configuration is a nondegenerate critical point of the navigation function; however, it can be shown that it is indeed nondegenerate if we think of the formation graph edge labels as configuration variables. Proposition 4.3 shows that there are no critical point on the boundary of the workspace, and then Proposition 4.4 ensures that we can push all critical points near the obstacles by selecting an appropriate k. Proposition 4.5 guarantees that with appropriate tuning, these critical points will not be local minima; in fact Proposition 4.6 will not only show that the navigation function is a Morse function but will also ensure that the critical points are saddles. We will also make use of the following Lemma from [11]:

Lemma 4.1 ([11]). *Let ν, δ be at least twice differentiable, and define $\rho \triangleq \frac{\nu}{\delta}$. At a critical point c of ρ,*

$$\nabla^2 \rho\Big|_c = \frac{1}{\delta^2}[\delta \nabla^2 \nu - \nu \nabla^2 \delta],$$

Proposition 4.2. *The projection of φ on the orthogonal complement of F_d has a nondegenerate local minimum at the origin.*

Proof. We define new configuration variables, which are linked to the formation graph labels: $\mathrm{vec}(w) \triangleq (B \otimes I)\mathrm{vec}(q) - c$, where B is the incidence matrix of the formation graph, I is the identity matrix of appropriate dimensions and c is the stack vector of label constants c_{ij}. The destination q_d is mapped to origin of the new configuration variable w, so that $\nabla^2_w \gamma_d = 2I$. Now the general expression of the gradient of φ is

$$\nabla \varphi = \frac{1}{e^{\beta^{1/k}}} \left[\nabla \gamma_d - \frac{\gamma_d}{k} \beta^{\frac{1}{k}-1} \nabla \beta \right], \tag{7}$$

from which we get that differentiating with respect to w and evaluating at the origin, $\nabla_w \varphi\Big|_0 = 0$. Using Lemma 4.1, the Hessian of φ, will be

$$\nabla^2 \varphi = \frac{1}{(e^{\beta^{1/k}})^2} [e^{\beta^{1/k}} \nabla^2 \gamma_d - \gamma_d \nabla^2 e^{\beta^{1/k}}] \tag{8}$$

Considering the derivatives in terms of w and since both γ_d and $\nabla_w \gamma_d$ vanish at q_d, we will have $\nabla^2_w \varphi\Big|_0 = \frac{1}{(e^{\beta^{1/k}})} 2I$, which implies that $w = 0$ is a nondegenerate minimum of $\varphi(w)$. □

Proposition 4.3. *All critical points of φ are in F_2.*

Proof. Consider a point q_0 close to the obstacle boundary. Then, as $q_0 \to \partial F$, meaning that $\beta \to 0$, and in view of (7), $\nabla \varphi(q_0) \to -\frac{1}{e^{\beta^{1/k}}} \frac{\gamma_d}{k} \beta^{1/k-1} \nabla \beta$. Since ∂F is given as the set where $\beta = 0$, then $\nabla \varphi\Big|_{\beta=0}$ will be normal to the surface and $-\nabla \varphi$ will point towards the interior of F. □

Proposition 4.4. *For all $\varepsilon > 0$, there is a lower bound for k_1 for which all critical points are in $F_1(\varepsilon)$.*

Proof. From (7), for critical points $c \neq q_d$,

$$k\beta^{(k-1)/k} = \frac{\gamma_d \|\nabla\beta\|}{\|\nabla\gamma_d\|}. \tag{9}$$

The right hand side remains bounded, since we can see that with β_{it} defined as in (4), we have

$$\nabla_q \beta_{it} = \frac{-2\lambda z}{(1+z_{it}^2)^2}\nabla_q(z_{it}), \tag{10}$$

where $z_{it} = \|q_i - p_t\|^2 - d^2$, and thus $\nabla\beta$ is always bounded in F. By increasing k, the left hand side increases and the only way to compensate for that increase is for β to decrease. Thus, for every $\varepsilon > 0$, picking

$$k > k_1 \triangleq \frac{1}{\varepsilon^{\frac{k-1}{k}}} \sup\left\{\frac{\gamma_d \|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\},$$

ensures that the critical point is within $F_1(\varepsilon)$. □

Proposition 4.5. *There exists an $\varepsilon_0 > 0$ such that φ has no local minimum in $F_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_0$.*

Proof. We will show that at every critical point, the Hessian of φ has at least one negative eigenvalue. Manipulating (8) we obtain

$$\nabla^2\varphi\Big|_c = \frac{\beta^{(1-k)/k}}{k(e^{\beta^{1/k}})}[k\beta^{(k-1)/k}\nabla^2\gamma_d + (k-1-\beta^{1/k})\frac{\gamma_d}{k\beta}\nabla\beta\nabla\beta^T - \gamma_d\nabla^2\beta]. \tag{11}$$

Without loss of generality, we select β_{it}, $t \in \{1,\ldots,s\}$ as the obstacle function that attains the smallest value (the case where this function is a b_{ij} for some $i, j = 1,\ldots,N$ can be treated identically) and break β as $\beta = \beta_{it}\bar{\beta}_{it}$, where $\bar{\beta}_{it} = \frac{\beta}{\beta_{it}}$. Denoting $(A)_s$ the symmetric part of a matrix A,

$$\nabla^2\varphi\Big|_c = \frac{\beta^{(1-k)/k}}{k(e^{\beta^{1/k}})}\Big\{k\beta^{(k-1)/k}\nabla^2\gamma_d + \frac{\gamma_d(k-1-\beta^{1/k})}{k\beta}[\beta_{it}^2\nabla\bar{\beta}_{it}\nabla\bar{\beta}_{it}^T + 2\beta_{it}\bar{\beta}_{it}(\nabla\bar{\beta}_{it}\nabla\beta_{it}^T)_s$$
$$+\bar{\beta}_{it}^2\nabla\beta_i\nabla\beta_{it}^T] - \gamma_d[\beta_{it}\nabla^2\bar{\beta}_{it} + 2(\nabla\bar{\beta}_{it}\nabla\beta_{it}^T)_s + \bar{\beta}_{it}\nabla^2\beta_{it}]\Big\}$$

Defining $\hat{v} \triangleq (\frac{\nabla\beta_{it}(c)}{\|\nabla\beta_{it}(c)\|})^{\perp}$, and taking the quadratic form:

$$\frac{k(e^{\beta^{1/k}})}{\beta^{(1-k)/k}}\hat{v}^T(\nabla^2\varphi)\hat{v} = \underbrace{\hat{v}^T[k\beta^{(k-1)/k}\nabla^2\gamma_d - \gamma_d\bar{\beta}_{it}\nabla^2\beta_i]\hat{v}}_{A} + \hat{v}^T[(k-1-\beta^{1/k})\frac{\gamma_d}{k\beta}\beta_i^2\nabla\bar{\beta}_{it}\nabla\bar{\beta}_{it}^T - \gamma_d\beta_{it}\nabla^2\bar{\beta}_{it}]\hat{v}$$

The second term on the right hand side can be made arbitrarily small by decreasing ε ($\varepsilon > \beta_{it}$), so for the above to be negative the first term, (A), should be strictly negative. From (9) we have,

$$k\beta^{\frac{k-1}{k}}\nabla^2\gamma_d - \gamma_d\bar{\beta}_{it}\nabla^2\beta_{it} = \gamma_d\left[\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\nabla^2\gamma_d - \bar{\beta}_{it}\nabla^2\beta_i\right]$$
$$\Rightarrow A \leq \gamma_d\hat{v}^T\left[\sup\left\{\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\}\nabla^2\gamma_d - \bar{\beta}_{it}\nabla^2\beta_{it}\right]\hat{v}$$
$$= \gamma_d\hat{v}^T\left[\sup\left\{\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\}\nabla^2\gamma_d - \bar{\beta}_i\frac{2(1+d^4)}{d^3(1+d^2)^2}H_{\beta_{it}}\right]\hat{v}. \tag{12}$$

A straightforward derivation of $\nabla^2\beta_{it}$ yields

$$\left\{\frac{-2\lambda z_{it}}{(1+z_{it}^2)^2}\right\}\nabla_q^2(z_{it}) + \left\{\frac{2\lambda(3z_{it}^2-1)}{(1+z_{it}^2)^3}\right\}\nabla_q(z_{it})\nabla_q(z_{it})^T.$$

With $\beta_{it} \to 0$, we will have $q_i \to p_t$, and given (5),

$$\lim_{\varepsilon\to 0} \nabla_q^2 \beta_{it} = \frac{2(1+d^4)}{d^3(1+d^2)^2} H_{\beta_{it}}.$$

where $H_{\beta_{it}} = 2(L^{(i)} \otimes I)$, (alternatively $H_{b_{ij}} = 2(I^{(i)} \otimes I)$), L being the Laplacian of $\mathcal{G}$, and with $(\cdot)^{(i)}$ used to denote the matrix obtained by setting all elements of a given matrix to zero, except for those in the i^{th} row and column. Matrix $H_{\beta_{it}}$ is essentially $\nabla_q^2(z_{it})$. Using Kronecker product and the incidence matrix(B), γ_d can be written as

$$\begin{aligned}
\gamma_d &= (q^T(B\otimes I)^T - c^T)((B\otimes I)q - c) \\
&= q^T(B\otimes I)^T(B\otimes I)q - 2q^T(B\otimes I)^T c + 2c^2 \\
&= q^T(L\otimes I)q - 2q^T(B\otimes I)^T c + 2c^2
\end{aligned}$$

We use the following property of Kronecker product to evaluate $\nabla_q^2\gamma_d$

$$(B\otimes I)^T(B\otimes I) = (B^T\otimes I)(B\otimes I) = (B^T B)\otimes I = L\otimes I$$

$$\nabla_q \gamma_d = 2(L\otimes I)q - 2(B\otimes I)^T c$$

Writing $\gamma_d = (q^T(B\otimes I)^T - c^T)((B\otimes I)q - c)$ it follows that $\nabla^2\gamma_d = 2(L\otimes I)$ and (12) implies

$$A \geq 2\gamma_d \left(N \sup\left\{\tfrac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\} - \bar{\beta}_i \frac{2(1+d^4)}{d^3(1+d^2)^2} \right).$$

which can be made strictly negative by choosing $d < d_1$ sufficiently small.

$$d \triangleq \left[\frac{\sqrt{1+6\xi+\xi^2} - \xi - 1}{2\xi} \right]^{\frac{1}{2}}, \ \xi \triangleq \frac{N \sup_F\left\{\frac{\|\nabla\beta\|}{\|\nabla\gamma_d\|}\right\}}{\inf_{t,B_t(\varepsilon)} \bar{\beta}_{it}}$$

The following choice of $\varepsilon < \varepsilon_0$ guarantees a negative eigenvalue for (8):

$$\varepsilon_0 \triangleq \frac{\left[\frac{(k-1)\sup_F\{\Lambda(\nabla\bar{\beta}_{it}\nabla\bar{\beta}_{it}^T)\}}{k \inf_{t,B_t(\varepsilon_0)} \bar{\beta}_{it}} + \sup_F \Lambda(\nabla^2\bar{\beta}_{it})\right] \sup_F \gamma_d}{A(d)}$$

where $\Lambda(\cdot)$ denotes the largest eigenvalue of a matrix. □

Proposition 4.6. *There exists an $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$, all critical points are nondegenerate.*

Proof. Just as in [11], it suffices to show that $\omega^T(\nabla^2\varphi)\omega$, with $\omega \triangleq \frac{\nabla\beta_{it}}{\|\nabla\beta_{it}\|}$ can be made positive by choosing ε sufficiently small. Using (11), we have

$$\frac{k(e^{\beta^{1/k}})}{\beta^{\frac{1-k}{k}}} \omega^T(\nabla^2\varphi)\omega = \underbrace{\omega^T[k\beta^{\frac{k-1}{k}}\nabla^2\gamma_d]\omega}_{\{I\}} + \gamma_d\omega^T\left[\frac{(k-1-\beta^{\frac{1}{k}})}{k\beta}\nabla\beta\nabla\beta^T - \nabla^2\beta\right]\omega$$

Recalling that $\nabla_q^2\gamma_d = 2(L\otimes I)$ and that L is positive semidefinite,

$$\begin{aligned}
\frac{ke^{\beta^{1/k}}}{\gamma_d\beta^{\frac{1-k}{k}}}\omega^T\nabla^2\varphi\,\omega &\geq \omega^T\left[\frac{k-1-\beta^{\frac{1}{k}}}{k\beta}\nabla\beta\nabla\beta^T - \nabla^2\beta\right]\omega \\
&= \frac{(k-1-\beta^{1/k})}{k\beta}\langle\nabla\beta,\omega\rangle^2 - \omega^T(\nabla^2\beta)\omega \triangleq \Gamma
\end{aligned}$$

Without loss of generality, we assume that we are in $B_{it} \subset F_0$ and we write $\langle \nabla\beta, \omega\rangle^2 = \langle \nabla(\bar{\beta}_{it}\beta_{it}), \omega\rangle^2$. Then, with some algebraic manipulation, Γ becomes:

$$\frac{(k-1-\beta^{\frac{1}{k}})\beta_{it}}{k\bar{\beta}_{it}}\langle \omega, \nabla\bar{\beta}_{it}\rangle^2 - \frac{2(1+\beta^{\frac{1}{k}})}{k}\langle \nabla\beta_{it}, \nabla\bar{\beta}_{it}\rangle + \frac{(k-1-\beta^{\frac{1}{k}})\bar{\beta}_{it}}{k\beta_{it}}\|\nabla\beta_{it}\|^2 - \beta_{it}\omega^T\nabla^2\bar{\beta}_{it}\omega - \bar{\beta}_{it}\omega^T\nabla^2\beta_{it}\omega.$$

In the above, the first term is nonnegative, so

$$\begin{aligned}\Gamma \geq & \frac{(k-1-\beta^{\frac{1}{k}})\bar{\beta}_{it}}{k\beta_{it}}\|\nabla\beta_{it}\|^2 - \frac{2(1+\beta^{\frac{1}{k}})}{k}\langle \nabla\beta_{it}, \nabla\bar{\beta}_{it}\rangle - \beta_{it}\omega^T\nabla^2\bar{\beta}_{it}\omega - \bar{\beta}_{it}\omega^T\nabla^2\beta_{it}\omega \\ = & \frac{1}{\beta_{it}}\Big[-\underbrace{(\omega^T\nabla^2\bar{\beta}_{it}\omega)}_{a_1}(\beta_{it})^2 - \underbrace{\left(\frac{2(1+\beta^{\frac{1}{k}})}{k}\langle \nabla\beta_{it}, \nabla\bar{\beta}_{it}\rangle + \bar{\beta}_{it}\omega^T\nabla^2\beta_{it}\omega\right)}_{a_2}\beta_{it} + \underbrace{\frac{(k-1-\beta^{\frac{1}{k}})\bar{\beta}_{it}}{k\beta_{it}}\|\nabla\beta_{it}\|^2}_{a_3}\Big].\end{aligned}$$

In the above, the term inside the square brackets is a second order polynomial in β_{it}: $-a_1(\beta_{it})^2 - a_2\beta_{it} + a_3$. Having $a_3 > 0$ for $k > 2$, and assuming first that $a_1 > 0$, the polynomial has two real roots and is positive whenever $\beta_{it} < \frac{-a_2+\sqrt{a_2^2+4a_1}}{2a_1}$. On the other hand, if $a_1 \leq 0$, then $\Gamma \geq -a_2 + \frac{a_3}{\beta_{it}}$ which is positive whenever $\beta_{it} > \frac{a_3}{|a_2|}$. Selecting

$$\varepsilon < \varepsilon_1 \triangleq \min\left\{\frac{-a_2+\sqrt{a_2^2+4a_1}}{2a_1}, \frac{a_3}{|a_2|}\right\},$$

ensures that the quadratic form will be positive. □

Having shown so far that the only critical points of φ (besides the destination) are inside F_0, and that these points are not minima, we complete this section with the following Proposition:

Together, Propositions 4.2 – 4.6 ensure that (2) has navigation function properties, which means that the flows of (1) approach q_d asymptotically from almost every initial condition, (except for a set of measure zero which includes the critical points).

5 Simulation Results

Example of a triangular formation

We verify the case of three mobile robots moving in the planar world. The robots start from some initial position and move towards achieving a triangular formation while avoiding collision with obstacles and with each other.

Consider three robots each represented by configuration q_i, for $i = 1,2,3$ and $q_i = (x_i, y_i)$. Relative position vector between agents i and j is denoted by $q_{ij} = q_i - q_j$ and the configuration space is spanned by $q = [q_1^T \quad q_2^T \quad q_3^T]^T$

Digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{L}\}$, where $\mathcal{V} = \{1,2,3\}$ and $\mathcal{E} = \{(1,3), (3,2), (2,1)\}$

The constraint imposed on each edge is defined as follows,

$$\begin{aligned}\gamma_{13} &= (x_3 - x_1 - c_1)^2 + (y_3 - y_1 - c_2)^2 \\ \gamma_{32} &= (x_2 - x_3 - c_3)^2 + (y_2 - y_3 - c_4)^2 \\ \gamma_{21} &= (x_1 - x_2 - c_5)^2 + (y_1 - y_2 - c_6)^2\end{aligned}$$

where c_i, for $i = 1,2...6$ are constants. So, the formation constraint function γ_d is given by

$$\gamma_d = \gamma_{13} + \gamma_{32} + \gamma_{21} \tag{13}$$

Incidence Matrix(B) of the above graph is given by

$$B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \tag{14}$$

Laplacian $L = B^T B$

$$L = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{15}$$

Therefore,

$$\nabla^2 \gamma_d = 2(L \otimes I_2)$$

$$\nabla^2 \gamma_d = 2 \begin{bmatrix} 2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{bmatrix} \tag{16}$$

In the simulation cases, we have coordinated a group of three mobile agents into forming a equilateral triangle, pointing north. The environment is populated with 25 stationary point-obstacles, forming a Π configuration. The agents start at initial configurations denoted $1'$,$2'$, and $3'$ and reach their final configuration denoted by 1, 2, and 3 in Figure 2. The agents start from initial positions away (north-east) from the Π obstacle configuration and therefore stationary obstacle avoidance is not an issue here (in Figure 2 two of the obstacles in the upper right hand corner of the Π configuration are visible). What we want to test is the ability of the agents to avoid each other while trying to achieve the desired formation.

In the second case, we initially position one of the agents inside the Π obstacle configuration. Again, the starting positions are denoted $1'$, $2'$, and $3'$ and converge to 1, 2, and 3 in Figure 3. In this case we are testing not only the ability of the robots to form the desired triangle, but also their obstacle avoidance capabilities.

6 Conclusions

We have presented a navigation function that can be used for centralized multi-agent navigation and coordination. The potential field produced by this function ensures almost global asymptotic convergence of the agents to a particular oriented formation shape, while guaranteeing collision avoidance in the process. Formal analysis of the navigation function presented, shows that the topology of the interconnections in the multi-agent group affects its motion planning capabilities. This scheme is thought to be the first step towards constructing a provably correct and globally convergent decentralized scheme, and assessing the cost of decentralization in terms of performance.

References

[1] J. Barraquand, B. Langlois, and J. C. Latombe. Numerical potential field techniques for robot path planning. *IEEE Transactions on Systems, Man and Cybernetics*, 22(2):224–241, 1992.

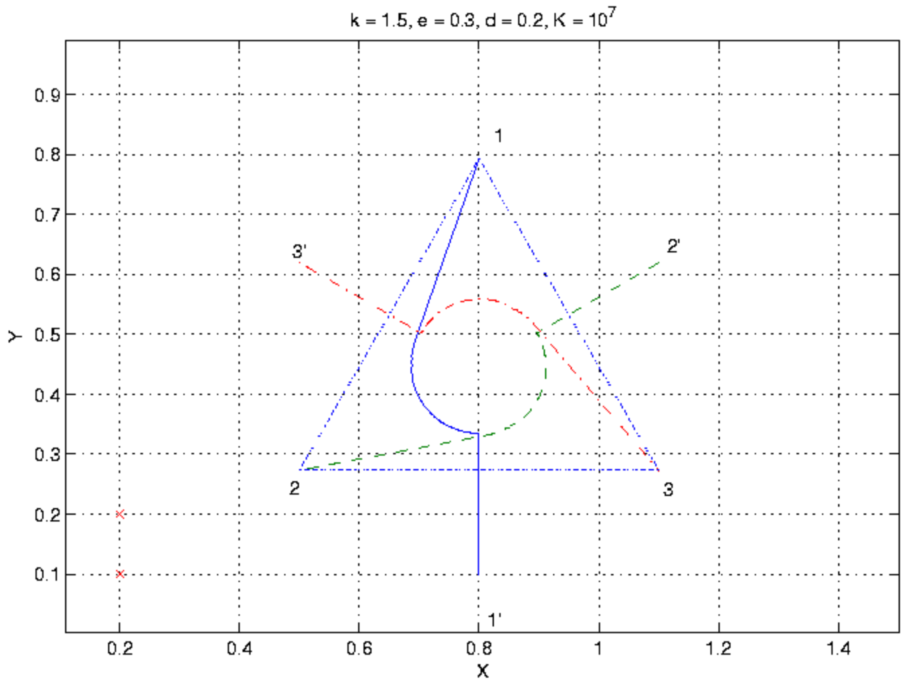

Figure 2: Inter-agent collision avoidance and convergence to desired formation.

[2] J. Barraquand and J. C. Latombe. A monte carlo algorithm for path planning with many degrees of freedom. In *Proceedings of the IEEE Conference on Robotics and Automation*, pages 1712–1717, 1990.

[3] D. V. Dimarogonas, M. M. Zavlanos, S. G. Loizou, and K. Kyriakopoulos. Decentralized motion control of multiple holonomic agents under input constraints. In *Proceedings of* 42^{nd} *IEEE Conf. on Decision and Control*, pages 3390–3395, Hawaii, December 2003.

[4] Magnus Egerstedt and Xiaoming Hu. Formation constrained multi-agent control. In *Proceedings of the IEEE Conference on Robotics and Automation*, pages 3961–3966, Seoul, Korea, May 2001.

[5] Jongrae Kim and Joao Hespanha. Cooperative radar jamming for groups of unmanned air vehicles. In *Proceedings of the IEEE Conference on Decision and Control*, December 2004. (to appear).

[6] Naomi Ehrich Leonard and Edward Fiorelli. Virtual leaders, artificial potentials and coordinated control of groups. In *Proceedings of the IEEE International Conference on Decision and Control*, pages 2968–2973, Orlando, FL, December 2001.

[7] Y. Liu, K. M. Passino, and M. M. Polycarpou. Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology. *IEEE Transactions on Automatic Control*, 48(1):76–95, February 2003.

[8] S. G. Loizou and K. Kyriakopoulos. Closed loop navigation for multiple holonomic vehicles. In *Proceedings of IEEE/RSJ Int. Conf. on Intelligent Robots and Systems*, pages 2861–2866, Switzerland, October 2002.

[9] Petter Ogren and Naomi Ehrich Leonard. Obstacle avoidance in formation. In *Proceedings of the IEEE Conference on Robotics and Automation*, pages 2492–2497, Taipei, September 2003.

[10] Wei Ren and Randal W. Beard. A decentralized scheme for spacecraft formation flying via the virtual structure approach. *AIAA Journal of Guidance Control and Dynamics*, 27(1):73–82, January 2004.

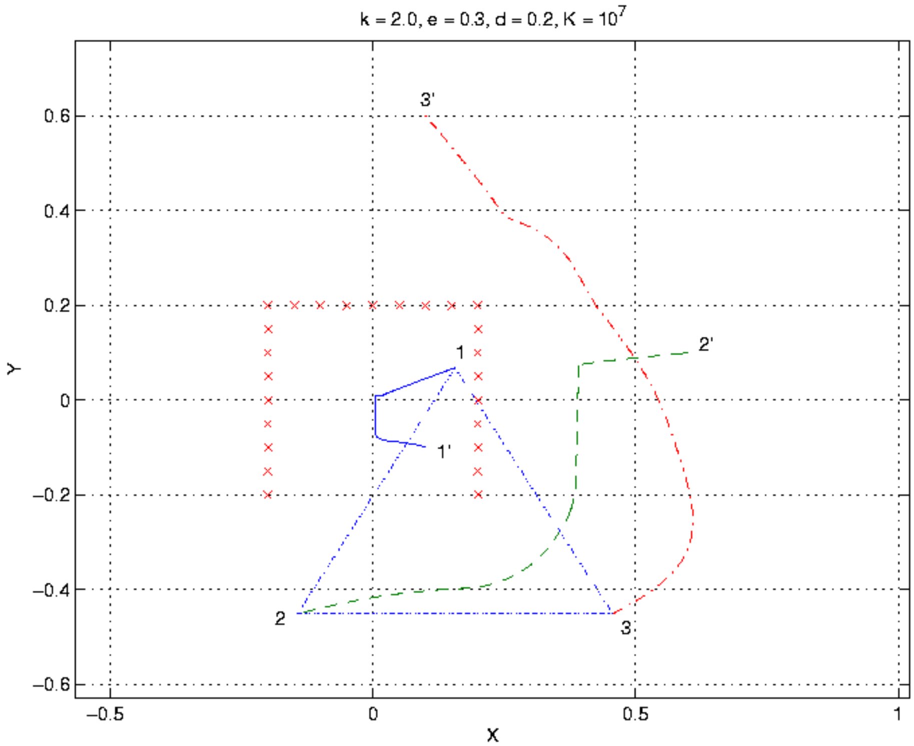

Figure 3: Simultaneous obstacle avoidance and formation stabilization.

[11] E. Rimon and D. E. Koditschek. Robot navigation functions on manifolds with boundary. *Advances Appl. Math.*, 11:412–442, 1990.

[12] E. Rimon and D. E. Koditschek. Exact robot navigation using artificial potential functions. *IEEE Trans. on Robotics and Automation*, 8(5):501–518, 1992.

[13] R. O. Saber and Richard M. Murray. Flocking with obstacle avoidance: cooperation with limited communication in mobile networks. In *Proceedings of the IEEE International Conference on Decision and Control*, 2003.

[14] Rafael Sandoval-Rodriguez and Chaouki T. Abdallah. Internet-like protocols for the control and coordination of multiple agents with time delay. In *Proceedings of the IEEE International Symposium on Intelligent Control*, Houston, TX, October 2003.

[15] H. G. Tanner. Flocking with obstacle avoidance in switching networks of interconnected vehicles. In *Proceedings of the IEEE International Conference on Robotics and Automation*, volume 3, pages 3006–3011, April 2004.

[16] H. G. Tanner, S. G. Loizou, and K. J. Kyriakopoulos. Nonholonomic navigation and control of multiple mobile manipulators. *IEEE Transactions on Robotics and Automation*, 19(1):53–64, February 2003.

[17] Herbert G. Tanner, George J. Pappas, and A. Jadbabaie. Stable flocking of mobile agents, part i: fixed topology. In *Proceedings of the IEEE International Conference on Decision and Control*, volume 2, pages 2010 –2015, December 2003.

[18] Herbert G. Tanner, George J. Pappas, and Vijay Kumar. Leader-to-formation stability. *IEEE Transactions on Robotics and Automation*, 20(3):443–455, June 2004.