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The Hilbert Transform as an Average of Dyadic Shift Operators

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BY

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Dedication

I would like to dedicate this work to my parents Melahat and Kazim Atasever and to my siblings Ilknur, Nurcan and Umit.

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Abstract

In this thesis we discuss Petermichl's characterization of the Hilbert transform as an average of dyadic shift operators following the presentation by Thomas Hytönen [Hyt].

A linear and bounded operator T in $L^2(\mathbb{R})$ that commutes with translations, dilations and anticommutes with reflections must be a constant multiple of Hilbert transform; $T = cH$. Using this principle Stefanie Petermichl showed that we can write H as a suitable average of dyadic operators [Pet]. Each Dyadic Shift Operator does not have the symmetries that characterize the Hilbert transform, but an average over all random dyadic grids do.

Contents

Chapter 1

Introduction

This thesis is about an extremely important operator in harmonic analysis, the Hilbert Transform. The Hilbert transform was defined in 1905 by David Hilbert, during his work on a problem posed by Riemann, concerning analytic functions. This is known as the Riemann-Hilbert problem for holomorphic functions in complex analysis. Hilbert was mainly interested in the Hilbert transform for functions defined on the circle, and with the Discrete Hilbert Transform. Schur added to and refined Hilbert's results regarding the discrete Hilbert transform and extended them to the integral case. These results were confined to the spaces L^2 and l^2. In 1928, Marcel Riesz proved that the Hilbert transform can be defined for f in $L^p(\mathbb{R})$ for $1 \leq p < \infty$, that the Hilbert transform is a bounded operator on $L^p(\mathbb{R})$ for $1 < p < \infty$, and that similar results hold for the Hilbert transform on the circle, and the discrete Hilbert transform. The Hilbert transform was an important example for Antoni Zygmund and Alberto Calderón during their study of singular integrals (Calderón and Zygmund 1952). Their investigations play a fundamental role in modern harmonic analysis. Several generalizations of the Hilbert transform, such as the bilinear and trilinear Hilbert transforms are still active areas of research.

A linear and bounded operator T in $L^2(\mathbb{R})$ that commutes with translations, dilations and anticommutes with reflections must be a constant multiple of Hilbert transform; $T = cH$ [Ste]. Using this principle Stefanie Petermichl showed in 2000 that we can write H as a suitable average of dyadic operators [Pet]. Each dyadic shift operator does not have the symmetries that characterize the Hilbert transform, but an average over all random dyadic grids does.

A comparable approach works for the Beurling-Ahlfors transform and Riesz transforms, higher dimensional analogues of the Hilbert transform. Adequately smooth one dimensional Calderón-Zygmund convolution operators are averages of Haar shift operators of bounded complexity [Vag]. Finally Hytönen's representation theorem (which utilizes and builds upon the current work of several researchers) for all Calderón-Zygmund singular integral operators as averages of Haar shift operators of arbitrary complexity are the apex [Hyt2]. Here, a shift operator that moves Haar coefficients up by m generations and down by n generations is said to have complexity (m, n). For instance, Petermichl's Ш operator is a Haar shift operator of complexity $(0, 1)$, while the martingale transform is a Haar shift operator of complexity $(0, 0)$. Hytönen used this representation to demonstrate a well known conjecture in the theory of weights (the A_2 conjecture) which is beyond the scope of this thesis.

In this thesis, we discuss carefully Petermich's characterization of the Hilbert transform as an average of dyadic shift operators of complexity $(0, 1)$: The Petermichl's Ш operator, following the presentation in Hytönen's paper [Hyt].

In Chapter 2, we provide the basic definitions and basic results that will be used throughout this manuscript.

In Chapter 3, we define the Hilbert transform in two equivalent ways: first by its action on the Fourier domain as a Fourier multiplier, second by its action on the time domain as a singular integral. We also prove that the Hilbert transform commutes with translations and dilations and anticommutes with reflections. These invariance properties characterize the Hilbert transform up to a multiplicative constant.

In Chapter 4, we define the dyadic grids, random dyadic grids in $\mathbb{R}$ and Haar functions. Then we give the definition of the Petermichl's dyadic shift operator which is called Sha. We prove that Petermichl's shift operators $Ш_{r,\beta}$ associated to the dyadic grids $\mathcal{D}_{r,\beta}$ do not commute with translations, dilations, nor do they anticommute with reflections, but some symmetries for the family of shift operators $\{Ш_{r,\beta}\}_{(r,\beta)\in\Omega}$ hold. We will see that the operations of translations, dilation, and reflection not only pass to the argument f but also affect the translation, dilation and reflection parameters (r, β).

In Chapter 5, we show that we can represent the Hilbert Transform as an average of Petermichl's shift operator over dyadic random grids.

In Chapter 6, we discuss possible future research.

Chapter 2

Preliminaries

In this chapter, we will review some basic definitions and introduce the notation that we will use throughout this thesis. We will work on the Euclidean space $\mathbb{R}$, but most of the results presented here will also hold for the Euclidean space $\mathbb{R}^n$. All functions will be real or complex valued.

Unless specified, p and q represent real numbers with $1 \leq p, q < \infty$. A function $f : \mathbb{R} \to \mathbb{C}$ is in $L^p(\mathbb{R})$ if $\int_{\mathbb{R}} |f(x)|^p dx < \infty$. The L^p norm of such f is defined by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

The L^p spaces are complete normed spaces which are called Banach spaces. We denote

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} dx,$$

the standard L^2- inner product on $\mathbb{R}$.

2.1 Fourier Transform

The Fourier transform is named after French mathematician Joseph Fourier (1768-1830). It is a mathematical transform that describes a mathematical function of time as a function of frequency. The function of time is usually referred to as the domain representation. The function of frequency is referred to as the frequency domain representation.

A Fourier series can be expressed as a periodic function f in terms of an infinite sum of trigonometric functions sines and cosines. The significance of the Fourier transform comes from the study of Fourier series. In the exploration of Fourier series, sophisticated and often difficult periodic functions are expressed as the sum of simple waves mathematically represented by sines and cosines. The Fourier transform is a continuation of the Fourier series that is brought about when the period of the represented function is expanded and approaches infinity.

Definition 2.1. *The Fourier transform* $\widehat{f}$ *of an integrable function* $f : \mathbb{R} \to \mathbb{R}$ *is defined by*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx.$$

The space of functions which are infinitely differentiable and the function and all of its derivatives decrease faster than any polynomial increases is called Schwartz class and is denoted $S(\mathbb{R})$. Fourier integral formulas work for the functions which are in this class.

	Time	Frequency
(a)	linear property $af + bg$	linear property $a\widehat{f} + b\widehat{g}$
(b)	translation $\tau_h f(x) := f(x-h)$	modulation $\widehat{\tau_h f}(\xi) = M_{-h}\widehat{f}(\xi)$
(c)	modulation $M_h f(x) := e^{2\pi i h x} f(x)$	translation $\widehat{M_h f}(\xi) = \tau_h \widehat{f}(\xi)$
(d)	dilation $\delta_s f(x) := f(sx)$	inverse dilation $\widehat{\delta_s f}(\xi) = \delta_{s^{-1}}\widehat{f}(\xi)$
(e)	reflection $\widetilde{f}(x) := f(-x)$	reflection $\widehat{\widetilde{f}}(\xi) = \widetilde{\widehat{f}}(\xi)$

A more complete Time-Frequency dictionary appears in [PW].

The Schwartz class is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$. The Fourier transform can be extended by continuity to $L^2(\mathbb{R})$ and there it is an isometry, this is known as Plancherel's identity: $\|f\|_{L^2(\mathbb{R})} = \|\widehat{f}\|_{L^2(\mathbb{R})}$.

2.2 Convolution

Convolution is a way to create new functions from old ones.

Definition 2.2. *When $f, g \in S(\mathbb{R})$, the convolution of these two functions is defined by*

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Properties of convolution : Let $f, g, h : \mathbb{R} \to \mathbb{C}$ be 2π periodic integrable functions and let $a \in \mathbb{C}$ be constant. Then

i) (commutative) $f * g = g * f$

ii) (distributive) $f * (g + h) = (f * g) + (f * h)$

iii) (associative) $(f * g) * h = f * (g * h)$

iv) (homogeneous) $(af) * g = a(f * g) = f * (ag)$

v) $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

Property (v) is another instance of the time-frequency dictionary.

Definition 2.3. *The characteristic function $\chi_I(x)$ of an interval I is defined by*

$$\chi_I(x) = \begin{cases} 1, & \text{if} \quad x \in I, \\ 0, & \text{if} \quad x \notin I. \end{cases}$$

Lemma 2.4. *If $I = (a, a + |I|]$ and $J = [b, b + |I|)$, $|I| = |J|$, where $|I|$ is the length of the interval. Then*

$$\chi_I * \chi_J(x) = \begin{cases} x - a - b, & \text{if} \quad x \in [a + b, a + b + |I|], \\ -x + a + b + 2|I|, & \text{if} \quad x \in [a + b + |I|, a + b + 2|I|], \\ 0, & \text{otherwise}. \end{cases}$$

which is a piece-wise linear function.

Proof. From the definition of the convolution we have

$$\chi_I * \chi_J(x) = \int_{-\infty}^{\infty} \chi_I(x - y)\chi_J(y)dy.$$

When $x - y \in I$, $y \in [-a + x - |I|, x - a)$. Then

$$\chi_I * \chi_J(x) = \int_{-\infty}^{\infty} \chi_{[-a+x-|I|,x-a)}(y)\chi_{[b,b+|I|)}(y)dy.$$

By using the definition of the characteristic function

$$\chi_I(x-y)\chi_J(y) = \begin{cases} 1, & \text{if} \quad y \in [-a + x - |I|, x - a) \cap [b, b + |I|), \\ 0, & \quad \text{otherwise.} \end{cases}$$

which is a piece-wise constant function. When we integrate this function

$$\chi_I * \chi_J(x) = \begin{cases} x - a - b, & \text{if} \quad x \in [a + b, a + b + |I|], \\ -x + a + b + 2|I|, & \text{if} \quad x \in [a + b + |I|, a + b + 2|I|], \\ 0, & \quad \text{otherwise.} \end{cases}$$

As we can see $\chi_I * \chi_J(x)$ is a piece-wise linear and continuous function. □

Definition 2.5. *A function f defined on $\mathbb{R}$ has compact support if there is a closed interval $[a, b] \subset \mathbb{R}$ such that $f(x) = 0$ for all $x \notin [a, b]$. We say that such a function is compactly supported.*

Lemma 2.6. *If f has a compact support in $[-a, a]$ and g has compact support in $[-b, b]$. In other words supp$f \subseteq [-a, a] = A$ and supp$g \subseteq [-b, b] = B$, then supp$(f * g) \subseteq [-(a + b), a + b] = A + B$.*

2.3 Binary Representation

If $x \in \mathbb{R}$ and $x > 0$, then the binary representation of x is $x = \sum_{i<-j} \beta_i 2^i$ where $\beta_{-j-1} = 1$, $\beta_i = 0$ or 1 and j is the unique integer such that $2^{-j-1} \leq x < 2^{-j}$.

When we have a real number x we define a binary number x_i as
$x = x_i + p2^{-i}$ where $0 \leq x_i = \sum_{k<-i} \alpha_k 2^k < 2^{-i}$, $\alpha_k = 0$ or 1 and $p \in \mathbb{Z}$. (Note that there exists $k < -i$ such that $\alpha_k = 0$.)

Note that when $x > 0$ and its binary representation is $x = \sum_{i<-j} \beta_i 2^i$, if $\sum_{k<-i} \beta_k 2^k < 2^{-i}$, then $x_i = \sum_{k<-i} \beta_k 2^k < 2^{-i}$ and $\alpha_k = \beta_k$ for all $k < -i$. If $\sum_{k<-i} \beta_k 2^k = 2^{-i}$, then $x_i = 0$ and $\alpha_k = 0$ for all $k < -i$.

Lemma 2.7. *The relation of $x, y \in \mathbb{R}$, $x \equiv_i y$ when $x - y = m2^{-i}$, $m \in \mathbb{Z}$ is an equivalence relation.*

Proof. To prove this lemma we should show that the relation is reflexive, symmetric and transitive.

To prove the reflexivity we should show that $x \equiv_i x$.

$$x - x = 0.2^{-i}, \quad 0 \in \mathbb{Z}.$$

From the definition of the relation $x \equiv_i x$.

To prove the symmetry we should show that if $x \equiv_i y$ then $y \equiv_i x$.

If $x \equiv_{-i} y$, by the definition of the relation $x - y = m2^{-i}$, $m \in \mathbb{Z}$. When we multiply both sides of the equation by -1 we have

$$y - x = -m2^{-i} \quad \text{where} \quad -m \in \mathbb{Z}.$$

From the definition of the relation we can see that $y \equiv_i x$.

To prove the transitivity we should show that if $x \equiv_i y$ and $y \equiv_i z$ then $x \equiv_i z$.

If $x \equiv_i y$ and $y \equiv_i z$, by the definition of the relation

$$x - y = m2^{-i}, \quad m \in \mathbb{Z}$$

and

$$y - z = k2^{-i}, \quad k \in \mathbb{Z}.$$

When we add these two equation above, we have $x - z = (m + k)2^{-i}$ and since $m, k \in \mathbb{Z}$, $m + k \in \mathbb{Z}$. Hence $x \equiv_i z$. □

2.4 Approximation Theorems and Inequalities

In this section, we present several approximation theorems that will be used in the proof of the main theorem in Chapter 5. Let us remind the reader several classical inequalities that we will need to prove the Mollification Theorems.

Theorem 2.8 (Hölder's Inequality). *[Fol] Let* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p,q>1$. *Then Hölder's inequality for integrals is*

$$\int_{\mathbb{R}} |f(x)g(x)|dx \leq \left(\int_{\mathbb{R}} |f(x)|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x)|^q\right)^{\frac{1}{q}},$$

where $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$.

Theorem 2.9 (Minkowski's Inequality). *[PW] Let* S *be a measure space ,* $1 \leq p \leq \infty$ *and let* f *and* g *be elements of* $L^p(S)$. *Then the Minkowski's Inequality is*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p,$$

where the norm is given by $\|f\|_p = \left(\int |f|^p dx\right)^{\frac{1}{p}}$ *if* $1 \leq p < \infty$ *and* $\|f\|_\infty = esssup_{x\in S}|f(x)|$ *when* $p=\infty$.

Theorem 2.10 (Minkowski's Integral Inequality). *[PW] Let* $F(x,y)$ *be a function of two variables such that for a.e* $y \in \mathbb{R}$, *assume that the function* $F_y(x) := F(x,y)$ *belongs to* $L^p(\mathbb{R}, dx)$ *for some* $1 \leq p$ *and that the function* $G(y) = \|F_y\|_{L^p(dx)}$ *is in* $L^1(\mathbb{R})$. *Then the function* $F_x(y) := F(x,y)$ *is in* $L^1(\mathbb{R}, dy)$ *for a.e.* $x \in \mathbb{R}$, *and the function* $H(x) := \int_{\mathbb{R}} F(x,y)dy$ *is in* $L^p(\mathbb{R}, dx)$. *Moreover*

$$\| \int_{\mathbb{R}} F(.,y)dy\|_{L^p(dx)} \leq \int_{\mathbb{R}} \|F(.,y)\|_{L^p(dx)}dy.$$

We will use the fact that the L^p norm is invariant under translation and reflection and that the L^p norm of the translation by h of f is a continuous function at $h=0$.

Theorem 2.11. *Given* $f \in L^p(\mathbb{R})$, $h \in \mathbb{R}$, *then*

$$\|f\|_p = \|\tau_h f\|_p = \|\widetilde{f}\|_p \quad \text{and} \quad \lim_{h\to 0} \|\tau_h f\|_p = \|f\|_p.$$

We now state and prove two approximation theorems that involve convolution with mollifier $\Phi_t(x) = \frac{1}{t}\Phi(\frac{x}{t})$ as $t \to 0$. The first theorem discusses convergence in $L^p(\mathbb{R})$ and the uniform norm.

Theorem 2.12. *[Fol, Theorem 8.14] Suppose $\Phi \in L^1(\mathbb{R})$ and let $a := \int \Phi(x)dx$.*

i) If $f \in L^p(\mathbb{R})$ $(1 \le p < \infty)$, then $f * \Phi_t \to af$ in the L^p norm as $t \to 0$.

ii) If f is bounded and uniformly continuous, then $f * \Phi_t \to af$ uniformly as $t \to 0$.

iii) If $f \in L^\infty(\mathbb{R})$ and f is continuous on an open set U, then $f * \Phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

Proof. [Fol] Setting $y = tz$ we have, using hypothesis $a = \int \Phi(y)dy = \int \Phi_t(y)dy$,

$$\begin{aligned} f * \Phi_t(x) - af(x) &= \int \big(f(x-y) - f(x)\big)\Phi_t(y)dy \\ &= \int \big(f(x-tz) - f(x)\big)\Phi(z)dz \\ &= \int \big(\tau_{tz}f(x) - f(x)\big)\Phi(z)dz. \end{aligned}$$

When we apply Minkowski's integral inequality for integrals

$$\|f * \Phi_t - af\|_p \le \int \|\tau_{tz}f - f\|_p |\Phi(z)|dz.$$

By Theorem 2.11 $\|\tau_{tz}f - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $t \to 0$ for each z. Assertion (i) therefore follows from the Lebesgue dominated convergence theorem. The proof of (ii) is exactly the same, with $\|.\|_p$ replaced by $\|.\|_\infty$. The estimate for $\|f*\Phi_t - af\|_\infty$ is another standard application of the Lebesgue dominated convergence theorem, since $|\tau_{tz}f(x) - f(x)|\,|\Phi(z)| \le 2\|f\|_\infty|\Phi(z)| \in L^1(\mathbb{R})$ and $\lim_{t\to 0}|\tau_{tz}f(x) - f(x)||\Phi(z)| = 0$ since f is continuous, finally, $\|\tau_{tz}f - f\|_\infty \to 0$ as $t \to 0$ by the uniform continuity of f. As for (iii), given $\epsilon > 0$ let us choose a compact set $E \subset \mathbb{R}^n$ such that $\int_{E^c} |\Phi| < \epsilon$. Also, let K be a compact subset of U. If t is sufficiently small, then, we will have $x - tz \in U$ for all $x \in K$ and $z \in E$, so from the compactness of K it follows that

$$sup_{x\in K, z\in E}|f(x-tz) - f(x)| < \epsilon$$

for small t. But then

$$\begin{aligned} sup_{x\in K}|f * \Phi_t(x) - af(x)| &\leq sup_{x\in K}\left(\int_E + \int_{E^c}\right)|f(x-tz) - f(x)||\Phi(z)|dz \\ &\leq \epsilon \int |\Phi| + 2\|f\|_\infty \epsilon, \end{aligned}$$

from which (iii) follows. □

The second approximation theorem involving convolutions with mollifiers $\Phi_t(x) = \frac{1}{t}\Phi(\frac{x}{t})$ as $t \to 0$ discusses pointwise almost everywhere, convergence under some additional decay condition on Φ.

Theorem 2.13. *[Fol, Theorem 8.15] Suppose $|\Phi(x)| \leq C(1+|x|)^{-1-\epsilon}$ for some $C, \epsilon > 0$ (which implies that $\Phi \in L^1(\mathbb{R})$), and let $a := \int \Phi(x)dx$. If $f \in L^p(\mathbb{R})$ $(1 \leq p \leq \infty)$, then $f * \Phi_t \to af(x)$ as $t \to 0$ for every x in the Lebesgue set of f - in particular, for almost every x, and for every x at which f is continuous.*

We borrow the proof from [Fol].

Proof. If x is in the Lebesgue set[1] of f, for any $\delta > 0$ there exists $\eta > 0$ such that

$$\int_{|y|<r} |f(x-y) - f(x)|dy \leq \delta r \quad \text{for} \quad r \leq \eta.$$

Let us set

$$I_1 = \int_{|y|<\eta} |f(x-y) - f(x)||\Phi_t(y)|dy,$$

[1]The Lebesgue set of a measurable function f is the set of points $x \in \mathbb{R}$ such that $\lim_{r\to 0} \frac{1}{r}\int_{|y|<r} |f(x-y) - f(x)|dy = 0$. It is well known that for $f \in L^p$ $1 \leq p$, the complement of the Lebesgue set has measure zero [Fol].

$$I_2 = \int_{\eta \leq |y|} |f(x-y) - f(x)||\Phi_t(y)|dy.$$

We claim that I_1 is bounded by $A\delta$ where A is independent of t, whereas $I_2 \to 0$ as $t \to 0$. Since

$$|f * \Phi_t(x) - af(x)| \leq I_1 + I_2,$$

we will have

$$\limsup_{t\to 0}|f * \Phi_t(x) - af(x)| \leq A\delta,$$

and since δ is arbitrary, this will complete the proof. To estimate I_1, let K be the integer such that $2^K \leq \frac{\eta}{t} < 2^{K+1}$ if $1 \leq \frac{\eta}{t}$, and $K = 0$ if $\frac{\eta}{t} < 1$. We view the interval $|y| < \eta$ as the union of the annuli $2^{-k}\eta \leq |y| < 2^{1-k}\eta$ $(1 \leq k \leq K)$ and the interval $|y| < 2^{-K}\eta$. On the kth annulus we use the estimate $\frac{2^{-k}\eta}{t} \leq \frac{|y|}{t} < 2^{1-k}\frac{\eta}{t}$ to get,

$$|\Phi_t(y)| = \frac{1}{t}|\Phi(\frac{y}{t})| \leq \frac{C}{t(1+\frac{|y|}{t})^{1+\epsilon}} \leq Ct^{-1}|\frac{y}{t}|^{-1-\epsilon} \leq Ct^{-1}\left(\frac{2^{-k}\eta}{t}\right)^{-1-\epsilon},$$

and on the interval $|y| < 2^{-K}\eta$ we use the estimate $|\Phi_t(y)| \leq Ct^{-1}$. Thus

$$I_1 \leq \sum_{k=1}^{K} Ct^{-1}\left(\frac{2^{-k}\eta}{t}\right)^{-1-\epsilon} \int_{2^{-k}\eta \leq |y| < 2^{1-k}\eta} |f(x-y) - f(x)|dy$$
$$+ Ct^{-1}\int_{|y|<2^{-K}\eta} |f(x-y) - f(x)|dy.$$

Since x is in the Lebesque set of f, and $r = 2^{1-K}\eta \leq \eta$, $\int_{|y|<2^{1-K}\eta} |f(x-y) - f(x)|dy \leq 2^{1-K}\eta\delta$,

$$I_1 \leq C\delta\sum_{k=1}^{K}(2^{1-k}\eta)t^{-1}\left(\frac{2^{-k}\eta}{t}\right)^{-1-\epsilon} + C\delta t^{-1}(2^{-K}\eta)$$
$$= 2C\delta\left(\frac{\eta}{t}\right)^{-\epsilon}\sum_{k=1}^{K}2^{k\epsilon} + C\delta\left(\frac{2^{-K}\eta}{t}\right)$$
$$= 2C\delta\left(\frac{\eta}{t}\right)^{-\epsilon}\frac{2^{(K+1)\epsilon} - 2^{\epsilon}}{2^{\epsilon} - 1} + C\delta\left(\frac{2^{-K}\eta}{t}\right).$$

Therefore by the fact that $2^K \leq \frac{\eta}{t} < 2^{K+1}$,

$$I_1 \leq 2C\delta 2^{-K\epsilon}\left(\frac{2^{(K+1)\epsilon} - 2^{\epsilon}}{2^{\epsilon} - 1}\right) + 2C\delta$$

$$= 2C\delta\left(\frac{2^\epsilon - 2^{\epsilon(1-K)}}{2^\epsilon - 1} + 1\right)$$
$$\leq 2C\delta(2^\epsilon(2^\epsilon - 1)^{-1} + 1).$$

As for I_2, if p' is the conjugate exponent to p, that is $\frac{1}{p} + \frac{1}{p'} = 1$ and where χ is the characteristic function of $\{y : \eta \leq |y|\}$. By Hölder's inequality we have

$$\begin{aligned} I_2 &\leq \int_{\eta\leq|y|} (|f(x-y)| + |f(x)|)|\Phi_t(y)|dy \\ &= \int |f(x-y)||\Phi_t(y)\chi_{\{y:\eta\leq|y|\}}(y)|dy + |f(x)| \int |\Phi_t(y)\chi_{\{y:\eta\leq|y|\}}(y)|dy \\ &\leq \|f\|_p \|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_{p'} + |f(x)|\|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_1, \end{aligned}$$

So it suffices to show that for $1 \leq q \leq \infty$, and in particular for $q = 1$ and $q = p'$, $\|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_q \to 0$ as $t \to 0$. If $q = \infty$, and if $\frac{\eta}{t} \leq \frac{|y|}{t}$ then $|\Phi(\frac{y}{t})| \leq C(1 + \frac{|y|}{t})^{-1-\epsilon} \leq C(1 + \frac{\eta}{t})^{-1-\epsilon}$. Therefore

$$\|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_\infty \leq Ct^{-1}(1 + \frac{\eta}{t})^{-1-\epsilon} = Ct^\epsilon(t+\eta)^{-1-\epsilon} \leq C\eta^{-1-\epsilon}t^\epsilon.$$

If $q < \infty$, we have if $0 < \frac{\eta}{t} \leq |z|$ then $|\Phi(z)| \leq C_0(1+|z|)^{-1-\epsilon} \leq C_0|z|^{-1-\epsilon}$. Therefore

$$\begin{aligned} \|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_q^q &= \int_{\eta\leq|y|} t^{-q}|\Phi(t^{-1}y)|^q dy = t^{(1-q)} \int_{\frac{\eta}{t}\leq|z|} |\Phi(z)|^q dz \\ &\leq C_0^q t^{1-q} \int_{\frac{\eta}{t}\leq|z|} |z|^{-(1+\epsilon)q} dz \\ &= 2C_0^q t^{(1-q)} \int_{\frac{\eta}{t}}^\infty z^{-(1+\epsilon)q} dz = C_1 t^{(1-q)} \left(\frac{\eta}{t}\right)^{1-(1+\epsilon)q} = C_2 t^{\epsilon q}. \end{aligned}$$

In either case, $\|\chi_{\{y:\eta\leq|y|\}}\Phi_t\|_q$ is dominated by t^ϵ, so we are done. □

When $a = 1$ then the family $\{\Phi_t\}$ as $t \to 0$ is an approximation of the identity because $\Phi_t * f \to f$. We will use the case $a = 0$, then $f * \Phi_t \to 0$ almost everywhere and in L^p norm for $1 \leq p$.

The last approximation theorem discusses convolution with mollifier $\Phi_t(x) = \frac{1}{t}\Phi(\frac{x}{t})$ as $t \to \infty$, both in L^p and uniformly.

Theorem 2.14. *Suppose $\Phi : \mathbb{R} \to \mathbb{R}$ is measurable, has compact support in $[-1,1]$ and is bounded by M (hence it is in $L^q(\mathbb{R})$ for all $1 \le q$). Then*

i) $\Phi_t * f(x) \to 0$ uniformly on $\mathbb{R}$ as $t \to \infty$ if $f \in L^p(\mathbb{R})$, $1 \le p$.

ii) $\Phi_t * f(x) \to 0$ in $L^p(\mathbb{R})$ if $f \in L^p(\mathbb{R})$ as $t \to \infty$, $1 < p$.

Proof. i) Proof is a direct application of Hölder's inequality. From the definition of the convolution we have

$$|\Phi_t * f(x)| = \left| \int_{-\infty}^{\infty} \Phi_t(y) f(x-y) dy \right|.$$

Recall that $\Phi_t := \frac{1}{t}\Phi(\frac{x}{t})$. Then

$$|\Phi_t * f(x)| = \left| \frac{1}{t} \int_{-\infty}^{\infty} \Phi(\frac{y}{t}) f(x-y) dy \right|.$$

Since Φ has a compact support in $[-1,1]$, $\Phi(\frac{x}{t})$ has a compact support in $[-t,t]$. In other words $\Phi(\frac{x}{t}) \neq 0$ if and only if $-t \le x \le t$. So we have that

$$|\Phi_t * f(x)| = \left| \frac{1}{t} \int_{-t}^{t} \Phi(\frac{y}{t}) f(x-y) dy \right|.$$

We know that Φ is bounded by M. By using the triangle inequality for integrals,

$$|\Phi_t * f(x)| \le \frac{M}{t} \int_{-t}^{t} |f(x-y)| dy.$$

From Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$

$$|\Phi_t * f(x)| \le \frac{M}{t} \left(\int_{-t}^{t} |f(x-y)|^p dy \right)^{\frac{1}{p}} \left(\int_{-t}^{t} 1^q dy \right)^{\frac{1}{q}}.$$

Note that $f(x-y) = \widetilde{f}(y-x) = \tau_x \widetilde{f}(y)$. From reflection and shift invariance of L^p norms $||\tau_x f||_p = ||f||_p = ||\widetilde{f}||_p = ||\tau_x \widetilde{f}||_p$. Then

$$|\Phi_t * f(x)| \le \frac{M}{t} ||\tau_x \widetilde{f}||_{L^p} (2t)^{\frac{1}{q}}$$

$$\leq \frac{2^{\frac{1}{q}} M}{t^{1-\frac{1}{q}}} ||f||_{L^p}$$
$$= \frac{2^{\frac{1}{q}} M}{t^{\frac{1}{p}}} ||f||_{L^p}.$$

Hence $|\Phi_t * f(x)| \leq \frac{C}{t^{\frac{1}{p}}}$ where C is a constant. We can see that as $t \to \infty$, $|\Phi_t * f(x)| \to 0$ uniformly in $x \in \mathbb{R}$. Therefore as $t \to \infty$, $\Phi_t * f(x) \to 0$ uniformly in $x \in \mathbb{R}$.

ii) We will first prove the statement for $g \in L^p$ with compact support, say $supp g \subseteq [-k, k]$.

Since $supp \Phi_t \subseteq [-t, t]$ and $supp g \subseteq [-k, k]$ from Lemma 2.6 we have that

$$supp \Phi_t * g \subseteq [-(t+k), t+k].$$

So $\Phi_t * g(x) = 0$ if $x \notin [-(t+k), t+k]$. Then from the definition of the convolution we have

$$|\Phi_t * g(x)| = \frac{1}{t} \left| \int_{-t}^{t} \Phi(\frac{y}{t}) g(x-y) dy \right|.$$

Let $s = x - y$. We know that Φ is bounded by M. Then from triangle inequality of the integration we have

$$|\Phi_t * g(x)| \leq \frac{M}{t} \int_{x-t}^{x+t} |g(s)| ds.$$

Since $supp g \subseteq [-k, k]$ we can replace $\int_{x-t}^{x+t} |g|$ by the possibly larger integral over $[-k, k]$,

$$|\Phi_t * g(x)| \leq \frac{M}{t} \int_{-k}^{k} |g(s)| ds.$$

From Hölder's Inequality, $\frac{1}{p} + \frac{1}{q} = 1$,

$$|\Phi_t * g(x)| \leq \frac{M}{t} (2k)^{\frac{1}{q}} \|g\|_{L^p}.$$

$$\|\Phi_t * g\|_{L^p}^p = \int_{-(t+k)}^{t+k} |\Phi_t * g(x)|^p dx$$

$$\leq \frac{M^p (2k)^{\frac{p}{q}}}{t^p} \|g\|_{L^p}^p 2(k+t)$$
$$= C \frac{2^{\frac{p}{q}} M^p k^{\frac{p}{q}}}{t^{p-1}} \|g\|_{L^p}^p.$$

Hence $\|\Phi_t * g\|_{L^p}^p \to 0$ as $t \to \infty$, since $1 < p$. Notice that when $p = 1$ the upper bound we just found no longer goes to 0.

The L^p functions with compact support, denoted $L^p_{cpct}(\mathbb{R})$, are dense in $L^p(\mathbb{R})$, so an approximation argument will complete (ii). $f \in L^p(\mathbb{R})$, given $\epsilon > 0$ there exists $g \in L^p_{cpct}(\mathbb{R})$ so that $\|f - g\|_p \leq \frac{\epsilon}{2\|\Phi\|_1}$. Then

$$\|f * \Phi_t\|_p \leq \|(f - g) * \Phi_t + g * \Phi_t\|_p.$$

From triangle inequality we have

$$\|f * \Phi_t\|_p \leq \|(f - g) * \Phi_t\|_p + \|g * \Phi_t\|_p.$$

Since $\|g * \Phi_t\|_p \to 0$ as $t \to \infty$, given $\epsilon > 0$ we can find T sufficiently large so that for all $T \leq t$.

$$\|f * \Phi_t\|_p \leq \|(f - g) * \Phi_t\|_p + \frac{\epsilon}{2}.$$

From Minkowski's integral inequality we have

$$\|F * \Phi_t\|_p \leq \int \|\tau_y F\|_{L^p(dx)} |\Phi_t(y)| dy$$
$$= \|\tau_y F\|_{L^p(dx)} \|\Phi_t\|_{L^1}$$
$$= \|\tau_y F\|_{L^p(dx)} \|\Phi\|_{L^1}$$
$$\leq \|\Phi\|_{L^1} \|F\|_{L^p}.$$

Therefore $TF := F * \Phi_t$ is bounded in $L^p(\mathbb{R})$ if Φ is in $L^1(\mathbb{R})$. In particular if $F = f - g$ then

$$\|(f - g) * \Phi_t\|_p \leq \|f - g\|_{L^p} \|\Phi\|_1$$
$$\leq \frac{\|\Phi\|_1 \epsilon}{2\|\Phi\|_1}$$

$$= \frac{\epsilon}{2}.$$

All together we conclude that given $\epsilon > 0$ there is $T > 0$ such that for all $T \leq t$

$$\|f * \Phi_t\|_p \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We conclude that for $f \in L^p(\mathbb{R})$, $\lim_{t\to\infty} \|f * \Phi_t\|_p = 0$.

$\square$

Chapter 3

Classical Theory for Hilbert Transform

There are several equivalent ways to introduce the Hilbert transform. We define the Hilbert transform in two equivalent ways: via its action in the frequency domain as a Fourier multiplier, second via its action in the domain as a singular integral [PW]. In Chapter 4, we will find a third representation via its action in the Haar domain as an average of Haar shift operators. In this chapter, we will also discuss symmetry invariances that characterize the Hilbert transform.

3.1 In the frequency domain

Here we define the Hilbert transform by its action on the Fourier domain.

Definition 3.1. *The Hilbert transform H of the functions $f \in L^2(\mathbb{R})$ is defined on the Fourier side as*

$$(Hf)^\wedge(\xi) := -isgn(\xi)\widehat{f}(\xi).$$

In this definition signum function sgn is defined by

$$sgn(\xi) = \begin{cases} 1, & \text{if} \quad \xi > 0, \\ 0 & \text{if} \quad \xi = 0, \\ -1 & \text{if} \quad \xi < 0. \end{cases}$$

Theorem 3.2. *The Hilbert transform is an isometry on $L^2(\mathbb{R})$; it preserves L^2 norms.*

$$\|Hf\|_2 = \|f\|_2.$$

Proof. From Plancherel's identity we know that when function $g \in L^2(\mathbb{R})$, then $\|\widehat{g}\|_2 = \|g\|_2$. When we apply Plancherel's identity twice, we have

$$\begin{aligned} \|Hf\|_2 &= \|(Hf)^\wedge\|_2 \\ &= \| - isgn(.)\widehat{f}(.)\|_2 \\ &= \|\widehat{f}\|_2 \\ &= \|f\|_2. \end{aligned}$$

□

Definition 3.3. *An operator T which is defined on the Fourier side is said to be the Fourier multiplier if*

$$(Tf)^\wedge(\xi) := m(\xi)\widehat{f}(\xi).$$

The function $m(\xi)$ is called the symbol of the operator.

The Hilbert transform is an example of a Fourier multiplier. From the definition of the Hilbert transform we can see that the symbol m_H of the Hilbert transform is $m_H(\xi) = -isgn(\xi)$ and it is a bounded function.

3.2 In the time domain

Definition 3.4. *The Hilbert transform H is defined on the time domain by the formula*

$$Hf(x) := p.v.\frac{1}{\pi}\int \frac{f(y)}{x-y}dy = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}dy \quad \text{for } f\in L^2(\mathbb{R}).$$

In the time domain Hilbert transform is given by convolution with the principal value distribution $k_H(x) := p.v\frac{1}{\pi x}$; $Hf = k_H * f$. Consider the well defined and nonsingular kernel

$$k_{\epsilon,R}(y) := \frac{1}{\pi y}\chi_{\{y\in\mathbb{R}:\epsilon<|y|<R\}}(y),$$

which is in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ for $0<\epsilon<R<\infty$. The function $k_{\epsilon,R}(y)$ is a truncated version of the kernel k_H. When we calculate the Fourier transform of $k_{\epsilon,R}$ for $\xi\neq 0$

$$\begin{aligned}(k_{\epsilon,R})^\wedge(\xi) &= \int_{\epsilon<|y|<R}\frac{1}{\pi y}e^{-2\pi iy\xi}dy\\ &= \int_\epsilon^R \frac{1}{\pi y}e^{-2\pi iy\xi}dy + \int_{-R}^{-\epsilon}\frac{1}{\pi y}e^{-2\pi iy\xi}dy.\end{aligned}$$

Substituting $y=-z$ in the second integral

$$\begin{aligned}(k_{\epsilon,R})^\wedge(\xi) &= \int_\epsilon^R \frac{1}{\pi y}e^{-2\pi iy\xi}dy - \int_\epsilon^R \frac{1}{\pi y}e^{2\pi iy\xi}dy\\ &= -2i\int_\epsilon^R \frac{\sin(2\pi y\xi)}{\pi y}dy.\end{aligned}$$

Substituting $t=2\pi y\xi$, $\frac{dt}{t}=\frac{dy}{y}$ we have

$$\begin{aligned}(k_{\epsilon,R})^\wedge(\xi) &= -\frac{2i}{\pi}\int_{2\pi\epsilon\xi}^{2\pi R\xi}\frac{\sin t}{t}dt\\ &= -isgn(\xi)\left(\frac{2}{\pi}\int_{2\pi\epsilon|\xi|}^{2\pi R|\xi|}\frac{\sin t}{t}dt\right).\end{aligned}$$

For any $\xi\neq 0$,

$$\frac{2}{\pi}\int_{2\pi\epsilon|\xi|}^{2\pi R|\xi|}\frac{\sin t}{t}dt \to \frac{2}{\pi}\int_0^\infty \frac{\sin t}{t}dt = 1,$$

as $\epsilon \to 0$ and $R \to \infty$. Therefore the truncated Hilbert transforms $H_{\epsilon,R}$ on $L^2(\mathbb{R})$, given by the equivalent formulas

$$H_{\epsilon,R}f = k_{\epsilon,R} * f, \quad (H_{\epsilon,R}f)^\wedge(\xi) = (k_{\epsilon,R})^\wedge(\xi)\widehat{f}(\xi),$$

can be seen to be uniformly bounded for all $f \in L^2(\mathbb{R})$, by Plancherel's identity. Also, when we take the limits as $R \to \infty$ and $\epsilon \to 0$, $(k_{\epsilon,R})^\wedge(\xi)$ converge pontwise to the symbol $m_H(\xi) = -isgn(\xi)$ of the Hilbert transform and for $f \in L^2(\mathbb{R})$ we have $H_{\epsilon,R}f \to Hf$. By the continuity of the Fourier transform, it follows that $(H_{\epsilon,R}f)^\wedge \to m_H\widehat{f}$ and $(Hf)^\wedge = m_H\widehat{f}$.

3.3 Symmetries for the Hilbert transform

The Hilbert transform has the following invariance properties on $L^2(\mathbb{R})$ [Graf].

Theorem 3.5. *Hilbert transform commutes with translations and dilations and it anticommutes with reflections.*

i) Let τ_h denote the translation operator: $\tau_h f(x) := f(x-h)$ for $h \in \mathbb{R}$. Then

$$\tau_h H = H\tau_h.$$

ii) Let δ_a denote the dilation operator: $\delta_a f(x) := f(ax)$ for $a > 0$. Then

$$\delta_a H = H\delta_a.$$

iii) Let $\widetilde{f}(x) := f(-x)$ denote the reflection of f across the origin. Then

$$(Hf)^\sim = -H(\widetilde{f}).$$

Proof. i) From time-frequency dictionary we know that $\widehat{\tau_h f}(\xi) = M_{-h}\widehat{f}(\xi)$ and by the definition of modulation

$$(\tau_h Hf)^\wedge(\xi) = e^{-2\pi i h\xi}(Hf)^\wedge(\xi).$$

From the definition of the Hilbert transform on the Fourier side we have

$$(\tau_h Hf)^\wedge(\xi) = -isgn(\xi)\widehat{f}(\xi)e^{-2\pi i h\xi}.$$

Since $\widehat{f}(\xi)e^{-2\pi i h\xi} = \widehat{\tau_h f}(\xi)$

$$\begin{aligned}(\tau_h Hf)^\wedge(\xi) &= -isgn(\xi)\widehat{\tau_h f}(\xi)\\ &= (H\tau_h f)^\wedge(\xi).\end{aligned}$$

Since Fourier Transform is a bijection on $L^2(\mathbb{R})$, $\tau_h H = H\tau_h$.

ii) From the time-frequency dictionary for a function f we have $(\delta_a f)^\wedge(\xi) = \delta_{a^{-1}}\widehat{f}(\xi)$. Then

$$(\delta_a Hf)^\wedge(\xi) = \delta_{a^{-1}}(Hf)^\wedge(\xi).$$

From dilation we have

$$(\delta_a Hf)^\wedge(\xi) = (Hf)^\wedge(a^{-1}\xi).$$

By using the definition of the Hilbert transform

$$(\delta_a Hf)^\wedge(\xi) = -isgn(a^{-1}\xi)\widehat{f}(a^{-1}\xi).$$

Since $a > 0$, $sgn(a^{-1}\xi) = sgn(\xi)$ and from the definition of the dilation $\widehat{f}(a^{-1}\xi) = \delta_{a^{-1}}\widehat{f}(\xi)$. When we substitute those in the equality above, we have

$$(\delta_a Hf)^\wedge(\xi) = -isgn(\xi)\delta_{a^{-1}}\widehat{f}(\xi).$$

From the time frequency dictionary and Hilbert transform we have

$$\begin{aligned}(\delta_a Hf)^\wedge(\xi) &= -isgn(\xi)(\delta_a f)^\wedge(\xi)\\ &= (H_{\delta_a} f)^\wedge(\xi).\end{aligned}$$

Since Fourier transform is a bijection on $L^2(\mathbb{R})$ we have $\delta_a H = H\delta_a$.

iii) From time-frequency dictionary we have

$$((Hf)^{\sim})^{\wedge}(\xi) = ((Hf)^{\wedge})^{\sim}(\xi).$$

From the definition of reflection

$$((Hf)^{\wedge})^{\sim}(\xi) = (Hf)^{\wedge}(-\xi).$$

By using the definition of the Hilbert transform on Fourier side we can find

$$(Hf)^{\wedge}(-\xi) = -isgn(-\xi)\widehat{f}(-\xi).$$

Since $\widehat{f}(-\xi) = \widetilde{\widehat{f}}(\xi)$ and from time-frequency dictionary $\widetilde{\widehat{f}}(\xi) = \widehat{\widetilde{f}}(\xi)$, we have

$$\begin{aligned} ((Hf)^{\sim})^{\wedge}(\xi) &= -(-isgn(\xi)\widehat{\widetilde{f}}(\xi)) \\ &= -(H\widetilde{f})^{\wedge}(\xi). \end{aligned}$$

Since Fourier Transform is linear $((Hf)^{\sim})^{\wedge}(\xi) = (-H\widetilde{f})^{\wedge}(\xi)$ and $(Hf)^{\sim} = -H\widetilde{f}$.

□

We just showed that Hilbert transform commutes with translations and dilations and anticommutes with reflections. These invariance properties characterize the Hilbert transform up to a multiplicative constant.

Theorem 3.6. *[Ste, Prop.1, Ch 3] Let T be a bounded operator on $L^2(\mathbb{R})$ that commutes with translations and dilations. If T commutes with reflections: $(Tf)^{\sim} = T(\widetilde{f})$ for all $f \in L^2(\mathbb{R})$, then T is a constant multiple of the identity operator: $T = cI$ for some $c \in \mathbb{R}$. If T anticommutes with reflections: $(Tf)^{\sim} = -T(\widetilde{f})$ for all $f \in L^2(\mathbb{R})$, then T is a constant multiple of the Hilbert transform: $T = cH$ for some $c \in \mathbb{R}$.*

Chapter 4

Random Grids and Sha operators

In this Chapter first we give the definitions of Random grids, Haar functions and Petermichl's Dyadic shift operator which is called "Sha". Then we prove some symmetries for the family of shift operators.

4.1 Dyadic Intervals and Random Dyadic Grids in $\mathbb{R}$

In this section we define the dyadic grids, mirror dyadic grids and prove a lemma about the dyadic grids.

Definition 4.1. *The standard dyadic grid $\mathcal{D}^0$ is the collection of intervals in $\mathbb{R}$ of the form $I = [k2^{-j}, (k+1)2^{-j})$, for all $k, j \in \mathbb{Z}$.*

$$\mathcal{D}^0 = \cup_{j\in\mathbb{Z}}\mathcal{D}^0_j.$$

The standard dyadic grid $\mathcal{D}^0$ is organized by generations $\mathcal{D}^0_j$. An interval $I \in \mathcal{D}^0_j$ if and only if $|I| = 2^{-j}$. Each generation is a partition of $\mathbb{R}$.

Properties of the standard dyadic grids:

- (Nestedness) If $I, J \in \mathcal{D}^0$, then exactly one of the three conditions $I \cap J = \emptyset$, $I \subseteq J$, or $J \subset I$ holds.
- (One Parent) If $I \in \mathcal{D}^0_j$, then there is a unique interval $\widetilde{I} \in \mathcal{D}^0_{j-1}$, the parent of I, such that $I \subset \widetilde{I}$.
- (Two Children) If $I \in \mathcal{D}^0_j$, then there are two intervals $I_l, I_r \in \mathcal{D}^0_{j+1}$, the left and right children of I, such that $I = I_l \cup I_r$, and $\mid I \mid= 2 \mid I_l \mid= 2 \mid I_r \mid$.
- (Strong Nestedness) If $I, J \in \mathcal{D}^0$ and $I \subsetneq J$, then $I \subseteq J_r$ or $I \subseteq J_l$.

Definition 4.2. *A general dyadic grid in $\mathbb{R}$ is a collection $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}_j$, where $\mathcal{D}_j = \mathcal{D}^0_j + x_j$ for some $x_j = \sum_{k<-j} \beta_k 2^k$, $\beta = \{\beta_k\}_{k\in\mathbb{Z}}$, $\beta_k \in \{0, 1\}$.*

Each generation $\mathcal{D}_j$ is a partition of $\mathbb{R}$. Dyadic grids have nestedness, unique parent and two children per interval properties.

Definition 4.3. *For each scaling or dilation parameter r with $1 \leq r < 2$ and the random parameter β with $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i \in \{0, 1\}$, let $x_j = \sum_{i<-j} \beta_i 2^i$, the collection of intervals $\mathcal{D}^{r,\beta} = \cup_{j\in\mathbb{Z}}\mathcal{D}^{r,\beta}_j$ is a scaled random dyadic grid where,*

$$\mathcal{D}^{\beta}_j = x_j + \mathcal{D}^0_j \quad \textit{and} \quad \mathcal{D}^{r,\beta}_j = r\mathcal{D}^{\beta}_j.$$

An interval $I \in \mathcal{D}^{r,\beta}_j$ has the form $I = r(x_j + J)$ where $J \in D^0_j$. Naturally I_l and I_r will denote left and right halves of I which are intervals in $\mathcal{D}^{r,\beta}_{j+1}$. For example

$$I_r = r(x_j + J_r) = r(x_{j+1} + \beta_{-j-1}2^{-(j+1)} + J_r),$$

where $\beta_{-j-1}2^{-(j+1)} + J_r \in \mathcal{D}^0_{j+1}$.

Remark 4.4. *Every interval in $\mathbb{R}$ of the form $I = [a, b)$ belongs to infinitely many random dyadic grids $\mathcal{D}^{r,\beta}$. The scaling parameter is unique, and the random parameter $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$ is determined for all integers i less than $-m$, where m is a fixed integer to be determined below. For $i \geq -m$, β_i can be 0 or 1, since there is a binary choice for the parent of I: the parent could be the interval $[a, b+|I|)$ so that I is the right child, or it could be $[a-|I|, b)$ so that I is the left child, likewise I will have four choices for grandparent (each parent has two choices for their respective parent), etc.*

To be more precise, let $|I| = b - a > 0$ and let m be the unique integer such that $2^{-m} \leq |I| < 2^{-m+1}$. Let $r := 2^m|I| = 2^m(b-a)$, by our choice of m, $1 \leq r < 2$ and this will be the scaling parameter for all grids that contain the interval I. There is also a unique integer k such that $k2^{-m} \leq a/r < (k+1)2^{-m}$. Let $x_m = a/r - k2^{-m}$, by our choice of k then $0 \leq x_m < 2^{-m}$. Moreover $a = rx_m + rk2^{-m}$ and $|I| = r2^{-m}$ therefore

$$I = a + [0, |I|) = r(x_m + k2^{-m} + [0, 2^{-m})) = r(x_m + [k2^{-m}, (k+1)2^{-m}).$$

So that $I = r(x_m + J)$ where $J := [k2^{-m}, (k+1)2^{-m}) \in \mathcal{D}^0_m$. The non-negative real number x_m has a binary expansion of the form $x_m = \sum_{i<-m} \beta'_i x^i$, hence $I \in \mathcal{D}^{r,\beta}_m$ for all $\beta = \{\beta_i\}_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ such that $\beta_i = \beta'_i$ for all $i \leq -m$.

The random dyadic grids include but are not restricted to ordinary translation grids.

Lemma 4.5. *For any $t > 0$ (there exists a $j_t \in \mathbb{Z}$ such that $2^{-j_t-1} \leq t < 2^{-j_t}$ and $t = \sum_{i<-j_t} t_i 2^i$, $t_i \in \{0,1\}$ with $t_{-j_t-1} = 1$)*
$D^0_j + t = D^0_j + x_j, \quad \forall j$ where $x_j = \sum_{i<-j} \beta_i 2^i$ and

$$\beta_i := \begin{cases} t_i & \text{for} \quad i < -j_t, \\ 0 & \text{for} \quad i \geq -j_t. \end{cases}$$

For any $t < 0$ ($t = -\sum_{i<-j_t} t_i 2^i$, $t_i \in \{0,1\}$)
$D^0_j + t = D^0_j + x_j, \quad \forall j$ where $x_j = \sum_{i<-j}(1-\beta_i)2^i$ for the same β_i as above.

Proof. For $t > 0$, we know that $D_j^0 = \{[k2^{-j}, (k+1)2^{-j}) : k \in \mathbb{Z}\}$, $t = \sum_{i<-j_t} t_i 2^i$ for $t_i \in \{0, 1\}$ and there are two cases.

Case 1: If $j \leq j_t$. Then $t = x_j$ and $D_j^0 + t = D_j^0 + x_j$.

Case 2: If $j > j_t$. Then

$$\begin{aligned} t &= \sum_{i<-j} \beta_i 2^i + \sum_{i=-j}^{-j_t-1} \beta_i 2^i \\ &= x_j + \beta_{-j}2^{-j} + 2\beta_{-j+1}2^{-j} + \ldots + 2^{-j_t+j-1}\beta_{-j_t-1}2^{-j} \\ &= x_j + (\beta_{-j} + 2\beta_{-j+1} + \ldots + 2^{-j_t+j-1}\beta_{-j_t-1})2^{-j}. \end{aligned}$$

In this equation $\beta_{-j} + 2\beta_{-j+1} + \ldots + 2^{-j_t+j-1}\beta_{-j_t-1}$ is an integer. If we call it $m = m(j, t)$, we have

$$\begin{aligned} D_j^0 + t &= \{[k2^{-j}, (k+1)2^{-j}) + x_j + m2^{-j} : k \in \mathbb{Z}\} \\ &= \{[(k+m)2^{-j}, (k+m+1)2^{-j}) + x_j : k \in \mathbb{Z}\}. \end{aligned}$$

Note that $k + m$ is an integer, and as you vary $k \in \mathbb{Z}$, $k + m$ ranges over all of $\mathbb{Z}$. Moreover the map $f : \mathbb{Z} \to \mathbb{Z}$, given by $f(k) = k + m$ is a bijection. Denote $s = f(k) = k + m \in \mathbb{Z}$. Then

$$\begin{aligned} D_j^0 + t &= \{[s2^{-j}, (s+1)2^{-j}) + x_j : s \in \mathbb{Z}\} \\ &= D_j^0 + x_j. \end{aligned}$$

For $t < 0$, we know $t = -\sum_{i<-j_t} t_i 2^i$, $t_i \in \{0, 1\}$ and this shows that $-2^{-j_t} \leq t < 0$ and $0 \leq t + 2^{-j_t} < 2^{-j_t}$. Then

Case 1: If $j > j_t$

$$\begin{aligned} D_j^0 + t &= \{[k2^{-j}, (k+1)2^{-j}) + t : k \in \mathbb{Z}\} \\ &= \{[k2^{-j} - 2^{-j_t}, (k+1)2^{-j} - 2^{-j_t}) + 2^{-j_t} + t : k \in \mathbb{Z}\} \\ &= \{[(k - 2^{-j_t+j})2^{-j}, (k+1-2^{-j_t+j})2^{-j}) + 2^{-j_t} + t : k \in \mathbb{Z}\}. \end{aligned}$$

In this equation $k - 2^{-j_t+j}$ is an integer, since $-j_t + j > 0$. The map $f : \mathbb{Z} \to \mathbb{Z}$, $f(k) = k - 2^{-j_t+j} = m$ is a bijection. Then

$$D_j^0 + t = \{[m2^{-j}, (m+1)2^{-j}) : m \in \mathbb{Z}\} + 2^{-j_t} - \sum_{i<-j_t} \beta_i 2^i.$$

When we substitute $\sum_{i<-j_t} 2^i$ for 2^{-j_t}

$$\begin{aligned} D_j^0 + t &= D_j^0 + \sum_{i<-j_t} 2^i - \sum_{i<-j_t} \beta_i 2^i \\ &= D_j^0 + \sum_{i<-j_t} (1-\beta_i) 2^i \\ &= D_j^0 + x_j. \end{aligned}$$

Case 2: If $j \leq j_t$. Then $-j_t \leq -j$ and

$$2^{-j} + t = \sum_{i<-j} 2^i - \sum_{i<-j_t} \beta_i 2^i = \sum_{i<-j} 2^i - \sum_{i<-j} \beta_i 2^i = \sum_{i<-j} (1-\beta_i) 2^i = x_j.$$

Then

$$\begin{aligned} D_j^0 + t &= \{[k2^{-j}, (k+1)2^{-j}) + t : k \in \mathbb{Z}\} \\ &= \{[k2^{-j} - 2^{-j}, (k+1)2^{-j} - 2^{-j}) + t + 2^{-j} : k \in \mathbb{Z}\} \\ &= \{[(k-1)2^{-j}, k2^{-j}) + x_j : k \in \mathbb{Z}\} \\ &= D_j^0 + x_j. \end{aligned}$$

□

Definition 4.6. *The mirror standard dyadic grid $\breve{\mathcal{D}}^0$ is the collection of intervals in $\mathbb{R}$ of the form $I = (k2^{-j}, (k+1)2^{-j}]$, for all $k, j \in \mathbb{Z}$.*

$$\breve{\mathcal{D}}^0 = \cup_{j\in\mathbb{Z}} \breve{\mathcal{D}}_j^0.$$

Notice that $I \in \breve{\mathcal{D}}^0$ if and only if $-I \in \mathcal{D}^0$.

Definition 4.7. *A general mirror dyadic grid in* $\mathbb{R}$ *is a collection* $\breve{\mathcal{D}} = \cup_{j\in\mathbb{Z}}\breve{\mathcal{D}}_j$, *where* $\breve{\mathcal{D}}_j = \breve{\mathcal{D}}_j^0 + x_j$ *for some* $x_j = \sum_{k<-j}\beta_k 2^k$, $\beta = \{\beta_k\}_{k\in\mathbb{Z}}$, $\beta_k \in \{0, 1\}$.

Definition 4.8. *For each scaling or dilation parameter* r *with* $1 \leq r < 2$ *and the random parameter* β *with* $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i \in \{0, 1\}$, *let* $x_j = \sum_{i<-j}\beta_i 2^i$, *the collection of intervals* $\breve{\mathcal{D}}^{r,\beta} = \cup_{j\in\mathbb{Z}}\breve{\mathcal{D}}_j^{r,\beta}$ *is a mirror scaled random dyadic grid, where*

$$\breve{\mathcal{D}}_j^{\beta} = x_j + \breve{\mathcal{D}}_j \quad \text{and} \quad \breve{\mathcal{D}}_j^{r,\beta} = r\breve{\mathcal{D}}_j^{\beta}.$$

4.2 Haar functions and mirror Haar functions

Haar functions were presented in 1909 by Alfréd Haar. Haar utilized these functions to demonstrate an instance of a complete orthonormal system for the space of square-integrable functions on the unit interval [0, 1]. In addition, this was an unconditional basis in $L^p[0, 1]$. In this section, we define Haar and mirror Haar functions and give some properties of the Haar functions.

Definition 4.9. *The Haar function on a dyadic interval* I *is defined as*

$$h_I(x) = \frac{1}{\sqrt{|I|}}\left(\chi_{I_r}(x) - \chi_{I_l}(x)\right).$$

From this definition we can see that the Haar function on a dyadic interval I is

$$h_I(x) = \begin{cases} \frac{1}{\sqrt{|I|}} & \text{for} \quad x \in I_r, \\ -\frac{1}{\sqrt{|I|}} & \text{for} \quad x \in I_l, \\ 0 & \text{otherwise.} \end{cases}$$

which is a step function.

Definition 4.10. *The mirror Haar function on a mirror dyadic interval* J *is a step function and defined as*

$$\breve{h}_J(x) = \frac{1}{\sqrt{|J|}}\left(\chi_{J_r}(x) - \chi_{J_l}(x)\right).$$

Theorem 4.11. *[PW, Chapter 9] The Haar functions $\{h_I\}_{I\in\mathcal{D}}$ form an orthonormal basis for $L^2(\mathbb{R})$ so do the mirror Haar functions. In particular for all $f \in L^2(\mathbb{R})$*

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I\rangle h_I,$$

and Plancherel holds, that is

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2.$$

Definition 4.12. *The modified Haar function on a dyadic interval I is defined by*

$$H_I(x) := \frac{1}{\sqrt{2}}\left(h_{I_r}(x) - h_{I_l}(x)\right).$$

Definition 4.13. *The modified mirror Haar function on a mirror dyadic interval J is defined by*

$$\breve{H}_I(x) := \frac{1}{\sqrt{2}}\left(\breve{h}_{J_r}(x) - \breve{h}_{J_l}(x)\right).$$

Proposition 4.14. *For the Haar function associated to the dyadic interval I we have the following:*

i) $h_I(x+h) = h_{I-h}(x)$.

ii) $h_I(\frac{x}{a}) = \sqrt{a}h_{aI}(x)$, *where* $a > 0$.

iii) $h_I(-x) = -\breve{h}_{-I}(x)$.

Proof. i) From the definition of Haar function we have that

$$h_I(x+h) = \frac{1}{\sqrt{|I|}}\left(\chi_{I_r}(x+h) - \chi_{I_l}(x+h)\right).$$

If $x+h \in I_r$, $x \in I_r - h = (I-h)_r$. From the definition of characteristic function $\chi_{I_r}(x+h) = \chi_{(I-h)_r}(x)$ and similarly $\chi_{I_l}(x+h) = \chi_{(I-h)_l}(x)$. When we substitute these two equations in the equation above, we have

$$h_I(x+h) = \frac{1}{\sqrt{|I|}}\left(\chi_{(I-h)_r}(x) - \chi_{(I-h)_l}(x)\right).$$

The intervals I and $I-h$ have the same size, so

$$\begin{aligned} h_I(x+h) &= \frac{1}{\sqrt{|I-h|}} \left(\chi_{(I-h)_r}(x) - \chi_{(I-h)_l}(x)\right) \\ &= h_{I-h}(x). \end{aligned}$$

ii) From the definition of Haar function we have

$$h_I(\frac{x}{a}) = \frac{1}{\sqrt{|I|}} \left(\chi_{I_r}(\frac{x}{a}) - \chi_{I_l}(\frac{x}{a})\right).$$

If $\frac{x}{a} \in I_r$, $x \in aI_r = (aI)_r$. Then, from the definition of the characteristic function $\chi_{I_r}(\frac{x}{a}) = \chi_{(aI)_r}(x)$ and similarly $\chi_{I_l}(\frac{x}{a}) = \chi_{(aI)_l}(x)$. When we substitute these equations in the equation above, we have

$$h_I(\frac{x}{a}) = \frac{1}{\sqrt{|I|}} \left(\chi_{(aI)_r}(x) - \chi_{(aI)_l}(x)\right).$$

From the size of the intervals we have $|a||I| = |aI|$. So

$$\begin{aligned} h_I(\frac{x}{a}) &= \sqrt{a}\frac{1}{\sqrt{|aI|}} \left(\chi_{(aI)_r}(x) - \chi_{(aI)_l}(x)\right) \\ &= \sqrt{a}h_{aI}(x). \end{aligned}$$

iii) From the definition of the Haar function we have

$$h_I(-x) = \frac{1}{\sqrt{|I|}} \left(\chi_{I_r}(-x) - \chi_{I_l}(-x)\right).$$

If $-x \in I_r$, $x \in -I_r = (-I)_l$ where $-I$ is a mirror dyadic interval. From the definition of the characteristic function $\chi_{I_r}(-x) = \chi_{(-I)_l}(x)$ and similarly $\chi_{I_l}(-x) = \chi_{(-I)_r}(x)$. Then we have

$$\begin{aligned} h_I(-x) &= \frac{1}{\sqrt{|I|}} \left(\chi_{(-I)_l}(x) - \chi_{(-I)_r}(x)\right) \\ &= \frac{1}{\sqrt{|-I|}} \left(\chi_{(-I)_l}(x) - \chi_{(-I)_r}(x)\right). \end{aligned}$$

From the definition of the mirror Haar function we can see that

$$h_I(-x) = -\breve{h}_{-I}(x).$$

□

Recall that $\tau_h f(x) = f(x-h)$, $\delta_a f(x) = f(ax)$ and $\widetilde{f}(x) = f(-x)$.

Corollary 4.15. *For the Haar function associated to the interval I we have the following:*

i) Let τ_h denote the translation operator. Then $\tau_h h_I = h_{I-h}$.

ii) Let δ_a denote the dilation operator. Then $\delta_a h_I = \sqrt{a} h_{aI}$.

iii) Let $\widetilde{f}(x) := f(-x)$ denote the reflection of f across the origin. Then $\widetilde{h}_I = -\breve{h}_{-I}$.

Similar to the Haar function, for the modified Haar function we have the proposition below. We can prove it easily when we use the definition of the modified Haar function and Proposition 4.14.

Proposition 4.16. *For the modified Haar function associated to the dyadic interval I we have the following*

i) $H_I(x+h) = H_{I-h}(x)$

ii) $H_I(ax) = \frac{1}{\sqrt{a}} H_{\frac{I}{a}}(x)$, *where* $a > 0$.

iii) $H_I(-x) = \breve{H}_{-I}(x)$

Proof. i) From the definition of the modified Haar function we have

$$H_I(x+h) = \frac{1}{\sqrt{2}} \left(h_{I_r}(x+h) - h_{I_l}(x+h) \right).$$

By Proposition 4.14 we can find that $h_{I_r}(x+h) = h_{(I-h)_r}(x)$ and $h_{I_l}(x+h) = h_{(I-h)_l}(x)$. Then

$$H_I(x+h) = \frac{1}{\sqrt{2}}\left(h_{(I-h)_r}(x) - h_{(I-h)_l}(x)\right).$$

From the definition of the modified Haar function we can see that the right side of the above equality is $H_{I-h}(x)$. So

$$H_I(x+h) = H_{I-h}(x).$$

ii) From the definition of the modified Haar function we have

$$H_I(ax) = \frac{1}{\sqrt{2}}\left(h_{I_r}(ax) - h_{I_l}(ax)\right).$$

By Proposition 4.14 and $\frac{I_r}{a} = (\frac{I}{a})_r$, $\frac{I_l}{a} = (\frac{I}{a})_l$ we have $h_{I_r}(ax) = \frac{1}{\sqrt{a}} h_{(\frac{I}{a})_r}(x)$ and $h_{I_l}(ax) = \frac{1}{\sqrt{a}} h_{(\frac{I}{a})_l}(x)$. Then

$$\begin{aligned} H_I(ax) &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{a}}\left(h_{(\frac{I}{a})_r}(x) - h_{(\frac{I}{a})_l}(x)\right) \\ &= \frac{1}{\sqrt{a}} H_{(\frac{I}{a})}(x). \end{aligned}$$

iii) From the definition of the modified Haar function

$$H_I(-x) = \frac{1}{\sqrt{2}}(h_{I_r}(-x) - h_{I_l}(-x)).$$

Since $-I_r = (-I)_l$ and $-I_l = (-I)_r$, from Proposition 4.14 we have $h_{I_r}(-x) = -\breve{h}_{(-I)_l}(x)$ and $h_{I_l}(-x) = -\breve{h}_{(-I)_r}(x)$. Then

$$H_I(-x) = \frac{1}{\sqrt{2}}(-\breve{h}_{(-I)_l}(x) + \breve{h}_{(-I)_r}(x)).$$

From the definition of the mirror Haar function

$$H_I(-x) = \breve{H}_{-I}(x).$$

□

Lemma 4.17. *For the dyadic interval I*

$$h_{t(I+v)}(x) = \frac{1}{\sqrt{t}} h_I(\frac{x}{t} - v),$$

where $t > 0$ and v is a real number.

Proof. From the definition of the Haar function we have

$$h_{t(I+v)}(x) = \frac{1}{\sqrt{|t(I+v)|}} \big(\chi_{(t(I+v))_r}(x) - \chi_{(t(I+v))_l}(x)\big).$$

When $x \in (t(I+v))_r$, $\frac{x}{t} - v \in I_r$. From the definition of the characteristic function $\chi_{(t(I+v))_r}(x) = \chi_{I_r}(\frac{x}{t} - v)$ and similarly $\chi_{(t(I+v))_l}(x) = \chi_{I_l}(\frac{x}{t} - v)$. Then

$$h_{t(I+v)}(x) = \frac{1}{\sqrt{|t(I+v)|}} \big(\chi_{I_r}(\frac{x}{t} - v) - \chi_{I_l}(\frac{x}{t} - v).$$

The size of the interval $t(I+v)$ is $|t(I+v)| = t|I|$. So

$$\begin{aligned} h_{t(I+v)}(x) &= \frac{1}{\sqrt{t}\sqrt{|I|}} \big(\chi_{I_r}(\frac{x}{t} - v) - \chi_{I_l}(\frac{x}{t} - v) \\ &= \frac{1}{\sqrt{t}} h_I(\frac{x}{t} - v). \end{aligned}$$

□

Corollary 4.18. $H_{t(I+v)}(x) = \frac{1}{\sqrt{t}} H_I(\frac{x}{t} - v)$ *when* $t > 0$*.*

4.3 Petermichl's Dyadic Shift Operator (Sha)

Petermichl's dyadic shift operator associated to $r\mathcal{D}^\beta$ is denoted by the Russian letter Ш , pronounced "sha", and is defined for $f \in L^2(\mathbb{R})$ by

$$Ш^{\beta,r} f := \sum_{I \in r\mathcal{D}^\beta} \langle h_I, f \rangle H_I = \sum_{j \in \mathbb{Z}} \sum_{I \in r\mathcal{D}^\beta_j} \langle h_I, f \rangle H_I,$$

where $H_I = \frac{1}{\sqrt{2}}(h_{I_r} - h_{I_l})$ is a modified Haar function.

Theorem 4.19. *Petermichl's shift operator is an isometry in $L^2(\mathbb{R})$, that is, it preserves the L^2-norms, i.e. $||Шf||_2 = ||f||_2$.*

Proof. From the definition of Sha and modified Haar function we have

$$\begin{aligned} Шf &= \sum_{I\in\mathcal{D}} \langle f, h_I\rangle H_I \\ &= \frac{1}{\sqrt{2}} \sum_{I\in\mathcal{D}} \langle f, h_I\rangle (h_{I_r} - h_{I_l}). \end{aligned}$$

Let J is the left or right child of I and $\tilde{J}$ is the parent of J. Then

$$Шf = \frac{1}{\sqrt{2}} \sum_{J\in\mathcal{D}} \langle f, h_{\tilde{J}}\rangle h_J sgn(J, \tilde{J}),$$

where

$$sgn(J,\tilde{J}) = \begin{cases} 1 & \text{if} \quad J = \tilde{J}_r, \\ -1 & \text{if} \quad J = \tilde{J}_l. \end{cases}$$

The L^2 norm of $Шf$ is the sum of the squares of the coefficients in the Haar basis. Then

$$\|Шf\|_2^2 = \frac{1}{2}\sum_{J\in\mathcal{D}} |\langle f, h_{\tilde{J}}\rangle|^2 = \frac{1}{2}\sum_{I\in\mathcal{D}} |\langle f, h_I\rangle|^2 2 = \|f\|_2^2.$$

Hence

$$\|Шf\|_2 = \|f\|_2.$$

□

Unlike the Hilbert transform, Petermichl's dyadic shift operator associated to a dyadic grid $\mathcal{D}^{r,\beta}$ does not commute with translations and dilations, nor does it anti-commute with reflections. However it obeys closely related properties to be described in Theorem 2.4. Before we state the Theorem 2.4 we need to introduce an auxiliary family of operators that we denote $\breve{Ш}^{\beta,r}$ indexed by (β, r) the same parameters used $Ш^{\beta,r}$.

Definition 4.20. *Mirror Sha operator associated to $\breve{\mathcal{D}}^{r,\beta}$ is denoted by $\breve{Ш}^{\beta,r}$ and is defined for $f \in L^2(\mathbb{R})$ by*

$$\breve{Ш}^{\beta,r} f := \sum_{I \in r\breve{\mathcal{D}}^{\beta}} \langle \breve{h}_I, f \rangle \breve{H}_I = \sum_{j \in \mathbb{Z}} \sum_{I \in r\breve{\mathcal{D}}^{\beta}_j} \langle \breve{h}_I, f \rangle \breve{H}_I,$$

where $\breve{h}_I(y) = h_{-I}(-y)$ is a mirror Haar function and $\breve{H}_I = \frac{1}{\sqrt{2}}(\breve{h}_{I_r} - \breve{h}_{I_l})$ is a modified mirror Haar function associated to the interval $I \in r\breve{\mathcal{D}}^{\beta}$.

Recall that $\tau_h f(x) := f(x-h)$, $\delta_a f(x) := f(\frac{x}{a})$ and $\widetilde{f}(x) := f(-x)$. We will need some notation and familiarity with binary decomposition of real numbers. If $x \in \mathbb{R}$, $x > 0$, then $x = \sum_{i<-j} x_i 2^i$ where $x_i = 0$ or 1, $x_{-j-1} = 1$ and $2^{-j-1} \leq x \leq 2^{-j}$, $(x)_j := \sum_{i<-j} x_i 2^i$.

Theorem 4.21. *The following symmetries for the family of shift operators $\{Ш_{r,\beta}\}$ hold where $1 \leq r < 2$, $\beta \in \{0,1\}^{\mathbb{Z}}$.*

i) Translation: $\tau_h(Ш^{r,\beta} f)(x) = Ш^{\tau_h(r,\beta)}(\tau_h f)(x)$ where $\tau_h(r,\beta) = (s,\alpha)$ for $s = r$ and $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$, $\alpha_k \in \{0,1\}$ is defined by the equation $(x_i + \frac{h}{r})_i = y_i = \sum_{k<-i} \alpha_k 2^k$ for all $i \in \mathbb{Z}$.

ii) Dilation ($a > 0$): $\delta_a(Ш^{r,\beta} f) = Ш^{\delta_a(r,\beta)}(\delta_a f)(x)$ for $\delta_a(r,\beta) = (s,\alpha)$ where $s = \frac{r}{a}2^{-m}$ and $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$, $\alpha_k \in \{0,1\}$ defined as $\alpha_k = \beta_{k-m}$ for the unique $m \in \mathbb{Z}$ such that $2^m \leq \frac{r}{a} < 2^{m+1}$.

iii) Reflection: $(Ш^{r,\beta} f)^{\sim} = -\breve{Ш}^{(r,\beta)^{\sim}}(\tilde{f})$ where $(r,\beta)^{\sim} = (s,\alpha)$ for $s = r$ and $\widetilde{\beta}_i = 1 - \beta_i$ for $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$, $\beta_i \in \{0,1\}$.

Proof. i) We want to show $\tau_h(Ш^{r,\beta} f)(x) = Ш^{s,\alpha}(\tau_h f)(x)$ for appropriate $1 \leq s < 2$ and $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$, $\alpha_i \in \{0,1\}$. We want to find a relation between r, β, s and α.

From the definition of the operator Sha, we have

$$Ш^{s,\alpha}(\tau_h f)(x) = \sum_{J\in\mathcal{D}^{s,\alpha}} \langle \tau_h f, h_J\rangle H_J(x).$$

By using the definition of the inner product and the translation

$$\langle \tau_h f, h_J\rangle = \int_{-\infty}^{\infty} f(x-h)h_J(x)dx.$$

From substitution, $y = x - h$, we have

$$\langle \tau_h f, h_J\rangle = \int_{-\infty}^{\infty} f(y)h_J(y+h)dy.$$

When $y+h \in J$, J is an interval in $\mathcal{D}^{s,\alpha}$, $y \in J-h$ where $J-h = \{y : y = x - h \quad \text{for some} \quad x \in J\}$. By Proposition 4.14 we have that $h_J(y+h) = h_{J-h}(y)$, Hence

$$\langle \tau_h f, h_J\rangle = \int_{-\infty}^{\infty} f(y)h_{J-h}(y)dy = \langle f, h_{J-h}\rangle.$$

Then

$$Ш^{s,\alpha}(\tau_h f)(x) = \sum_{J\in\mathcal{D}^{s,\alpha}} \langle f, h_{J-h}\rangle H_J(x).$$

From the definition of translation and the operator Sha we have

$$\begin{aligned}\tau_h(Ш^{r,\beta} f)(x) &= Ш^{r,\beta} f(x-h)\\ &= \sum_{I\in\mathcal{D}^{r,\beta}} \langle f, h_I\rangle H_I(x-h).\end{aligned}$$

From Proposition 4.16 we have that $H_I(x-h) = H_{I+h}(x)$. By substituting this equation in the equation above we have

$$\tau_h(Ш^{r,\beta} f)(x) = \sum_{I\in\mathcal{D}^{r,\beta}} \langle f, h_I\rangle H_{I+h}(x).$$

If we had the equality of $\tau_h(Ш^{r,\beta} f)(x)$ and $Ш^{s,\alpha}(\tau_h f)(x)$ we will get

$$\sum_{I\in\mathcal{D}_{r,\beta}} \langle f, h_I\rangle H_{I+h}(x) = \sum_{J\in\mathcal{D}_{s,\alpha}} \langle f, h_{J-h}\rangle H_J(x).$$

We can get this equality if we can find $1 \leq s < 2$ and $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}},\ \alpha_i \in \{0,1\}$ such that for each $I \in \mathcal{D}^{r,\beta}$ there is exactly one $J \in \mathcal{D}^{s,\alpha}$ such that $J = I + h$, and for all $J \in \mathcal{D}^{s,\alpha}$ there is an $I \in \mathcal{D}^{r,\beta}$ such that $J = I+h$. Since $J \in \mathcal{D}^{s,\alpha} = \cup_{j\in\mathbb{Z}}\mathcal{D}^{s,\alpha}_j$ and $I \in \mathcal{D}^{r,\beta} = \cup_{i\in\mathbb{Z}}\mathcal{D}^{r,\beta}_i$ there are i, j, k and $m \in \mathbb{Z}$ such that

$$I = r\{x_i + [k2^{-i}, (k+1)2^{-i})\},\ \ x_i = \sum_{h<-i} \beta_h 2^h \ \ \beta_h \in \{0,1\},$$

$$J = s\{y_j + [m2^{-j}, (m+1)2^{-j})\},\ \ y_j = \sum_{k<-j} \alpha_k 2^k \ \ \alpha_k \in \{0,1\}.$$

If $J = I + h$, their sizes are the same, $s2^{-j} = r2^{-i}$. From this equality we can see that if $i \neq j$ then either r or s will not be in between 1 and 2, so $i = j$ and $r = s$. Then we have that

$$J = r\{y_i + [m2^{-i}, (m+1)2^{-i})\},$$

and

$$J = I + h = r\{x_i + \frac{h}{r} + [k2^{-i}, (k+1)2^{-i})\}.$$

From these two equalities and Section 2.3 we can see that for every $i \in \mathbb{Z}$ we have that $x_i + \frac{h}{r} = y_i + p_i 2^{-i}$ where $0 \leq y_i = (x_i + \frac{h}{r})_i = \sum_{k<-i} \alpha_k 2^k < 2^{-i}$, $p_i \in \mathbb{Z}$, so $(x_i + \frac{h}{r}) \equiv_i y_i$. That is, $(x_i + \frac{h}{r})$ and y_i are in the same equivalence class. (Recall $a \equiv_i b$ iff $\exists p \in \mathbb{Z}$ such that $a - b = p2^{-i}$, this is an equivalence relation.)

When we substract $\frac{h}{r}$ from y_i and write it in base 2, we have

$y_i - \frac{h}{r} = u_i + k_i 2^{-i}$ where $0 \leq u_i = (y_i - \frac{h}{r})_i < 2^{-i}$, $k_i \in \mathbb{Z}$ and $(y_i - \frac{h}{r}) \equiv_i u_i$. When we add the equalities $x_i + \frac{h}{r} = y_i + p_i 2^{-i}$ and $y_i - \frac{h}{r} = u_i + k_i 2^{-i}$ we have

$$x_i + \frac{h}{r} + y_i - \frac{h}{r} = y_i + p_i 2^{-i} + u_i + k_i 2^{-i}$$

$$x_i = u_i + (p_i + k_i)2^{-i}$$

As we can see from this equation x_i and $u_i = (y_i - \frac{h}{r})_i$ are in the same equivalence class, $x_i \equiv_i u_i$. This shows that we can undo this calculation. In other words given y_i, J, h and r we can find $I = J - h$, $x_i = (y_i - \frac{h}{r})_i = \sum_{k<-i} \beta_k 2^k$.

Since every step can be reversed we conclude that

$$\tau_h(\text{Ш}^{r,\beta} f)(x) = \text{Ш}^{s,\alpha}(\tau_h f)(x).$$

where $s = r$ and $\{\alpha_k\}_{k\in\mathbb{Z}}$ is defined by the equation $(x_i+\frac{h}{r})_i = y_i = \sum_{k<-i} \alpha_k 2^k$ for all $i \in \mathbb{Z}$.

ii) We want to show $\delta_a(\text{Ш}^{r,\beta} f)(x) = \text{Ш}^{s,\alpha}(\delta_a f)(x)$ for appropriate $1 \leq s < 2$ and $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$, $\alpha_i \in \{0, 1\}$. We want to find a relation between r, β, s and α.

From the definition of the operator Sha, we have

$$\text{Ш}^{s,\alpha}(\delta_a f)(x) = \sum_{J\in D^{s,\alpha}} \langle \delta_a f, h_J\rangle H_J(x).$$

By using the definition of the inner product and the dilation

$$\langle \delta_a f, h_J\rangle = \int_{-\infty}^{\infty} f(ax)h_J(x)dx.$$

From substitution, $y = ax$, we have

$$\langle \delta_a f, h_J\rangle = \frac{1}{a}\int_{-\infty}^{\infty} f(y)h_J(\frac{y}{a})dy.$$

$h_J(\frac{y}{a})$ is a standard Haar function and from Proposition 4.14 we have that $h_J(\frac{y}{a}) = \sqrt{a}h_{aJ}(y)$ where $aJ := \{y : y = ax \text{ for some } x \in J\}$.

By substituting this in our equation we have

$$\langle \delta_a f, h_J\rangle = \frac{1}{\sqrt{a}}\langle f, h_{aJ}\rangle.$$

Then

$$\text{Ш}^{s,\alpha}(\delta_a f)(x) = \frac{1}{\sqrt{a}} \sum_{J\in D^{s,\alpha}} \langle f, h_{aJ}\rangle H_J(x).$$

From the definition of the dilation and the operator Sha

$$\delta_a(\text{Ш}^{r,\beta} f)(x) = \text{Ш}^{r,\beta} f(ax)$$

$$= \sum_{I \in D^{r,\beta}} \langle f, h_I \rangle H_I(ax).$$

$H_I(ax)$ is a modified Haar function and from Proposition 4.16 $\frac{1}{\sqrt{a}} H_{\frac{I}{a}}(x)$. By substituting this in the equation above we have

$$\delta_a(\text{Ш}^{r,\beta} f)(x) = \frac{1}{\sqrt{a}} \sum_{I \in D_{r,\beta}} \langle f, h_I \rangle H_{\frac{I}{a}}(x).$$

If we had equality of $\delta_a(\text{Ш}^{r,\beta} f)(x)$ and $\text{Ш}^{s,\alpha}(\delta_a f)(x)$ we will get

$$\sum_{I \in D^{r,\beta}} \langle f, h_I \rangle H_{\frac{I}{a}}(x) = \sum_{J \in D^{s,\alpha}} \langle f, h_{aJ} \rangle H_J(x).$$

We can get this equality if we can find $1 \leq s < 2$ and $\alpha \in \{0,1\}^{\mathbb{Z}}$ such that $aJ = I$ where $J \in \mathcal{D}^{s,\alpha}$ and $I \in \mathcal{D}^{r,\beta}$. Since $J \in \mathcal{D}^{s,\alpha} = \cup_{j \in \mathbb{Z}} \mathcal{D}^{s,\alpha}_j$ and $I \in \mathcal{D}^{r,\beta} = \cup_{i \in \mathbb{Z}} \mathcal{D}^{r,\beta}_i$ there are i, j, k and $m \in \mathbb{Z}$ such that

$$I = r\{x_i + [k2^{-i}, (k+1)2^{-i})\}, \quad x_i = \sum_{h < -i} \beta_h 2^h$$

and

$$J = s\{y_j + [m2^{-j}, (m+1)2^{-j})\}, \quad y_j = \sum_{k < -j} \alpha_k 2^k.$$

Since $aJ = I$, $J = \frac{1}{a} I$. Which is

$$J = \frac{r}{a}\{x_i + [k2^{-i}, (k+1)2^{-i})\}.$$

The size of the dyadic interval J is $|J| = s2^{-j} = \frac{r}{a} 2^{-i}$. This equality shows that there is a relation between s and $\frac{r}{a}$. We know the r and a so there is a unique $m \in \mathbb{Z}$ such that $2^m \leq \frac{r}{a} < 2^{m+1}$.

We want an s which is between 1 and 2 so divide both sides of the equality by 2^m, to get $1 \leq \frac{r}{a} 2^{-m} < 2$. Hence, if we let $s := \frac{r}{a} 2^{-m}$ such that $2^m \leq \frac{r}{a} < 2^{m+1}$. Then $1 \leq s < 2$ as required.

Then

$$\begin{aligned} J &= \frac{r}{a}\{x_i + [k2^{-i}, (k+1)2^{-i})\} \\ &= \frac{r}{a}2^{-m}\{x_i 2^m + [k2^{-i+m}, (k+1)2^{-i+m})\} \\ &= s\{x_i 2^m + [k2^{-i+m}, (k+1)2^{-i+m})\}. \end{aligned}$$

From this equality we can define $y_j = x_i 2^m$ and $j := i - m$. Then

$$\begin{aligned} y_j &= x_i 2^m \\ &= \sum_{h<-i} \beta_h 2^h 2^m \\ &= \sum_{k<-i+m} \beta_{k-m} 2^k, \quad h+m=k \\ &= \sum_{k<-j} \beta_{k-m} 2^k. \end{aligned}$$

Set now $\alpha_k = \beta_{k-m}$. Then $y_j = \sum_{k<-j} \alpha_k 2^k$.

Since every step can be reversed we conclude that $\delta_a(Ш^{r,\beta} f) = Ш^{\delta_a(r,\beta)}(\delta_a f)(x)$ where $s = \frac{r}{a}2^{-m}$ and for all $k \in \mathbb{Z}$ $\alpha_k = \beta_{k-m}$ for the unique $m \in \mathbb{Z}$ such that $2^m \leq \frac{r}{a} < 2^{m+1}$.

iii) We want to show $(Ш^{r,\beta} f)^{\sim} = -\breve{Ш}^{s,\alpha}(\widetilde{f})$ for appropriate s and α, where $1 \leq s < 2$ and $\alpha = \{\alpha_i\}_{i\in\mathbb{Z}}$, $\alpha_i \in \{0,1\}$. We want to find a relation between r, s, β and α.

From the definition of the Sha and reflection,

$$\begin{aligned} (Ш^{r,\beta} f)^{\sim}(x) = Ш^{r,\beta} f(-x) \\ = \sum_{I \in D^{r,\beta}} \langle f, h_I \rangle H_I(-x). \end{aligned}$$

From the definition of the operator Mirror Sha, we have

$$\breve{Ш}^{s,\alpha}(\widetilde{f})(x) = \sum_{J \in \breve{D}^{s,\alpha}} \langle \widetilde{f}, \breve{h}_J \rangle \breve{H}_J(x).$$

By using the definition of the norm and reflection

$$\langle \widetilde{f}, \breve{h}_J \rangle = \int_{-\infty}^{\infty} f(-x)\breve{h}_J(x)dx$$

From substitution, $y = -x$, we have

$$\begin{aligned}\langle \widetilde{f}, \breve{h}_J \rangle &= -\int_{\infty}^{-\infty} f(y)\breve{h}_J(-y)dy \\ &= \int_{-\infty}^{\infty} f(y)\breve{h}_J(-y)dy.\end{aligned}$$

From the Proposition 4.14 and proposition 4.16 we have $\breve{h}_J(-y) = -h_{-J}(y)$ and $\breve{H}_J(x) = H_{-J}(-x)$. Then

$$\langle \widetilde{f}, \breve{h}_J \rangle = -\int_{-\infty}^{\infty} f(y)h_{-J}(y)dy,$$

and

$$\breve{Ш}_{s,\alpha}(\widetilde{f})(x) = -\sum_{J\in\breve{D}_{s,\alpha}} \langle f, h_{-J}\rangle H_{-J}(-x).$$

From the desired equality of $(Ш^{r,\beta} f)^{\sim}$ and $-\breve{Ш}^{s,\alpha}(\widetilde{f})$ we can see that if we can show $J = -I$ for $J \in \breve{\mathcal{D}}^{s,\alpha}$ and $I \in \mathcal{D}^{r,\beta}$ when $s = r$ and $\alpha = \widetilde{\beta}$, then we are done.

The dyadic interval $I \in D^{r,\beta}$ so there are $k, i \in \mathbb{Z}$ such that

$$I = r\{x_i + [k2^{-i}, (k+1)2^{-i})\},\ x_i = \sum_{n<-i} \beta_n 2^n.$$

Since $J \in \breve{D}^{s,\alpha}$, so there are $m, j \in \mathbb{Z}$ such that

$$J = s\{y_j + (m2^{-j}, (m+1)2^{-j}]\},\ y_j = \sum_{t<-j} \beta_t 2^t.$$

When we multiply I by -1, we want to have the interval $J = -I \in \breve{\mathcal{D}}^{s,\alpha}$. So

$$\begin{aligned}J &= -r\{x_i + [k2^{-i}, (k+1)2^{-i})\} \\ &= -r\{\sum_{n<-i} \beta_n 2^n + [k2^{-i}, (k+1)2^{-i})\}\end{aligned}$$

$$= r\{\sum_{n<-i} -\beta_n 2^n + (-(k+1)2^{-i}, -k2^{-i}]\}.$$

By adding and subtracting $r \sum_{n<-i} 1.2^n = r2^{-i}$ from the right side of the equation we have

$$J = r\{\sum_{n<-i} -\beta_n 2^n + (-(k+1)2^{-i}, -k2^{-i}] + \sum_{n<-i} 1.2^n - 2^{-i}\}$$
$$= r\{\sum_{n<-i} (1-\beta_n)2^n + (-(k+2)2^{-i}, -(k+1)2^{-i}]\}.$$

From this equality we can see that if $s = r$ and $\alpha = \widetilde{\beta} = 1-\beta$, $J \in \breve{\mathcal{D}}^{s,\alpha} = \breve{\mathcal{D}}^{r,\tilde{\beta}}$ if and only if $-J \in \mathcal{D}^{r,\beta}$. Since every step can be reversed we conclude that $(\text{Ш}^{r,\beta} f)^{\sim} = -\breve{\text{Ш}}^{r,\tilde{\beta}}(\tilde{f})$.

$\square$

Remark 4.22. *When $r = 1$ and $\beta = \{0\}_{i\in\mathbb{Z}}$, the initial grid is the standard grid $\mathcal{D}^{1,0} = \mathcal{D}^0$. Then $\mathcal{D}^{\tau_h(1,0)} = \mathcal{D}^0 + h$, the standard grid translated by h; $\mathcal{D}^{\delta_a(1,0)} = (1/a)\mathcal{D}^0$, the standard grid dilated by the reciprocal of a; and $\mathcal{D}^{(1,0)^{\sim}} = \breve{D}^0$, the mirror standard grid.*

Chapter 5

Hilbert Transform as an average of Petermichl's shift Operator

In this chapter we will show that we can represent the Hilbert transform as an average of Petermichl's shift operator (Sha operator) over dyadic random grids. Let us define the average Sha operator.

Definition 5.1. *[Hyt] The average of Petermichl's shift operator is defined as*

$$\langle Ш \rangle f = \lim_{n,m\to\infty} \sum_{j=-m}^{n} \int_1^2 \int_{\beta\in\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^{\beta}} \langle h_I, f\rangle H_I d\mu(\beta) \frac{dr}{r},$$

where $d\mu_j(\beta)$ is the canonical probability product measure associated to the space $X = \{0,1\}^{\mathbb{Z}}$. More specifically given by: for each $i \in \mathbb{Z}$ $\mu(\beta_i = 0) = \mu(\beta_i = 1) = \frac{1}{2}$.

Remark 5.2. *[Tao] For each $j \in \mathbb{Z}$, let (X_j, β_j, μ_j) be two-element set $X_j = \{0,1\}$ with the discrete metric (and thus discrete σ-algebra) and the uniform probability measure μ_j. Theorem 2.4.4 [Tao] gives a probability measure μ on the infinite discrete cube $\{0,1\}^{\mathbb{Z}}$, the product measure known as the (uniform) Bernoulli measure on this cube. The coordinate functions $\pi_j : \{0,1\}^{\mathbb{Z}} \to \{0,1\}$ can then be interpreted as a* *countable sequence of random variables taking values in* $\{0,1\}$*. Also this topic is beyond the scope of this thesis.*

We first show that the average of Petermichl's Shift operator can be reinterpreted in terms of the Lebesque measure.

Lemma 5.3. *[Hyt]* $\langle Ш \rangle f(x) = \int_0^\infty \int_{-\infty}^\infty H_{t([0,1)+v)}(x)\langle h_{t([0,1)+v)}, f\rangle dv\frac{dt}{t}$.

Proof. By definition,

$$\langle Ш \rangle f(x) = \lim_{n,m\to\infty} \sum_{j=-m}^{n} \int_1^2 \int_{\beta\in\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I(x) d\mu(\beta)\frac{dr}{r}.$$

Notice that when $I \in r\mathcal{D}_j^\beta$, $I = r(x_j + J)$ where $J \in \mathcal{D}^0$, $J = [k2^{-j}, (k+1)2^{-j})$, $k \in \mathbb{Z}$ and $x_j = \sum_{i<-j} \beta_i 2^i$. Let $u_j := 2^j x_j = \sum_{i<-j} \beta_i 2^{i+j}$. From the change of summation variable $s = i + j$, $u_j = \sum_{s<0} \beta_{s-j} 2^s$. From this equality we can see that $0 \le u_j \le 1$.

As a random variable $u_j = u_j(\beta)$, u_j is uniformly distributed on interval $[0,1]$. So Lebesgue measure is its corresponding probability measure.

$$I = r(x_j + J) = r2^{-j}\left(2^j x_j + [k, k+1)\right) = r2^{-j}(u_j + k + [0,1)).$$

Substituting in the sum we get,

$$\sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I(x) = \sum_{k\in\mathbb{Z}} \langle f, h_{r2^{-j}(u_j+k+[0,1))}\rangle H_{r2^{-j}(u_j+k+[0,1))}(x).$$

The variable $u_j(\beta)$ is uniformly distributed on the interval $[0,1]$.

$$\int_{\beta\in\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I(x) d\mu(\beta) = \int_0^1 \sum_{k\in\mathbb{Z}} \langle f, h_{r2^{-j}(u_j+k+[0,1))}\rangle H_{r2^{-j}(u_j+k+[0,1))}(x) du_j.$$

There is exactly one k so that $x \in r2^{-j}(u_j + k + [0,1))$, so the series is truly just one term, and we can interchange integral and sum.

$$\int_{\beta\in\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I(x) d\mu(\beta) = \sum_{k\in\mathbb{Z}} \int_0^1 \langle f, h_{r2^{-j}(u_j+k+[0,1))}\rangle H_{r2^{-j}(u_j+k+[0,1))}(x) du_j.$$

For each $k \in \mathbb{Z}$, let $v := u_j + k$ and $t = r2^{-j}$. Then

$$\int_{\beta\in\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I(x) d\mu(\beta) = \sum_{k\in\mathbb{Z}} \int_k^{k+1} \langle f, h_{t(v+[0,1))}\rangle H_{t(v+[0,1))}(x) dv$$
$$= \int_{-\infty}^{\infty} \langle f, h_{t(v+[0,1))}\rangle H_{t(v+[0,1))}(x) dv,$$

and since $\frac{dt}{t} = \frac{dr}{r}$ and if $r \in [1,2)$, then for each j, $t \in [2^{-j}, 2^{-j+1})$

$$\langle Ш\rangle f(x) = \lim_{n,m\to\infty} \sum_{j=-m}^{n} \int_{2^{-j}}^{2^{-j+1}} \int_{-\infty}^{\infty} \langle f, h_{t(v+[0,1))}\rangle H_{t(v+[0,1))}(x) dv \frac{dt}{t}.$$

By additivity of the integral, $\int \sum_{\cup_{j=-m}^n A_j} = \sum_{j=-m}^n \int_{A_j}$ where $A_j = [2^{-j}, 2^{-j+1})$ are pairwise disjoint and measurable, we have

$$\langle Ш\rangle f(x) = \lim_{n,m\to\infty} \int_{2^{-n}}^{2^{m+1}} \int_{-\infty}^{\infty} \langle f, h_{t(v+[0,1))}\rangle H_{t(v+[0,1))}(x) dv \frac{dt}{t}$$
$$= \int_0^{\infty} \int_{-\infty}^{\infty} \langle f, h_{t(v+[0,1))}\rangle H_{t(v+[0,1))}(x) dv \frac{dt}{t}.$$

□

We are now ready to state and prove the main Theorem in the thesis.

Theorem 5.4. *[Hyt] For $r \in [1,2)$ and $\beta \in \{0,1\}^{\mathbb{Z}}$, let $Ш^{\beta,r}$ be the dyadic shift associated to the dyadic system $r\mathcal{D}^\beta$. Let μ stand for the canonical probability measure on $\{0,1\}^{\mathbb{Z}}$ which makes the coordinate functions β_j independent with $\mu(\beta_j = 0) = \mu(\beta_j = 1) = \frac{1}{2}$. Then for all $p \in (1,\infty)$ and $f \in L^p(\mathbb{R})$.*

$$Hf(x) = \frac{8}{\pi}\langle Ш\rangle f(x),$$

where

$$\langle Ш\rangle f(x) := \lim_{n,m\to\infty} \sum_{j=-m}^{n} \int_1^2 \int_{\{0,1\}^{\mathbb{Z}}} \sum_{I\in r\mathcal{D}_j^\beta} \langle h_I, f\rangle H_I d\mu(\beta) \frac{dr}{r},$$

and convergence is both pointwise for almost every $x \in \mathbb{R}$, and in $L^p(\mathbb{R})$ for $1 < p < \infty$.

Proof. From Lemma 5.3

$$\langle Ш\rangle f(x)=\int_0^\infty\int_{-\infty}^\infty H_{t([0,1)+v)}(x)\langle h_{t([0,1)+v)},f\rangle dv\frac{dt}{t}.$$

From the definition of the inner product, we have

$$\langle Ш\rangle f(x)=\int_0^\infty\int_{-\infty}^\infty H_{t([0,1)+v)}(x)\int_{-\infty}^\infty h_{t([0,1)+v)}(y)f(y)dydv\frac{dt}{t}.$$

From Lemma 4.17 $h_{t([0,1)+v)}(y)=\frac{1}{\sqrt{t}}h_{[0,1)}(\frac{y}{t}-v)$. Similarly from Corollary 4.18 $H_{t([0,1)+v)}(x)=\frac{1}{\sqrt{t}}H_{[0,1)}(\frac{x}{t}-v)$. By using these equalities above and Fubini's theorem we can transform the above equality of $\langle Ш\rangle f(x)$ into the form

$$\langle Ш\rangle f(x)=\int_0^\infty\int_{-\infty}^\infty\frac{1}{t}\left(\int_{-\infty}^\infty H_{[0,1)}(\frac{x}{t}-v)h_{[0,1)}(\frac{y}{t}-v)dv\right)f(y)dy\frac{dt}{t}.$$

Let $\frac{x}{t}=x_0$ and $\frac{y}{t}=y_0$. Then the innermost integral is

$$\int_{-\infty}^\infty H_{[0,1)}(\frac{x}{t}-v)h_{[0,1)}(\frac{y}{t}-v)dv=\int_{-\infty}^\infty H_{[0,1)}(x_0-v)h_{[0,1)}(y_0-v)dv.$$

Let $s=v-y_0$, then

$$\int_{-\infty}^\infty H_{[0,1)}(\frac{x}{t}-v)h_{[0,1)}(\frac{y}{t}-v)dv=\int_{-\infty}^\infty H_{[0,1)}((x_0-y_0)-s)h_{[0,1)}(-s)ds.$$

From the definition of the reflection $h_{[0,1)}(-s)=\widetilde{h}_{[0,1)}(s)$. Then, by the definition of the convolution

$$\int_{-\infty}^\infty H_{[0,1)}(\frac{x}{t}-v)h_{[0,1)}(\frac{y}{t}-v)dv=\widetilde{h}_{[0,1)}*H_{[0,1)}(x_0-y_0).$$

Notice that we can write both $\widetilde{h}_{[0,1)(s)}$ and $H_{[0,1)}(s)$ as linear combinations of characteristic functions of 4 disjoint intervals of length $\frac{1}{4}$.

$$\widetilde{h}_{[0,1)(s)}=h_{[0,1)}(-s)=\chi_{(-1,\frac{-3}{4}]}(s)+\chi_{(\frac{-3}{4},\frac{-1}{2}]}(s)-\chi_{(\frac{-1}{2},\frac{-1}{4}]}(s)-\chi_{(\frac{-1}{4},0]}(s).$$

$$H_{[0,1)}(s)=\chi_{[\frac{3}{4},1)}(s)-\chi_{[\frac{1}{2},\frac{3}{4})}(s)-\chi_{[\frac{1}{4},\frac{1}{2})}(s)+\chi_{[0,\frac{1}{4})}(s).$$

Let $u=x_0-y_0$. By using the two equalities above and the distributive property of the convolution we have

$$\widetilde{h}_{[0,1)}*H_{[0,1)}(u)=\chi_{(-1,\frac{-3}{4}]}*\chi_{[\frac{3}{4},1)}(u)+\chi_{(\frac{-3}{4},\frac{-1}{2}]}*\chi_{[\frac{3}{4},1)}(u)-\chi_{(\frac{-1}{2},\frac{-1}{4}]}*\chi_{[\frac{3}{4},1)}(u)$$

$$- \chi_{(\frac{-1}{4},0]} * \chi_{[\frac{3}{4},1)}(u) - \chi_{(-1,\frac{-3}{4}]} * \chi_{[\frac{1}{2},\frac{3}{4})}(u) - \chi_{(\frac{-3}{4},\frac{-1}{2}]} * \chi_{[\frac{1}{2},\frac{3}{4})}(u)$$
$$+ \chi_{(\frac{-1}{2},\frac{-1}{4}]} * \chi_{[\frac{1}{2},\frac{3}{4})}(u) + \chi_{(\frac{-1}{4},0]} * \chi_{[\frac{1}{2},\frac{3}{4})}(u) - \chi_{(-1,\frac{-3}{4}]} * \chi_{[\frac{1}{4},\frac{1}{2})}(u)$$
$$- \chi_{(\frac{-3}{4},\frac{-1}{2}]} * \chi_{[\frac{1}{4},\frac{1}{2})}(u) + \chi_{(\frac{-1}{2},\frac{-1}{4}]} * \chi_{[\frac{1}{4},\frac{1}{2})}(u) + \chi_{(\frac{-1}{4},0]} * \chi_{[\frac{1}{4},\frac{1}{2})}(u)$$
$$+ \chi_{(-1,\frac{-3}{4}]} * \chi_{[0,\frac{1}{4})}(u) + \chi_{(\frac{-3}{4},\frac{-1}{2}]} * \chi_{[0,\frac{1}{4})}(u) - \chi_{(\frac{-1}{2},\frac{-1}{4}]} * \chi_{[0,\frac{1}{4})}(u)$$
$$- \chi_{(\frac{-1}{4},0]} * \chi_{[0,\frac{1}{4})}(u).$$

Let $\widetilde{h}_{[0,1)} * H_{[0,1)}(u) = k(u)$. As we remember from Lemma 2.4 convolution of two characteristic functions of intervals with the same length is a continuous and piecewise linear function. Since addition or substruction of two continuous piecewise linear function is still a continuous piecewise linear function, k(u) is a piecewise linear function and it is continuous. A direct calculation yields,

$$k(u) = \begin{cases} u+1, & \text{if} \quad -1 \leq u \leq -\frac{3}{4}, \\ -u-\frac{1}{2} & \text{if} \quad -\frac{3}{4} \leq u \leq -\frac{1}{2}, \\ -3u-\frac{3}{2} & \text{if} \quad -\frac{1}{2} \leq u \leq -\frac{1}{4}, \\ 3u & \text{if} \quad -\frac{1}{4} \leq u \leq 0, \\ 3u & \text{if} \quad 0 \leq u \leq \frac{1}{4}, \\ -3u+\frac{3}{2} & \text{if} \quad \frac{1}{4} \leq u \leq \frac{1}{2}, \\ -u+\frac{1}{2} & \text{if} \quad \frac{1}{2} \leq u \leq \frac{3}{4}, \\ u-1 & \text{if} \quad \frac{3}{4} \leq u \leq 1, \\ 0 & \text{if} \quad u > 1 \quad \text{or} \quad u < -1. \end{cases}$$

Figure 5.1 shows the graph of $k(u)$. So,

$$\langle \text{III} \rangle f(x) = \int_0^\infty \left(\int_{-\infty}^\infty \frac{1}{t} k(\frac{x}{t} - \frac{y}{t}) f(y) dy \right) \frac{dt}{t}.$$

By using the definition of the convolution we have,

$$\langle \text{III} \rangle f(x) = \int_0^\infty k_t * f(x) \frac{dt}{t},$$

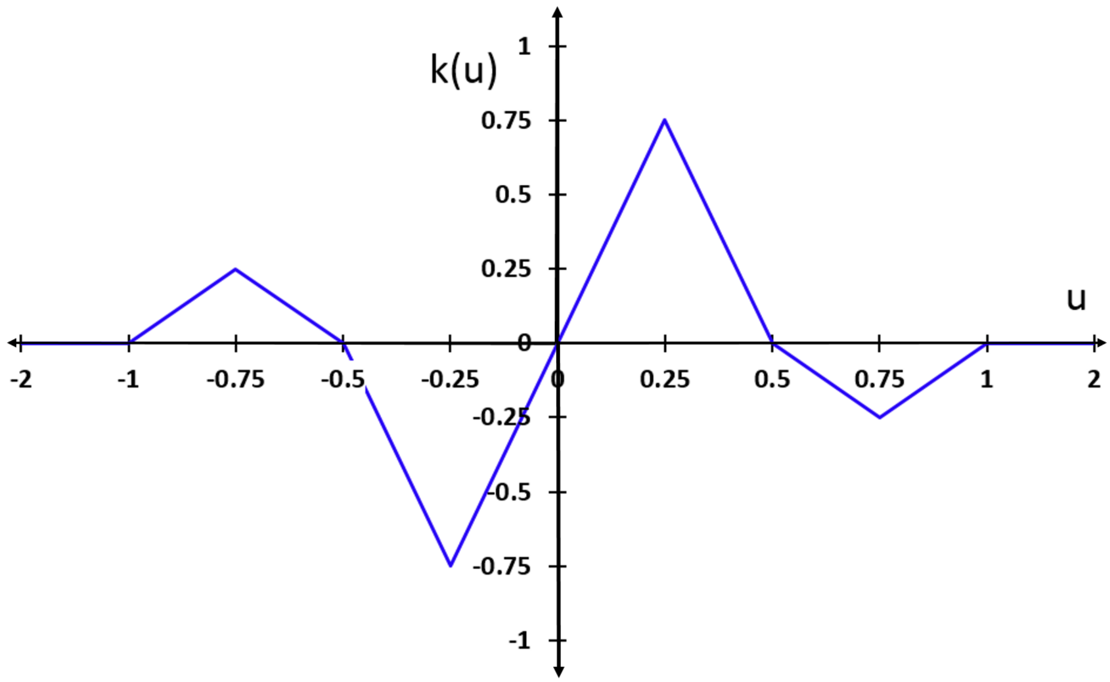

Figure 5.1: Graph of the continuous piecewise linear function k(u)

where $k_t(x) := t^{-1}k(t^{-1}x)$ is the usual L^1-dilation and $k_t * f(x)$ is convolution. Denote the antiderivative of k by K more precisely, $K(x) := \int_0^x k(u)du$. Since k is continuous and piecewise linear its antiderivative will be continuous and piecewise quadratic, moreover $K(0) = 0$. With this notation and substituting $b = t^{-1}x$, $db = -\frac{dt}{t^2}$ we have

$$\begin{aligned}\int_\epsilon^R k_t(x)\frac{dt}{t} &= \int_\epsilon^R \frac{1}{t}k(t^{-1}x)\frac{dt}{t} \\ &= -\frac{1}{x}\int_{\frac{x}{\epsilon}}^{\frac{x}{R}} k(b)d(b) \\ &= -\frac{1}{x}\Big(\int_0^{\frac{x}{R}} k(b)d(b) - \int_0^{\frac{x}{\epsilon}} k(b)db\Big) \\ &= \frac{1}{x}\big(K(\frac{x}{\epsilon}) - K(\frac{x}{R})\big).\end{aligned}$$

From the definition of the characteristic function

$$\begin{aligned}\int_\epsilon^R k_t(x)\frac{dt}{t} &= \frac{1}{x}\Big\{K(\frac{x}{\epsilon})\big(\chi_{(-\infty,-R)}(x)+\chi_{[-R,-\epsilon)}(x)+\chi_{[-\epsilon,\epsilon]}(x)+\chi_{(\epsilon,R]}(x)\\ &\quad +\chi_{[R,\infty)}(x)\big)-K(\frac{x}{R})\big(\chi_{(-\infty,-R)}(x)+\chi_{[-R,R]}(x)+\chi_{(R,\infty)}(x)\big)\Big\}\\ &= \frac{1}{x}\Big\{K(\frac{x}{\epsilon})\chi_{[-\epsilon,\epsilon]}(x)-K(\frac{x}{R})\chi_{[-R,R]}(x)\Big\}\\ &\quad +\frac{1}{x}\Big\{K(\frac{x}{\epsilon})\big(\chi_{(-\infty,-R)}(x)+\chi_{[-R,-\epsilon)}(x)+\chi_{(\epsilon,R]}(x)+\chi_{(R,\infty)}(x)\big)\\ &\quad -K(\frac{x}{R})\big(\chi_{(-\infty,-R)}(x)+\chi_{(R,\infty)}(x)\big)\Big\}.\end{aligned}$$

Define $\Phi(x) := x^{-1}K(x)\chi_{[-1,1]}(x)$. Notice that by definition Φ is supported on the interval $[-1,1]$. Since K is continuous piecewise quadratic and $K(0)=0$, necessarily $x^{-1}K(x)$ is continuous at $x=0$ and compactly supported. Hence Φ is bounded. From the definition of Φ we can find that

$$\frac{1}{x}K(\frac{x}{\epsilon})\chi_{[-\epsilon,\epsilon]}(x)=\Phi_\epsilon(x),$$

and

$$\frac{1}{x}K(\frac{x}{R})\chi_{[-R,R]}(x)=\Phi_R(x).$$

Then we have

$$\begin{aligned}\int_\epsilon^R k_t(x)\frac{dt}{t} &= \Phi_\epsilon(x)-\Phi_R(x)+\frac{1}{x}\Big\{K(\frac{x}{\epsilon})\big(\chi_{(-\infty,-R)}(x)+\chi_{[-R,-\epsilon)}(x)\\ &\quad +\chi_{(\epsilon,R]}(x)+\chi_{(R,\infty)}(x)\big)-K(\frac{x}{R})\big(\chi_{(-\infty,-R)}(x)+\chi_{(R,\infty)}(x)\big)\Big\}.\end{aligned}$$

From the definition of $K(x)$ we can find that $K(x)=\frac{1}{8}$ when $x<-1$ or $x>1$. When we call

$$\begin{aligned}A = \frac{1}{x}\Big\{&K(\frac{x}{\epsilon})\big(\chi_{(-\infty,-R)}(x)+\chi_{[-R,-\epsilon)}(x)+\chi_{(\epsilon,R]}(x)+\chi_{(R,\infty)}(x)\big)\\ &-K(\frac{x}{R})\big(\chi_{(-\infty,-R)}(x)+\chi_{(R,\infty)}(x)\big)\Big\},\end{aligned}$$

we have five different cases for A such that

i) If $-\infty<x<-R$, $A=\frac{1}{x}\{K(\frac{x}{\epsilon})-K(\frac{x}{R})\}=0$.

ii) If $-R \leq x < -\epsilon$, $A = \frac{1}{x}K(\frac{x}{\epsilon}) = \frac{1}{8x}$.

iii) If $-\epsilon \leq x \leq \epsilon$, $A = 0$.

iv) If $\epsilon \leq x \leq R$, $A = \frac{1}{x}K(\frac{x}{\epsilon}) = \frac{1}{8x}$.

v) If $R < x < \infty$, $A = \frac{1}{x}\{K(\frac{x}{\epsilon}) - K(\frac{x}{R})\} = 0$.

Then we can see that for any $x \in \mathbb{R}$ we have

$$A = \frac{1}{8x}\chi_{\{x:\epsilon<|x|<R\}}(x).$$

So,

$$\int_\epsilon^R k_t(x)\frac{dt}{t} = \Phi_\epsilon(x) - \Phi_R(x) + \frac{1}{8x}\chi_{\{x:\epsilon<|x|<R\}}(x),$$

and

$$\int_\epsilon^R k_t * f\frac{dt}{t} = \Phi_\epsilon * f - \Phi_R * f + \frac{\pi}{8}H_{\epsilon,R}f,$$

where $H_{\epsilon,R}f(x) = \frac{1}{\pi}\int_{\epsilon\leq|y|\leq R}\frac{f(y)}{x-y}dy$. As k is and odd function, $K(x) = \int_0^x k(u)du$ is an even function and Φ is an odd function, so $\int_{\mathbb{R}}\Phi = 0$. Also Φ is bounded and compactly supported. Using Mollifications Theorems 2.12, 2.13 and Theorem 2.14 $\lim_{\epsilon\to 0}\Phi_\epsilon * f = 0$ and $\lim_{R\to\infty}\Phi_R * f = 0$ in L^p for $1 < p < \infty$ and almost everywhere. Hence

$$\langle\text{Ш}\rangle f = \frac{\pi}{8}\lim_{n,m\to\infty} H_{2^{-m},2^n}f = \frac{\pi}{8}Hf.$$

□

Note: Hytönen defines his Haar functions differently; $h_I^* = -h_I = \frac{1}{\sqrt{|I|}}(\chi_{I_l} - \chi_{I_r})$, his Haar function is the negative of ours. He apparently defines the modified Haar function in a alternate manner, but in the end his Haar function coincide with ours. $H_I^* = \frac{1}{\sqrt{2}}(h_{I_l}^* - h_{I_r}^*) = \frac{1}{\sqrt{2}}(-h_{I_l} + h_{I_r}) = H_I$. This explains why we are getting $\frac{\pi}{8}Hf$ when he got $-\frac{\pi}{8}Hf$.

Chapter 6

Conclusions and Future Research

In this thesis we showed that the Hilbert transform in $\mathbb{R}$ can be represented as an average of dyadic shift operators. There are discrete and finite analogues of the Hilbert transform. For vectors $x = (x_{-N+1}, ..., x_{-1}, x_0, x_1, ..., x_N)$ in $\mathbb{R}^{2N}$, the finite Hilbert transform H_N is defined by

$$(H_N x)(i) := \sum_{-N \leq j, j \neq i}^{N} \frac{x_j}{i-j} \quad \text{for} \quad |i| \leq N.$$

Discrete Hilbert transform H^d is defined by

$$H^d x(m) = \sum_{n=-\infty}^{\infty} h_{m,n} x_n,$$

where

$$h_{m,n} := \begin{cases} \frac{1}{m-n}, & \text{if} \quad m \neq n, \\ 0, & \text{if} \quad m = n. \end{cases}$$

and $x \in l^2(\mathbb{Z})$, $x = (x_n)_{n \in \mathbb{Z}}$ and $\sum |x_n|^2 < \infty$.

There are Haar basis in $\mathbb{R}^d$ for $d = 2^k$ and in $l^2(\mathbb{Z})$. Haar shift operators can be defined in both the finite and sequential settings. Can we write the finite and/or the discrete Hilbert transform as averages of finite and/or sequential Haar shift operators?

First one should stablish some symmetry theorem analogue to Steins's theorem. Second in the sequential case one can deal with random dyadic grids on a parameter $\beta \in \{0,1\}^{\mathbb{N}}$. In the finite case is unclear how to randomize the dyadic grid.

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