12-14-1994

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Recommended Citation
PHASE-CONVEX ARCS IN ROOT SPACE AND THEIR APPLICATION TO ROBUST SPR PROBLEM

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ABSTRACT
This paper considers the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. Applications of the results are also described.

1 Introduction
This short paper considers the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. More specifically, given a family of polynomials $P(z^{-1})$

$$P(z^{-1}) = \{p(z^{-1}) : p(z^{-1}) = 0 \Rightarrow z \in \Omega \}$$

where $\Omega$ is a set about the corresponding roots of a nominal polynomial $p_0(z^{-1})$. We are interested in finding the sets $\Omega$, such that the phase of every member in $P(z^{-1})$ is determined by that of a few extreme members. Our development parallels that of [1] and the motivation given there and elsewhere [2], [3], is that an SPR condition is frequently invoked to prove convergence of adaptive algorithms. We differ from [1] and [4] however, in that we do not assume any knowledge of the spectral density of the regressor. Section 2 presents a simple version of the problem, while Section 3 generalizes the results to roots in arbitrary domains. Some applications are given in Section 4, and our conclusions are given in Section 5.

2 Phase-Convex arcs
Let $\lambda \rightarrow F(\lambda)$, $\lambda \in [\lambda_0, \lambda_1]$ be a continuously differentiable parameterization of an arc in the complex plane. We write $F(\lambda) = X(\lambda) + jY(\lambda)$, where $X$ and $Y$ are two real functions of the real variable $\lambda$. We first consider polynomials with the following structure

$$p(z^{-1}, \lambda) = [1 - (X + jY)z^{-1}][1 - (X - jY)z^{-1}]$$

Throughout the text we will assume that $|F(\lambda)| < 1/r_0$, $\lambda \in [\lambda_0, \lambda_1]$ which will be necessary for stability of the polynomials considered. Let $\phi(\omega, \lambda) = \arg \{p_r e^{j\omega - \lambda} \}$ we seek conditions that guarantee that $\phi(\omega, \lambda)$ is bounded by $\phi(\omega, \lambda_0)$ and $\phi(\omega, \lambda_1)$, for all $\omega \in [0, 2\pi]$. We will call an arc satisfying this property a phase-convex arc or simply say that an extreme-point property holds for the phase. Next we drop the dependence on $\lambda$ wherever there is no possible confusion, and state our first result.

Theorem 1 An extreme-point result holds for $\phi(\omega, \lambda)$ if

for any $\lambda \in (\lambda_0, \lambda_1)$ the functions

$$g_1(\lambda) = (1 - \alpha \lambda) \frac{\partial X}{\partial \lambda} - 2\alpha \frac{\partial Y}{\partial \lambda}$$

$$g_{-1}(\lambda) = (1 - \beta \lambda) \frac{\partial X}{\partial \lambda} + 2\beta \frac{\partial Y}{\partial \lambda}$$

both have the same sign, where

$$\alpha = \frac{Y_0}{(1 - X_0)}, \quad \beta = \frac{Y_0}{(1 + X_0)}$$

Proof: See [5].

For a geometric interpretation, provided that $\partial X/\partial \lambda \neq 0$, we make $\gamma = \partial Y/\partial X$ so that the sign condition can be restated in terms of $h_1$ and $h_{-1}$

$$h_1(\lambda) = |\alpha + \gamma| - \sqrt{\beta^2 + 1}; \quad h_{-1}(\lambda) = |\beta - \gamma| - \sqrt{\alpha^2 + 1}$$

thus giving a feasible region of points $X + jY$ for which the condition holds for a fixed $\gamma$, which can be thought as the direction of the tangent to the arc. In [5] we show that the feasible region is a rectangle. Next, we discuss the particular cases of straight line segments and circular arcs.

2.1 Straight Line Segments
In this case, we can write $X(\lambda) = a_0 + a\lambda$, $Y(\lambda) = y_0 + b\lambda$ where $\lambda \in [0, 1]$, and $a$ and $b$ are real numbers. Now, whenever $a \neq 0$ we can write $\gamma = b/a$ and therefore a extreme point result will hold when $h_1$ and $h_{-1}$ have the same sign for $(a_0, y_0)$ and $(a_0 + a, y_0 + b)$. For the case $a = 0$ (vertical lines) it is not possible to use the sign condition. However, it is shown in [5] that with an adequate restructuring of the problem, an extreme-point result holds also for vertical segments. The case of real roots can be treated as a special case of horizontal segments for which we have proven that an extreme point result holds as long as $F(\lambda) \subset (-1/r_0, 1/r_0)$.

2.2 Circular Arcs
Circular arcs are described by $F(\psi) = (a_0 + jy_0) + r e^{j\psi}$, $\psi \in [\psi_0, \psi_1]$. Calculating $g_1$ and $g_{-1}$ in (2-3) and after some straightforward algebraic manipulations, it can be shown that the set of values of $\psi \in [0, 2\pi]$ satisfying the sign condition can be calculated from the roots of two equations of the form $A \sin \psi + B \cos \psi + C = 0$, with $A, B, C$ real numbers [5]. These solutions divide the interval $[0, 2\pi]$ into
subintervals where the sign condition holds or fails and, obviously, $\psi_0$ and $\psi_1$ must belong to one subinterval of the first type in order to have an extreme-point result. As a particularly simple case, we mention $\psi_0 = 0$, that gives an extreme point result as long as $x_0 \in (-1/r_0, 1/r_0)$.

3 Generalization to Domains

Let $\Omega$ be an open region, simply connected and symmetric with respect to the real axis. Let $P(z^{-1})$ be the family of second order polynomials with roots in $\Omega$ and assume that the boundary of $\Omega$, $\partial \Omega \subset C$, is given by

$$\partial \Omega = \bigcup_{i=1}^{N} F_i(\lambda_i), \lambda_i \in [\lambda^0_i, \lambda^1_i] \subset \mathbb{R} \quad (4)$$

where $F_i(\lambda_i)$ is a continuously differentiable phase-convex arc. Let $\phi(z) = \arg(p(z^{-1}))$ and define

$$\phi(z) = \sup_{p \in P} \phi_p(z), \quad \phi(z) = \inf_{p \in P} \phi_p(z) \quad (5)$$

These functions are respectively termed lead and lag functions of the family $P(z^{-1})$ [4]. Also define the extreme polynomials as $p_i(z^{-1}) = \left[1 - F_i(\lambda_i)^{-1} \right] \left[1 - F_i^{-1}(\lambda_i z^{-1}) \right], i = 1, \ldots, N$ and define the corresponding extreme phase functions $\phi_i(z) = \arg(p_i(z^{-1})), i = 1, \ldots, N$. Then, the following result can be stated

**Theorem 2** The lead and lag functions of $P(z^{-1})$ for a given $z = r_0 e^{j\omega}, \omega \in [0, 2\pi)$, are given by

$$\phi_{i}(r_0 e^{j\omega}) = \max_{i=1,\ldots,N} \phi_i(r_0 e^{j\omega}); \quad \phi_{i}(r_0 e^{j\omega}) = \min_{i=1,\ldots,N} \phi_i(r_0 e^{j\omega}) \quad (6)$$

**Proof:** See [5].

Obviously, it is possible that only a subset of the $N$ extremes is necessary to bound the phase, and a very important special case is that in which the lead and lag functions correspond to the same extreme-phase function for every $\omega \in [0, 2\pi)$, so that only two extremes are necessary. Note that the arcs considered here (except for vertical line segments) satisfy the conditions for this result.

Theorem 2 can be also used to produce conservative (but simple) bounds for families of the form

$$Q(z^{-1}) = \{q(z^{-1}); q(z^{-1}) = 0 \Rightarrow x_0 \in \Gamma \} \quad (7)$$

for which $\Gamma \subset \Omega$. The results given so far can be extended to families of polynomials of any degree $n$. Hereafter $n$ is an even number and the roots appear only in complex conjugate pairs, although the results can be easily extended to the remaining cases. In short, in [5] it is shown that a extreme-point result also holds for families of the form

$$P(z^{-1}) = \{p(z^{-1}) = \prod_{i=1}^{m} p_i(z^{-1}), \ p_i(z^{-1}) \in P(z^{-1})\} \quad (8)$$

where for $l = 1, \ldots, m$

$$p_l(z^{-1}) = \left\{p_l(z^{-1}) = \prod_{i=1}^{m} \left(1 - z^{-1} \right)(1 - z^{-1} z_i), z_i \in \Omega \right\} \quad (9)$$

and where the $F_i(\lambda_i)$ are continuously differentiable phase-convex arcs.

4 Application to SPR problems

The first important application has to do with SPR checking of families of polynomials. Suppose that the family $P(z^{-1})$ described in (7) is stable, then, the condition for SPRness on the complement of the circle of radius $1/r_0$ is

$$\pi/2 < \phi(r_0 e^{j\omega}) < \pi/2 \quad (8)$$

so if the root domains are bounded by closed curves that are piecewise phase-convex, the family will be SPR if and only if some adequately selected extreme polynomials are SPR.

The second application deals with filter design to enforce SPRness of a family of polynomials. In this case, we want to find a stable polynomial $f(z^{-1})$ such that $f(z^{-1})$ is SPR for every $p \in P$. Again, if the root domains for the family of polynomials satisfy the requirements for phase-convexity, it can be shown that the necessary and sufficient condition for the existence of a stable $f(z)$ can be stated in terms of a finite set of conditions involving the extreme polynomials [5].

Finally, if the problem is to find $f(z^{-1})$ to make $P(z^{-1})$ SPR, we can split the uncertain family into a product of even-order families, each having its roots inside a domain for which the phase can be bounded by two extremes. The optimal solution for the synthesis of $f(z)$ is then obtained from [5].

5 Conclusions

In this paper we have considered the problem of identifying regions in the complex-plane, such that the phase of polynomials having roots in those regions, is bounded by that of a few extreme polynomials. Our results are simple to check and generalize previously available ones. Applications of the results were also described.

References


