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Chaouki T. Abdallah

F. Perez-Gonzalez

D. Docampo

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BOUNDING THE FREQUENCY RESPONSE FOR DIGITAL TRANSFER FUNCTIONS: RESULTS AND APPLICATIONS

F. Pérez-González, D. Docampo* and C. Abdallah*†

* BSP Group, Departamento de Tecnologías de las Comunicaciones,
ETSI Telecomunicación, Universidad de Vigo, 36200-VIGO, SPAIN
† ICS Group, Department of Electrical and Computer Engineering
University of New Mexico, Albuquerque, NM 87131, USA.

ABSTRACT

This paper introduces robust stability techniques for the computation of exact bounds for the frequency response of FIR and IIR digital filters in which the l^∞ norm of the coefficients is bounded.

1. INTRODUCTION

In recent years there has been considerable research concerning the stability of uncertain systems, primarily oriented to control applications [1]. The potential of these methods has not yet been fully exploited in the Signal Processing area, although some well-known problems can be "robustified" with the new tools. The aim of the present paper is to introduce these techniques for the computation of exact bounds for the frequency response of digital filters in which the l^∞ norm of the coefficients is bounded. This setup is reasonable in a number of different situations: when the coefficients are regarded as independent and parametric identification methods are used, it is possible to bound the coefficients using confidence intervals. Similar-type bounds are obtained in a linear prediction context when bootstrap methods are used to calculate the extremes [2]. When *set membership* identification methods are used, the resulting "feasible" set is a polytope in the coefficients space [3], which can also be tackled with the methods that will be proposed. Finally, the most notorious problem appears from the quantisation of the filter parameters, when finite wordlength introduces errors in digital filter design.

The study of boundary implications in the analysis of the frequency response of FIR and IIR digital filters has been recently undertaken by Bose and Kim [4]. In their paper they show that the frequency response of an infinite family of linear-phase interval filters can be bounded by the frequency response of a **finite** number of filters. Specifically, they consider an FIR filter with transfer function

$$H(z) = \sum_{k=0}^{n_1} h(k) z^{-k}$$

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Instead of assigning fixed values to the coefficients $h(k)$, it is more realistic to expect that

$$h(k) \in [h^-(k), h^+(k)],\ k = 0, 1, \cdots, n_1 \qquad (1)$$

and the problem consists of finding

$$H^-(e^{j\omega}) = \min |H(e^{j\omega})|,\ H^+(e^{j\omega}) = \max |H(e^{j\omega})|$$

with the minimisation and maximisation taking place over the set of possible filter coefficients in (1).

Bose and Kim's paper imposes a severe linear-phase constraint on the filter response, although it is widely recognised that this property does not hold in many practical filters. In the same paper, this constraint is removed in an attempt to solve the problem for general IIR filters at the expense of overbounding the associated frequency response. It is then of interest to reformulate the problem, so that given the transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{n_1} b(k) z^{-k}}{\sum_{k=0}^{n_2} a(k) z^{-k}}$$

with similar bounds as in (1) for $a(k)$ and $b(k)$, we want to eliminate the conservativeness by calculating the following bounding functions

$$\begin{aligned} H^-(e^{j\omega}) &= \min |H(e^{j\omega})|, \\ H^+(e^{j\omega}) &= \max |H(e^{j\omega})| \qquad (2) \\ \phi_H^-(e^{j\omega}) &= \min arg\{H(e^{j\omega})\}, \\ \phi_H^+(e^{j\omega}) &= \max arg\{H(e^{j\omega})\} \qquad (3) \end{aligned}$$

Since the coefficients sets have infinite members it might appear at first that the solution to the problem above can only be approximated. Nevertheless, in the present paper we will solve it by showing that it can be transformed in such a way that the required maximisation and minimisation are performed over a finite number of discrete points, thus making it possible to compute the exact solution. Let $B^-(e^{j\omega})$, $B^+(e^{j\omega})$, $\phi_B^-(e^{j\omega})$, $\phi_B^+(e^{j\omega})$, $A^-(e^{j\omega})$, $A^+(e^{j\omega})$, $\phi_A^-(e^{j\omega})$, $\phi_A^+(e^{j\omega})$ denote the minimum and maximum of the magnitude and phase of respectively $B(e^{j\omega})$ and $A(e^{j\omega})$, defined as in (2),(3), we can immediately write

$$\begin{aligned} H^-(e^{j\omega}) &= B^-(e^{j\omega})/A^+(e^{j\omega}) \\ H^+(e^{j\omega}) &= B^+(e^{j\omega})/A^-(e^{j\omega}) \end{aligned}$$

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$$\phi_H^-(e^{j\omega}) = \phi_B^-(e^{j\omega}) - \phi_A^+(e^{j\omega})$$
$$\phi_H^+(e^{j\omega}) = \phi_B^+(e^{j\omega}) - \phi_A^-(e^{j\omega})$$

The remaining of this paper is organised as follows. Section 2 contains a brief discussion of the value set concept and its characterisation in the present setting. Section 3 discusses the ideas of finding the minimum and maximum of the phase, while section 4 deals with bounding the magnitude. A numerical example is provided in section 5, and different applications of the proposed methodology to signal processing problems are discussed in section 6. Finally, our conclusions and lines of future research are given section 7.

2. THE VALUE SET

From the discussion above, it becomes apparent that it is necessary to calculate the extremal values of the magnitude and phase of both the numerator and denominator functions. Let us concentrate then on obtaining the extremal values for the function $B(e^{j\omega})$ over the set of valid coefficients. It is customary to define $b^0(k) = [b^+(k) + b^-(k)]/2$, $\Delta_k = [b^+(k) - b^-(k)]/2$ and $q(k) = [b(k) - b^0(k)]/\Delta_k$, $k = 0, \cdots, n_1$. Then, the problem can be posed as finding the extremal values of magnitude and phase of

$$B(e^{j\omega}) = B^0(e^{j\omega}) + \sum_{k=0}^{n_1} q(k)\Delta_k e^{-jk\omega} \tag{4}$$

constrained to

$$\max_{k=0,\cdots,n_1} |q(k)| \leq 1 \tag{5}$$

where $B^0(z) = \sum_{i=0}^{n_1} b^0(k)z^{-k}$ is a *center* polynomial.

The fundamental idea of the value set allows us to solve the problem by reducing it to an optimisation in the complex plane. In order to see this, note that (4) is, for a fixed ω, simply a linear mapping from $\mathbf{R}^{n_1+1}$ to $\mathbf{C}$, so that the box in (5) (that contains all the feasible points) is transformed into a convex polygon in $\mathbf{C}$. It is important to characterise this polygon, due to the role it plays in the solution. The steps that lead to this geometrical characterisation are somewhat lengthy and may be found elsewhere [1].

First, note that the polygon has opposite sides which are parallel (parpolygon) and have directions $e^{-jk\omega}$, $k = 0, \cdots, n_1$ in the complex plane. The number of sides and vertices is at most $2(n_1+1)$ (since many of them are mapped inside the polygon). Now define $g_i(e^{j\omega}) \in \mathbf{C}$, $i = 0, \cdots, 2n_1+1$ such that $g_i(e^{j\omega}) = \pm e^{-jk\omega}$ for some $i, k \in \{0, \cdots, n_1+1\}$ and that

$$\begin{aligned} 0 &\leq Arg\{g_0(e^{j\omega})\} \leq Arg\{g_1(e^{j\omega})\} \leq \cdots \\ &\leq Arg\{g_{n_1}(e^{j\omega})\} \leq \cdots \leq Arg\{g_{2n_1+1}(e^{j\omega})\} < 2\pi \end{aligned}$$

This corresponds to reordering the terms $e^{-jk\omega}$ according to their argument principal value in $[0, 2\pi)$, for a fixed $\omega \in [0, 2\pi)$. Then, the $2(n_1+1)$ vertices of the polygon can be generated as $k = 1, \cdots, 2n_1+1$ by

$$\begin{aligned} v_0(e^{j\omega}) &= B^0(e^{j\omega}) - \sum_{k=0}^{n_1} g_k(e^{j\omega})\Delta_k \\ v_k(e^{j\omega}) &= v_{k-1}(e^{j\omega}) + 2g_{k-1}(e^{j\omega})\Delta_{k-1} \end{aligned} \tag{6}$$

Considering the above, the $2(n_1+1)$ edges of the frequency dependent polygon can be written as

$$e_k(\lambda_k, e^{j\omega}) = v_k(e^{j\omega}) + 2\lambda_k g_k(e^{j\omega})\Delta_k, \tag{7}$$

with $\lambda_k \in [0,1]$ and $k = 0, \cdots, 2n_1+1$. For a numerical example illustrating these concepts see Figure 1, in section 5.

3. BOUNDING THE PHASE

Once the polygon is fully described, phase bounding becomes quite simple. We have to distinguish two different cases:

1) <u>The polygon includes the origin</u>: There is at least one filter in $B(z)$ with a zero on the unit circle and, therefore, the phase is not well-defined for this frequency. Moreover, the total span of phases will be at least 2π.

2)<u>The polygon does not include the origin</u>: From the convexity of the polygon, it is easy to conclude that the maximum and minimum of the phase are always attained at two vertices of the polygon. We will restrict our study to this second case. Here,

$$\phi_B^-(e^{j\omega}) = \min_{k=0,\cdots,2n_1+1} arg\{v_k(e^{j\omega})\} \tag{8}$$

and a similar expression holds for $\phi_B^+(e^{j\omega})$, using the maximum. Instead of carrying the minimisation (maximisation) along the $2n_1+2$ vertices, it is possible to use the following result: the vertices at which the minimum and maximum are achieved are those for which the function $\text{Im}\{v_k(e^{j\omega})g_k^*(e^{j\omega})\}$ changes its sign as k take successive values modulo $2n_1+2$ starting from $k=0$.

Actually, it can be shown that monitoring this function at the $2n_1+2$ finite values is suitable for checking for the situation in which the polygon includes the origin (case 1) since then there is no change of sign at all.

4. BOUNDING THE MAGNITUDE

We have shown by means of the value set that $B^-(e^{j\omega})$ and $B^+(e^{j\omega})$ are respectively the minimum and maximum distances from the origin to any point of the frequency dependent polygon. Due to the convexity of the polygon, it is immediate to see that the maximum distance is always attained at a vertex, so that

$$|B^+(e^{j\omega})|^2 = \max_{k=0,\cdots,2n_1+1} |v_k|^2 \tag{9}$$

Also, the minimum distance is always attained at an edge of the polygon so that

$$|B^-(e^{j\omega})|^2 = \min_{k=0,\cdots,2n_1+1} \{\min_{\lambda_k \in [0,1]} |e_k(\lambda_k, e^{j\omega})|^2\} \tag{10}$$

The case of the minimum distance is, at first sight, the most tricky, since it implies $2n_1+2$ minimisations along one-dimensional edges. However, we will show how this number can be greatly reduced. First, we have to rule out the case for which the minimum is not at an edge or vertex. Clearly, this happens only at those frequencies for which the polygon includes the origin and, consequently, the minimum distance is 0. We have already developed a procedure for phase-bounding that is suitable for this purpose, so hereafter we will concentrate on the simplification of the minimisation along the edges. From (7) we can recognise the form of every edge as

$$e_k = v_k + 2\lambda_k g_k \Delta_k, \quad \lambda_k \in [0,1] \tag{11}$$

where we have dropped the dependency on $e^{j\omega}$ for the sake of conciseness. Differentiating $|e_k|^2$ with respect to λ_k, and setting the result to zero, we obtain the expression for the λ_k that produces the minimum as

$$\hat{\lambda}_k = -\frac{\mathrm{Re}\{v_k g_k^*\}}{2\Delta_k} \tag{12}$$

However, this value is only valid if $\hat{\lambda}_k \in [0,1]$, otherwise, the minimum for the particular edge is attained at a vertex. If $\hat{\lambda}_k < 0$ then the minimum corresponds to $\lambda_k = 0$ and if $\hat{\lambda}_k > 1$ then the maximum corresponds to $\lambda_k = 1$. This result shows that the edge-minimisation can be explicitly solved so that the overall minimisation depends on at most a finite set of $2n_1+2$ points. When $\hat{\lambda}_k \in [0,1]$, we substitute (12) into the expression for $|e_k|^2$ to obtain the expression for the squared distance as

$$|\mathrm{Im}\{v_k g_k^*\}|^2 \tag{13}$$

In order to reduce further the number of computations, we can make use of the concepts of supporting and separating lines. A straight line is a *supporting line* if it intersects the polygon and one of the closed half planes that it generates fully contains the polygon. A straight line is a *separating line* if one of the closed half planes that it generates fully contains the polygon and the complementary open half plane contains the origin. It should be clear that when the polygon does not include the origin, all its edges belong to a supporting line. Moreover, any of these lines is either a separating line or not. Let us call the edges contained in a separating supporting line SSE and those contained in a non-separating supporting line NSSE.

The relevant result for our work is that the minimum is always attained at an SSE. Moreover, it can be shown that there is at most one SSE for which $\hat{\lambda}_k$ in (12) is in $[0,1]$. Therefore, the search can be reduced to the SSEs. Indeed, the vertices that bound the phase (see previous section) partition the set of edges into SSEs and NSSEs. Now, to look for the edge with the minimum of $|B(e^{j\omega})|$, simply check for the change of sign in $\mathrm{Re}\{v_k(e^{j\omega})g_k^*(e^{j\omega})\}$ for two consecutive vertices. Then, use (12) to check if the minimum is at the segment. If so, the squared distance is obtained using (13). If $\hat{\lambda}_k < 0$ then the minimum is achieved at v_k, so $|B^-(e^{j\omega})|^2 = |v_k(e^{j\omega})|^2$. If $\hat{\lambda}_k > 1$ then the minimum is achieved at v_{k+1}, so $|B^-(e^{j\omega})|^2 = |v_{k+1}(e^{j\omega})|^2$. If there is no change of sign in $\mathrm{Re}\{v_k(e^{j\omega})g_k^*(e^{j\omega})\}$ for any of the vertices defining an SSE, then the minimum will be at the extreme SSEs, i.e., those edges closest to the vertices bounding the phase.

$|B^+(e^{j\omega})|^2$ can be computed by maximisation along the NSSEs. In this case, the simplest thing to do is to directly calculate $\max|v_k|^2$ along the vertices defining the NSSEs.

5. NUMERICAL EXAMPLE

Consider the following polynomial

$$B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}$$

where the following bounds for the coefficients are considered:

$$\begin{aligned} b_0 &\in [0.99, 1.01],\ b_1 \in [-1.0414, -1.0214], \\ b_2 &\in [0.2169, 0.2369], b_3 \in [0.3934, 0.4134], \\ b_4 &\in [-0.2020, -0.1820] \end{aligned}$$

This implies that $B^0(z)$ has the form

$$1 - 1.0314z^{-1} + 0.2269z^{-2} + 0.4034z^{-3} - 0.1920z^{-4}$$

and $\Delta_k = 0.01$ for all $k = 0, \cdots, 4$. We have used the proposed algorithm for computing the frequency response of every $B(e^{j\omega})$, $\omega = 2\pi l/64$, $l = 0, \cdots, 63$ and used linear interpolation for plotting the magnitude and phase. In figure 1 we plot the value set in the complex plane for $\omega = 2\pi 5/64$ together with an approximation obtained by means of gridding of Δ_k to values $\{-0.01, 0, 0.01\}$ for $k = 0, \cdots, 4$. It is

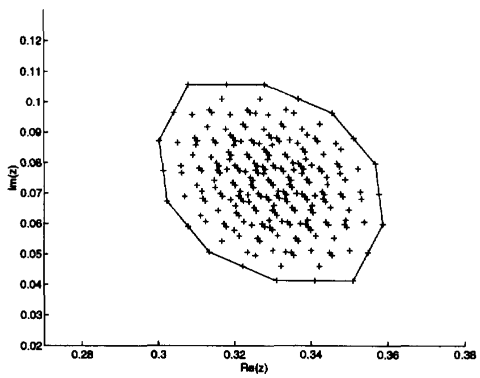

Figure 1: Value set for $\omega = 2\pi 5/64$. +: Points obtained with gridding

worth noting that whenever there is a term with no powers of z^{-1} (an independent term), the polygon will have a horizontal edge for every frequency, and for frequencies $\omega = 0$ and $\omega = \pi$ the polygon degenerates into a horizontal segment. Figures 2 and 3 represent the exact bounds obtained for the magnitude and the phase, respectively.

6. APPLICATIONS

The proposed solution allows to easily compute the frequency response for a set of transfer functions in an exact way, which can be very useful in a variety of situations where no exact information of the coefficients of the system is available. One immediate use of our result is the computation of the *robust periodogram* in which it is possible to take into account the existence of just an estimate of the autocorrelation sequence. Note that windowing of the data would basically change the size of Δ_k (see section 2). Of

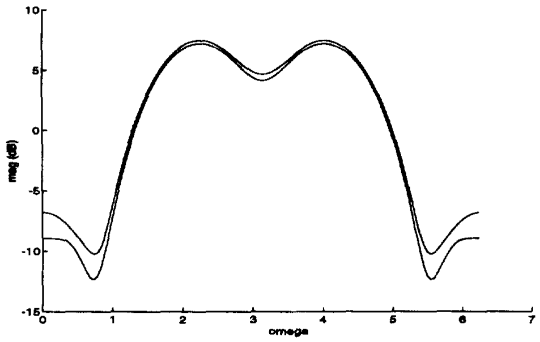

Figure 2: Exact bounds for the magnitude

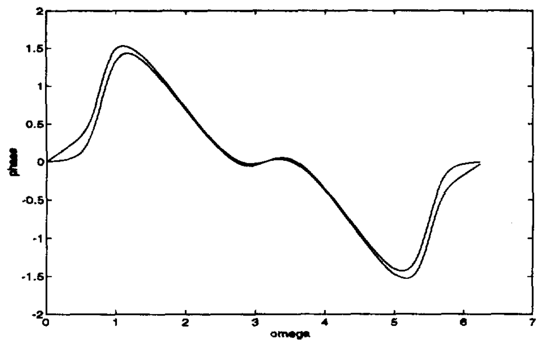

Figure 3: Exact bounds for the phase

course, it is possible to add some information about correlation between different samples, which just changes the mapping from the coefficient space to the complex plane, but keeps the discussion and algorithm valid. Moreover, the computational cost only increases linearly with the number of parameters.

Another important application appears in filter design, where this tool can be used to study the effect of coefficient quantisation measured in terms of the error produced in the frequency response. This allows us to calculate a worst case value for every frequency, which is useful whenever the coefficients are not known (e.g., adaptive) and finite precision arithmetic is used.

Finally, we will mention the application of robust stability, i.e., the stability of polynomials with uncertain coefficients. In this case, if $B^0(z^{-1})$ denotes the center denominator polynomial, it is enough to guarantee that this polynomial is minimum-phase and that the magnitude response does not take the value zero (we have provided simple ways of doing this). A similar approach was taken in [5] to analyse if a family of channel equalisers for digital communications met some dynamic specifications, including their existence.

7. CONCLUSIONS AND FUTURE WORK

We have shown how the frequency response of discrete-time transfer functions with interval coefficients can be exactly calculated with a simple and efficient algorithm that exploits the concept of value set and uses elementary geometry. Since the value set is frequency dependent, it is necessary to recompute it for each frequency, with no FFT-like algorithm presently available. This topic will be investigated in the future.

Since the computations depend on a set of vertices and edges of the resulting polygon, and since it is possible to obtain an explicit minimisation for these edges, it turns out that the magnitude and phase bounding functions for all the frequencies in $[0, 2\pi)$ can be computed by optimisation over a finite set of functions. Even though an l^{∞} norm approach has been taken in this paper, it is possible to extend it without much difficulty for l^1 and l^2 (weighted) norms.

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