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Florentin Smarandache

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Florentin Smarandache

University of New Mexico, Math & Science Division, 705 Gurley Ave., Gallup, NM 87301, USA, E-mail: smarand@unm.edu

Abstract. Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. We now introduce for the first time the notions of neutrosophic measure and neutrosophic integral. Neutrosophic Science means development and applications of neutrosophic logic/set/measure/integral/probability etc. and their applications in any field. It is possible to define the neutrosophic measure and consequently the neutrosophic integral and neutrosophic probability in many ways, because there are various types of indeterminacies, depending on the problem we need to solve. Indeterminacy is different from randomness. Indeterminacy can be caused by physical space materials and type of construction, by items involved in the space, or by other factors. Neutrosophic measure is a generalization of the classical measure for the case when the space contains some indeterminacy. Neutrosophic Integral is defined on neutrosophic measure. Simple examples of neutrosophic integrals are given.

Keywords: neutrosophy, neutrosophic measure, neutrosophic integral, indeterminacy, randomness, probability.

1 Introduction to Neutrosophic Measure

1.1 Introduction

Let <A> be an item. <A> can be a notion, an attribute, an idea, a proposition, a theorem, a theory, etc.

And let <antiA> be the opposite of <A>; while <neutA> be neither <A> nor <antiA> but the neutral (or indeterminacy, unknown) related to <A>.

For example, if <A> = victory, then <antiA> = defeat, while <neutA> = tie game.

If <A> is the degree of truth value of a proposition, then <antiA> is the degree of falsehood of the proposition, while <neutA> is the degree of indeterminacy (i.e. neither true nor false) of the proposition.

Also, if <A> = voting for a candidate, <antiA> = voting against that candidate, while <neutA> = not voting at all, or casting a blank vote, or casting a black vote. In the case when <antiA> does not exist, we consider its measure be null {m(antiA)=0}. And similarly when <neutA> does not exist, its measure is null { m(neutA) = 0}.

1.2 Definition of Neutrosophic Measure

We introduce for the first time the scientific notion of neutrosophic measure.

Let X be a neutrosophic space, and Σ a σ-neutrosophic algebra over X. A *neutrosophic measure* ν is defined by for neutrosophic set $A \in \Sigma$ by

$\nu : X \to R^3$,

$$\nu(A) = (m(A), m(neutA), m(antiA)), \quad (1)$$

with *antiA* = the opposite of A, and *neutA* = the neutral (indeterminacy) neither A nor anti A (as defined above);

for any $A \subseteq X$ and $A \in \Sigma$,

m(A) means *measure of the determinate part of A;*

m(neutA) means *measure of indeterminate part of A;*

and *m(antiA)* means *measure of the determinate part of antiA*;

where ν is a function that satisfies the following two properties:

a) Null empty set: $\nu(\Phi) = (0,0,0)$.

b) Countable additivity (or σ-additivity): For all countable collections $\{A_n\}_{n\in L}$ of disjoint neutrosophic sets in Σ, one has:

$$\nu\left(\bigcup_{n\in L} A_n\right) = \left(\sum_{n\in L} m(A_n),\ \sum_{n\in L} m(neutA_n), \sum_{n\in L} m(antiA_n) - (n-1)m(X)\right)$$

where X is the whole neutrosophic space,

and

$$\sum_{n\in L} m(antiA_n) - (n-1)m(X) = m(X) - \sum_{n\in L} m(A_n) = m(\bigcap_{n\in L} antiA_n). \quad (2)$$

1.3 Neutrosophic Measure Space

A neutrosophic measure space is a triplet (X, Σ, ν).

1.4 Normalized Neutrosophic Measure

A neutrosophic measure is called normalized if

$\nu(X) = (m(X), m(neutX), m(antiX)) = (x_1, x_2, x_3)$,

4 with $x_1 + x_2 + x_3 = 1$,

$$\text{and } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \quad (3)$$

Where, of course, X is the whole neutrosophic measure space.

1.5 Finite Neutrosophic Measure Space

Let $A \subset X$. We say that $\nu(A)=(a_1,a_2,a_3)$ is finite if all a_1, a_2, and a_3 are finite real numbers.

A *neutrosophic measure space* (X,Σ,ν) is called *finite* if $\nu(X)=(a,b,c)$ such that all *a, b,* and *c* are finite (rather than infinite).

1.6 σ-Finite Neutrosophic Measure

A neutrosophic measure is called σ-finite if *X* can be decomposed into a countable union of neutrosophically measurable sets of fine neutrosophic measure.

Analogously, a set *A* in *X* is said to have a σ-finite neutrosophic measure if it is a countable union of sets with finite neutrosophic measure.

1.7 Neutrosophic Axiom of Non-Negativity

We say that the neutrosophic measure ν *satisfies the axiom of non-negativity*, if:

$$\forall A \in \Sigma ,$$

$$\nu(A)=(a_1,a_2,a_3)\geq 0 \text{ if } a_1 \geq 0, a_2 \geq 0, \text{ and } a_3 \geq 0 . \quad (4)$$

While a neutrosophic measure ν, that satisfies only the null empty set and countable additivity axioms (hence not the non-negativity axiom), takes on at most one of the $\pm\infty$ values.

1.8 Measurable Neutrosophic Set and Measurable Neutrosophic Space

The members of Σ are called measurable neutrosophic sets, while (X,Σ) is called a measurable neutrosophic space.

1.9 Neutrosophic Measurable Function

A function $f:(X,\Sigma_X)\to(Y,\Sigma_Y)$, mapping two measurable neutrosophic spaces, is called *neutrosophic measurable function* if $\forall B \in \Sigma_Y, f^{-1}(B)\in\Sigma_X$ (the inverse image of a neutrosophic Y-measurable set is a neutrosophic X-measurable set).

1.10 Neutrosophic Probability Measure

As a particular case of neutrosophic measure ν is th neutrosophic probability measure, i.e. a neutrosophic measure that measures probable/possible propositions

$$^{-}0 \leq \nu(X) \leq 3^{+}, \quad (5)$$

where X is the whole neutrosophic probability sample space.

We use nonstandard numbers, such 1^+ for example, to denominate the absolute measure (measure in all possible worlds), and standard numbers such as *1* to denominate the relative measure (measure in at least one world). Etc.

We denote the neutrosophic probability measure by $\mathcal{NP}$ for a closer connection with the classical probability $\mathcal{P}$.

1.11 Neutrosophic Category Theory

The neutrosophic measurable functions and their neutrosophic measurable spaces form a neutrosophic category, where the functions are arrows and the spaces objects.

We introduce the neutrosophic category theory, which means the study of the neutrosophic structures and of the neutrosophic mappings that preserve these structures.

The classical category theory was introduced about 1940 by Eilenberg and Mac Lane.

A neutrosophic category is formed by a class of neutrosophic objects $X,Y,Z,...$ and a class of neutrosophic morphisms (arrows) $\nu,\xi,\omega,...$ such that:

a) If $Hom(X,Y)$ represent the neutrosophic morphisms from X to Y, then $Hom(X,Y)$ and $Hom(X',Y')$ are disjoint, except when $X=X'$ and $Y=Y'$;

b) The composition of the neutrosophic morphisms verify the axioms of

i) Associativity: $(\nu\circ\xi)\circ\omega=\nu\circ(\xi\circ\omega)$

ii) Identity unit: for each neutrosophic object X there exists a neutrosophic morphism denoted id_X, called *neutrosophic identity* of X such that $id_X\circ\nu=\nu$ and $\xi\circ id_X=\xi$

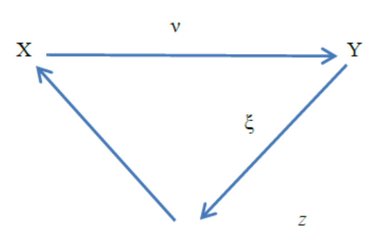

Fig. 2

1.12 Properties of Neutrosophic Measure

a) Monotonicity.

If A_1 and A_2 are neutrosophically measurable, with $A_1 \subseteq A_2$, where

$\nu(A_1)=(m(A_1),m(neutA_1),m(antiA_1))$,

and $\nu(A_2)=(m(A_2),m(neutA_2),m(antiA_2))$,

then

$$m(A_1)\leq m(A_2), m(neutA_1)\leq m(neutA_2), m(antiA_1)\geq m(antiA_2) \quad (6)$$

Let $\nu(X)=(x_1,x_2,x_3)$ and $\nu(Y)=(y_1,y_2,y_3)$. We say that $\nu(X)\leq\nu(Y)$, if $x_1 \leq y_1, x_2 \leq y_2$, and $x_3 \geq y_3$.

b) Additivity.

If $A_1 \cap A_2 = \Phi$, then $\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$, (7)

where we define

$$(a_1,b_1,c_1)+(a_2,b_2,c_2)=(a_1+a_2,b_1+b_2,a_3+b_3-m(X)) \quad (8)$$

where X is the whole neutrosophic space, and

$$a_3+b_3-m(X)=m(X)-m(A)-m(B)=m(X)-a_1-a_2 \\ =m(antiA \cap antiB). \quad (9)$$

1.13 Neutrosophic Measure Continuous from Below or Above

A *neutrosophic measure* ν *is continuous from below* if, for $A_1, A_2, \ldots$ neutrosophically measurable sets with $A_n \subseteq A_{n+1}$ for all n, the union of the sets A_n is neutrosophically measurable, and

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n) \quad (10)$$

And a *neutrosophic measure* ν *is continuous from above* if for $A_1, A_2, \ldots$ neutrosophically measurable sets, with $A_n \supseteq A_{n+1}$ for all n, and at least one A_n has finite neutrosophic measure, the intersection of the sets A_n and neutrosophically measurable, and

$$\nu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \nu(A_n). \quad (11)$$

1.14 Generalizations

Neutrosophic measure is a generalization of the *fuzzy measure*, because when $m(neutA)=0$ and *m(antiA)* is ignored, we get

$$\nu(A)=(m(A),0,0)\equiv m(A) \quad (12)$$

and the two fuzzy measure axioms are verified:

a) If $A=\Phi$, then $\nu(A)=(0,0,0)\equiv 0$

b) If $A \subseteq B$, then $\nu(A) \le \nu(B)$.

The neutrosophic measure is practically a triple classical measure: a classical measure of the determinate part of a neutrosophic object, a classical part of the indeterminate part of the neutrosophic object, and another classical measure of the determinate part of the opposite neutrosophic object. Of course, if the indeterminate part does not exist (its measure is zero) and the measure of the opposite object is ignored, the neutrosophic measure is reduced to the classical measure.

1.15 Examples

Let's see some examples of neutrosophic objects and neutrosophic measures.

a) If a book of *100* sheets (covers included) has *3* missing sheets, then

$$\nu(book)=(97,3,0) \quad (13)$$

where ν is the neutrosophic measure of the book number of pages.

b) If a surface of *5 × 5* square meters has cracks of *0.1 × 0.2* square meters, then $\nu(surface)=(24.98,0.02,0)$, (14), where ν is the neutrosophic measure of the surface.

c) If a die has two erased faces then

$$\nu(die)=(4,2,0), \quad (14)$$

where ν is the neutrosophic measure of the die's number of correct faces.

d) An approximate number N can be interpreted as a neutrosophic measure $N=\underline{d}+\underline{i}$, where $\underline{d}$ is its determinate part, and $\underline{i}$ its indeterminate part. Its anti part is considered *0*.

For example if we don't know exactly a quantity q, but only that it is between let's say $q\in[0.8,0.9]$, then $q=0.8+i$, where *0.8* is the determinate part of q, and its indeterminate part $i\in[0,0.1]$.

We get a negative neutrosophic measure if we approximate a quantity measured in an inverse direction on the *x*-axis to an equivalent positive quantity.

For example, if $r\in[-6,-4]$, then $r=-6+i$, where -6 is the determinate part of r, and $i\in[0,2]$ is its indeterminate part. Its anti part is also 0.

e) Let's measure the truth-value of the proposition

G = "through a point exterior to a line one can draw only one parallel to the given line".

The proposition is incomplete, since it does not specify the type of geometrical space it belongs to. In an *Euclidean geometric space* the proposition *G* is <u>true</u>; in a *Riemannian geometric space* the proposition *G* is <u>false</u> (since there is no parallel passing through an exterior point to a given line); in a *Smarandache geometric space* (constructed from mixed spaces, for example from a part of Euclidean subspace together with another part of Riemannian space) the proposition *G* is <u>indeterminate</u> (true and false in the same time).

$$\nu(G)=(1,1,1). \quad (15)$$

f) In general, not well determined objects, notions, ideas, etc. can become subject to the neutrosophic theory.

2 Introduction to Neutrosophic Integral

2.1 Definition of Neutrosophic Integral

Using the neutrosophic measure, we can define a neutrosophic integral.

The neutrosophic integral of a function *f* is written as:

$$\int_X f d\nu \quad (16)$$

where *X* is the a neutrosophic measure space, and the integral is taken with respect to the neutrosophic measure ν.

Indeterminacy related to integration can occur in multiple ways: with respect to value of the function to be integrated, or with respect to the lower or upper limit of integration, or with respect to the space and its measure.

2.2 First Example of Neutrosophic Integral: Indeterminacy Related to Function's Values

Let *fN: [a, b]* → *R* (17)

where the neutrosophic function is defined as:

fN (x) = g(x)+i(x) (18)

with *g(x)* the determinate part of *fN(x)*, and *i(x)* the indeterminate part of *fN(x)*, where for all *x* in *[a, b]* one has: $i(x)\in[0,h(x)], h(x)\geq 0$. (19)

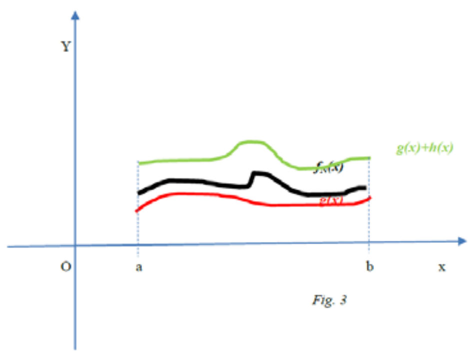

Fig. 3

Therefore the values of the function fN(x) are approximate, i.e. $f_N(x)\in[g(x),g(x)+h(x)]$. (20)

Similarly, the neutrosophic integral is an approximation:

$$\int_a^b f_N(x)d\nu = \int_a^b g(x)dx + \int_a^b i(x)dx \tag{21}$$

1.10 Second Example of Neutrosophic Integral: Indeterminacy Related to the Lower Limit

Suppose we need to integrate the function

f: X → *R* 22)

on the interval *[a, b]* from *X*, but we are unsure about the lower limit *a*. Let's suppose that the lower limit "*a*" has a

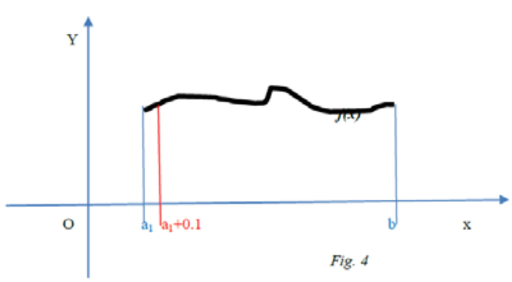

Fig. 4

determinant part "a_1" and an indeterminate part ε, i.e.

$$a = a_1+\varepsilon \tag{23}$$

where

$$\varepsilon\in[0,0.1]. \tag{24}$$

Therefore

$$\int_a^b {}_X f d\nu = \int_{a_1}^b f(x)dx - \mathrm{i}_1 \tag{25}$$

where the indeterminacy i_1 belongs to the interval:

$$i_1\in[0,\int_{a_1}^{a_1+0.1} f(x)dx]\cdot \tag{26}$$

Or, in a different way:

$$\int_a^b {}_X f d\nu = \int_{a_1+0.1}^b f(x)dx + \mathrm{i}_2 \tag{27}$$

where similarly the indeterminacy i_2 belongs to the interval:

$$i_2\in[0,\int_{a_1}^{a_1+0.1} f(x)dx]\cdot \tag{28}$$

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