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CR geometry and twisting type N vacuum solutions

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B.S., Physics, Shandong University, 2007

DISSERTATION

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Dedication

To my parents and old friends

One China

Real China

Free China

Republic of China

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Abstract

In the search for vacuum solutions, with or without the cosmological constant Λ, of the Einstein field equations for Petrov type N with twisting principal null directions, the CR structures, which describe the parameter space for the geodesic congruence tangent to such null vectors, provide a useful invariant approach. Work of Hill, Lewandowski and Nurowski has laid a solid foundation for this, reducing the field equations to a set of differential equations for two functions, one real, one complex, of three variables. Under the assumption of the existence of one special Killing vector, the infinite-dimensional classical symmetries of those equations are determined and group-invariant solutions are considered. This results in a single ODE of the third order which may easily be reduced to one of the second order. A one-parameter class of power series solutions, $g(w)$, of this second-order equation is realized, holomorphic in a neighborhood of the origin and behaving asymptotically as a simple quadratic function plus lower-order terms for large values of w, which constitutes new solutions of the twisting type N problem. The solution found by Leroy, and also later by Nurowski, is shown to be a special case in this class. Cartan's method for determining

local equivalence of CR manifolds is used to show that this class is indeed much more general. Also for the general metrics determined by this second-order ODE, two Killing vectors, including the one already assumed, can be found, both of which are inherited from symmetries of the underlying CR structures.

In addition, for a special choice of a parameter, this ODE may be integrated once, to provide a first-order Abel equation. It can also determine new solutions to the field equations although no general solution has yet been found for it.

Contents

Glossary

CR	Cauchy-Riemann or complex-real
PND	principal null direction
Λ	the cosmological constant
$\partial_x f$	$\frac{\partial f}{\partial x}$
$\mathcal{L}_X$	the Lie derivative along the vector field X
$k \lrcorner \mu$	the contraction of a vector field k with a 1-form μ
$\mathrm{Ric}(\mathbf{g})$	the Ricci tensor of a Lorentzian metric $\mathbf{g}$
$R_{ij} = \Lambda g_{ij}$	the Einstein equations, Einstein spaces
C_{ijkl}	the Weyl tensor
$\Psi_{0,1,2,3,4}$	the Weyl scalars
ζ, η	the first CR function, the second CR function

Chapter 1

Introduction

1.1 Overview

The 1993 Nobel Prize was awarded to Russel A. Hulse and Joseph H. Taylor for their 1974 discovery of the binary pulsar system PSR B1913+16, "a discovery that has opened up new possibilities for the studies of gravitation" by providing the first, and so far, still the only experimental evidence for gravitation radiations. Nowadays, high accuracy laser interferometer experiments, such as LIGO and Virgo, have been racing to a direct detection of the weak signals from gravitational waves, which is likely to be achieved in a foreseeable near future.

On the theoretical side, due to the essential nonlinearity of general relativity, the exact solutions have been playing an important role ever since the theory was born. Discoveries and analyses of various specific solutions, which effectively make the nonlinearity more tractable, have revealed most unforeseen features of the theory (black holes, gravitational waves, cosmology, etc.). Especially, studies of special exact solutions can provide useful guidance to approximative and numerical approaches, promote further questions concerning more general situations, and verify or modify conjectures (e.g., the cosmic censorship conjecture, the cosmic no-hair conjecture). Moreover, previously known solutions may turn out to be asymptotic states of general classes of models, hence becoming more interesting in physics (e.g., Robinson-Trautman and Schwarzschild metrics). In fact, as J. Bičák once commented [1], "if analyzed globally, almost any solutions can tell us something about the basic issues in general relativity, like the nature of singularities, or cosmic censorship," though such analyses are often very complicated to carry out. Although a large number of exact solutions have been discovered, relatively few exact solutions are known for real physical situations. Among those without exact solutions are, for example, the two-body problem and gravitational radiations from other realistic bounded sources.

Despite that non-twisting radiative solutions (e.g., *pp*-waves, Kundt solutions, Robinson-Trautman solutions, all of Petrov type N) have been extensively studied and are well-known, real physical situations, however, mostly generate gravitational waves with principal null rays that have a nonzero twist (with decay rate $1/r^2$, compared to field strength decay $1/r$), e.g., those emitted from binary black hole mergers [2] that are considered as key sources for gravitational wave detection. Hence there has been great interest in the twisting problem that hopefully may lead to more realistic radiative spacetimes. Nevertheless, finding a twisting vacuum solution, especially of Petrov type N (with its prominence signified by the peeling theorem, cf. Section 1.2), with or without the cosmological constant Λ, is one of the most difficult in the theory of algebraically special solutions and has remained largely unsolved for decades[1]. Besides that the field equations for twisting type N are strongly overdetermined, the difficulty also lies in that a nonzero twist itself is associated with a certain non-integrability (in the sense of Frobenius), which causes, for instance, a lack of 2-dimensional wavefronts (surfaces orthogonal to null congruences) in the

[1]In some sense, this prolonged stagnant situation without finding new solutions has elevated doubts about the physical relevance of twisting type N vacuums [3, 4], the issue of which nowadays is still somehow clouded by various inconclusive results from approximate approaches.

spacetimes that are always seen in non-twisting solutions. So far, before our new results published in early 2012 [5] which this dissertation shall discuss with more details, there had been, despite great efforts, only two known twisting type N vacuum solutions, i.e., that of Hauser (1974) [6, 7] with $\Lambda = 0$, and that of Leroy (1970) [8] with $\Lambda < 0$, which in the limit of $\Lambda \to 0$ degenerates to a flat solution rather than one of type N. Many different approaches have been used to attempt the finding of more solutions. With a requirement of one or more Killing vectors—not more than two can be allowed when $\Lambda = 0$— the problem can be reduced to the solution of a single, nonlinear ODE, which has been produced in several forms of various complexities by different authors [9, 10, 11, 12]; nonetheless, this approach has produced no new solutions. Looking at the problem as a reduction from complex-valued manifolds via Plebański's hyperheavenly equation [13] has produced no new solutions [14]. Therefore we were quite interested when we became aware of a different approach in a recent paper by Paweł Nurowski [15], looking for exact solutions of this type with nonzero cosmological constant.

Many of Nurowski's research articles use the fact that one can productively study (4-dimensional) Lorentz geometries which admit a shearfree geodesic null congruence of curves by viewing the 3-parameter space that picks out any particular curve in the congruence, as a (3-dimensional) CR structure [16]. In [17], he and his collaborators use the first CR function in such a structure to create a very appropriate choice of coordinates for a twisting type N Einstein space[2], and reduce the Einstein equations to a set of nonlinear PDEs for a couple of functions of three variables. Then in [15], he makes a clever ansatz depending only on a single variable and discovers a particular twisting solution; unfortunately that solution turns out to be the same as the one mentioned above and first found by Leroy, as he notes in a more recent paper

[2]We use the relatively common nomenclature "Einstein spaces" for solutions to the Einstein equations with a source of either the pure vacuum or that vacuum with a nonzero cosmological constant Λ.

[18]. However, we were quite intrigued by the approach and have made efforts to follow it through with the hopes of obtaining more general solutions of the equations in Nurowski's article.

A spacetime of type N allows one, and only one, congruence of twisting shearfree null geodesics, referred to as the principal null direction (PND); this (3-parameter) family of null geodesics allows the option to choose a single coordinate r along any such geodesic, and to associate the other three degrees of freedom in the parameter space as a model for a CR manifold. We first insist that our manifold admit a Killing vector in the real direction in this (3-dimensional) CR manifold, so that the remaining unknown functions depend only on the complex coordinates there, and then calculate the (infinite-dimensional) classical symmetries for the system. This allows us to derive a quite simple nonlinear third-order ODE which the invariant solutions of the classical symmetries must satisfy. Because this equation does not contain the independent variable explicitly, it can be immediately reduced to the following second-order ODE, for $g = g(w)$, with two slightly different forms that differ by a constant:

$$g'' = -\frac{(g' + 2w)^2}{2g} - \frac{2C}{g} - \frac{10}{3}, \qquad C = 0 \text{ or } 1. \tag{1.1}$$

We are then able to show that the Leroy-Nurowski solution is indeed a special solution for this equation. At this point it is worthwhile to enter into the question as to how one knows that the new solution of Nurowski does indeed describe locally[3] the same manifold as the solution found by Leroy. The method was originally created by E. Cartan [19, 20, 21] to prove equivalencies of CR structures, without the need of actually determining an explicit transformation between the two sets of representatives and coordinates on two CR manifolds. Instead, one determines the values of a set of invariant quantities for a CR structure, the same for all equivalent such structures.

[3]All our considerations are local, both in the Lorentz-signature spacetime and in the associated complex spaces we need to use.

Therefore it is necessary to calculate the invariants for Leroy's solution and compare them with the ones already known to Nurowski for his solution, noting that they are the same constants (see Section 8.1 for the explicit coordinate transformation between these two solutions). We have therefore also calculated these invariants for our class of solutions, which we find to be quite different in general.

For the case when the constant C in (1.1) is zero, an integral transformation may be performed to reduce that equation further, to a first-order ODE of Abel type for $f = f(t)$:

$$f' = \frac{4}{t}\left(t + \frac{3}{2}\right)\left(t + \frac{1}{3}\right) f^3 + \frac{5}{t}\left(t + \frac{2}{5}\right) f^2 + \frac{1}{2t} f, \tag{1.2}$$

which, quite unfortunately, we have not been able to identify as any of the known solvable types of Abel equations [22, 23]. Nonetheless, we believe that these two equations so far are the simplest ODEs available that determine nontrivial twisting type N Einstein spaces. Returning to the case when $C = 1$, our current examinations suggest the possibility that the solutions of this equation might define a new class of transcendental functions, which constitutes a major result of this work, establishing a new set of solutions to the type N problem with $\Lambda \neq 0$. The remainder of this work will describe the process involved in this, and our reasons for stating that these are indeed new solutions. In particular, we will present solutions to (1.1) in the forms of power series and Puiseux series, both shown to be locally convergent. Although prior to this time there were indeed only two known twisting type N Einstein spaces, it is always worth remembering that the solution space for the problem is in fact quite large. It has been shown by Sommers [24] that the full set of solutions for type N is given by two complex functions of two real variables. Surely the requirement of nonzero twist puts a very strong constraint on this, but it is expected that there should be a large number of new analytic functions involved in the full solution of the twisting type N problem.

The dissertation is organized as follows. For the rest of this chapter, we introduce basic concepts and theorems associated with algebraically special spacetimes that admit congruences of shearfree null geodesics (twisting or not). In Chapter 2, we give a brief review of the fundamentals of CR geometry and its lifting to spacetimes, as well as the derivation of the field equations for type N in this setting. Then we make a detailed comparison with another version of the field equations to reveal the invariant properties and advantages of using CR geometry. In addition, the Hauser solution is given as the first example of solutions. Chapter 3 describes, in a nutshell, classical symmetries of PDEs as an important technique for finding exact solutions. Then in Chapter 4, assuming the existence of one special Killing vector, we apply this technique to calculate the classical symmetries of a set of simplified field equations. From the invariant solutions of these symmetries, the aforementioned second-order and the first-order ODEs are derived. Moveover, we point out an important connection between CR equivalency and the classical symmetries we have found. For Chapter 5, we make our first attempt on solving those ODEs. Particularly for (1.1), all conformally flat solutions are found and the Leroy-Nurowski solution is also recovered with its form extended to include a free function. With these solutions in mind, in Chapter 6, we invoke the weak Painlevé test for those ODEs as a way to estimate the chance of success in finding new solutions to them. The test suggests the existence of Puiseux series solutions to (1.1), which are shown to be locally convergent and different from the Leroy-Nurowski solution. In Chapter 7, we construct for (1.1) a one-parameter series solution connecting a conformally flat solution and the Leroy-Nurowski solution, which constitutes a new family of type N solutions. As a supplement, in Chapter 8, Killing symmetries are discussed for the Leroy-Nurowski solution and for general solutions determined by (1.1). All together, the dissertation is largely based on our published paper of [5] with more background, details and new results added.

For an extra comment, we would like to point out that three different kinds of symmetries are discussed in this work. They are respectively symmetries of spacetime metrics (Section 1.3), symmetries of CR structures (Section 2.3) and symmetries of PDEs (Chapter 3). It should be clear from the names and context which symmetry we refer to in a particular circumstance.

1.2 Geometry of Spacetimes and Petrov Types

In this section, we will introduce various concepts involved with spacetimes that admit a congruence of shearfree null geodesics (twisting or not). Such spacetimes are closely related to algebraically special spacetimes classified according to the Petrov types, of which the type N is the most algebraically special one.

The approach to a 4-dimensional Lorentzian manifold $\mathcal{M}$ begins with the usual form for a spacetime metric [25] (see p. 31) in terms of a complex null tetrad of 1-forms $(\theta^1, \theta^2, \theta^3, \theta^4)$ and with the Lorentzian signature $(+,+,+,-)$:

$$\mathbf{g} = 2\left(\theta^1\theta^2 + \theta^3\theta^4\right), \qquad g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{1.3}$$

where θ^2 is the complex conjugate of θ^1, while θ^3 and θ^4 are real, and the product is the usual symmetric tensor product of two 1-forms, e.g., $\theta^1\theta^2 = \frac{1}{2}(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1)$. Now we assume that the spacetime admits a congruence of a 3-parameter family of curves that are tangent to a non-vanishing real null vector field k such that

$$\mathbf{g}(k,k) = 0, \qquad \theta^3 = \mathbf{g}(k,\cdot), \qquad k \lrcorner\, \theta^1 = k \lrcorner\, \theta^2 = 0.$$

With such a congruence, i.e., the null direction of k, fixed, one can only determine the metric (1.3) up to the following Lorentz transformations [25] (see p. 33)

$$\begin{aligned}
\theta^{1'} &= \mathrm{e}^{\mathrm{i}\varphi}\left(\theta^1 + \bar{B}\theta^3\right),\\
\theta^{2'} &= \mathrm{e}^{-\mathrm{i}\varphi}\left(\theta^2 + B\theta^3\right),\\
\theta^{3'} &= A\theta^3,\\
\theta^{4'} &= A^{-1}\left(\theta^4 - B\theta^1 - \bar{B}\theta^2 - B\bar{B}\theta^3\right).
\end{aligned} \tag{1.4}$$

with two real functions $A \neq 0$, φ and a complex function B on $\mathcal{M}$. This congruence associated with k is called a congruence of *shearfree* and null *geodesics* if the tetrad satisfies

$$\begin{aligned}
\mathrm{d}\theta^3 \wedge \theta^1 \wedge \theta^3 &= 0,\\
\mathrm{d}\theta^1 \wedge \theta^1 \wedge \theta^3 &= 0,
\end{aligned} \tag{1.5}$$

the vanishing of which is *invariant* under the transformation (1.4), i.e., that (1.5) together with (1.4) implies

$$\begin{aligned}
\mathrm{d}\theta^{3'} \wedge \theta^{1'} \wedge \theta^{3'} &= 0,\\
\mathrm{d}\theta^{1'} \wedge \theta^{1'} \wedge \theta^{3'} &= 0.
\end{aligned}$$

Hence this is a property of the congruence. In addition, one can show that (1.5) is equivalent to the condition [26]

$$\mathcal{L}_k \mathbf{g} = \Theta \mathbf{g} + 2\theta^3 \vartheta, \tag{1.6}$$

with a real function Θ, called the expansion, and a real 1-form ϑ on $\mathcal{M}$. This condition means that the conformal metric induced from $\mathbf{g}$ in the 2-dimensional quotient space $k^{\perp}/k$ is preserved along the congruence.

Assuming that (1.5) holds, we can go further to define a real function Ω by

$$\mathrm{d}\theta^3 \wedge \theta^3 = \mathrm{i}\Omega\, \theta^1 \wedge \theta^2 \wedge \theta^3,$$

which, by virtue of (1.4), implies

$$\mathrm{d}\theta^{3'} \wedge \theta^{3'} = \mathrm{i}A\Omega\, \theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'}.$$

Therefore the vanishing or not of Ω is also an invariant property of the congruence. We say that a shearfree geodesic null congruence is *twisting* (rotating) if $\Omega \neq 0$, and non-twisting if otherwise. For a geometric explanation, a congruence being twisting means that the associated vector field k cannot be proportional to a gradient ∇f for some real function f on $\mathcal{M}$.

To study the Einstein equations, we first introduce the Cartan structure equations for the metric (1.3):

$$\begin{aligned}
&\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = 0, \\
&\mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \frac{1}{2} R^i{}_{jkl} \theta^k \wedge \theta^l,
\end{aligned}$$

where $\Gamma^i{}_j$ is the Levi-Civita connection 1-form, satisfying $\Gamma_{ij} = g_{ik}\Gamma^k{}_j = -\Gamma_{ji}$. From these equations, one can determine the Riemann tensor $R^i{}_{jkl}$, and henceforth the Ricci tensor $R_{ij} = R^k{}_{ikj}$ as well as the Ricci scalar $R = R_{ij}g^{ij}$ with g^{ij} the inverse of g_{ij}. As conditions imposed on the Ricci tensor, we have the vacuum Einstein equations

$$R_{ij} = \Lambda g_{ij} \tag{1.7}$$

with the cosmological constant Λ zero or not. Spacetimes satisfying all ten individual equations of (1.7) are called *Einstein spaces.*

With all these quantities in hands, the Weyl tensor, i.e., the traceless part of the Riemann tensor, is given by

$$C_{ijkl} = R_{ijkl} + \tfrac{1}{6} R\,(g_{ik}g_{lj} - g_{il}g_{kj}) + \tfrac{1}{2}(g_{il}R_{kj} - g_{ik}R_{lj} + g_{jk}R_{li} - g_{jl}R_{ki}), \tag{1.8}$$

which has the property of being invariant under conformal transformations of the metric (1.3). Due to its symmetry properties, the Weyl tensor has only ten independent components all of which can be fully determined by the five complex-valued

Weyl scalars $\Psi_{0,1,2,3,4}$:

$$
\begin{aligned}
\Psi_0 &= C_{4141} = R_{4141}, \\
\Psi_1 &= C_{4341} = \tfrac{1}{2}(R_{4341} + R_{1421}), \\
\Psi_2 &= C_{4132} = R_{1423} - \tfrac{1}{6}(R_{12} + R_{34}), \\
\Psi_3 &= C_{3432} = \tfrac{1}{2}(R_{3432} + R_{2312}), \\
\Psi_4 &= C_{3232} = R_{3232},
\end{aligned}
$$

Here we have used (1.8) to express the Weyl scalars in terms of the Riemann and Ricci tensor components.

At a point of the spacetime $\mathcal{M}$, if the Weyl tensor is nonzero, then there exist at most *four* distinct null directions called the *principal null directions* (PNDs). All possible multiplicities of PNDs constitute the Petrov-Penrose classification of spacetimes [27], which can be enumerated by the five different partitions of the number 4, denoted respectively by [1111], [112], [22], [13] and [4]. For example, if a spacetime admits four distinct PNDs, i.e., the case [1111], then it is called *algebraically general* or of Petrov type I; otherwise it is *algebraically special.* Particularly for our interest, a type N spacetime (denoted by [4] above) has only one repeated PND of multiplicity 4, which means that all four PNDs coincide. To determine the Petrov type of a metric, one only needs to calculate the Weyl scalars in the null tetrad we are using:

$$
\begin{aligned}
\Psi_0 = 0,\ \Psi_1 \neq 0 &\iff \text{type I [1111] } (k \text{ is a PND}), \\
\Psi_0 = \Psi_1 = 0,\ \Psi_2 \neq 0 &\iff \text{type II [112] or D [22]}, \\
\Psi_0 = \Psi_1 = \Psi_2 = 0,\ \Psi_3 \neq 0 &\iff \text{type III [13]}, \\
\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0,\ \Psi_4 \neq 0 &\iff \text{type N [4]}, \\
\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 &\iff \text{type 0 (conformally flat)}.
\end{aligned}
$$

Here for the type 0, the Weyl tensor vanishes, in which case the spacetime is conformally flat and does not single out any null direction. To summarize, the Penrose diagram below signifies the successive growth in multiplicity of PNDs among different Petrov types with arrows pointing towards more special cases.

$$
\begin{tikzcd}
 & & \mathrm{I} \arrow[dl] \arrow[d] \\
 & \mathrm{II} \arrow[r] \arrow[dl] \arrow[d] & \mathrm{D} \arrow[dl] \arrow[d] \\
\mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & 0
\end{tikzcd}
$$

The importance of type N spacetimes is manifested from the *peeling* theorem [28, 29] of gravitational radiations, which, roughly speaking, shows the following asymptotic behaviours of the Weyl tensor C:

$$C = \frac{N}{r} + \frac{III}{r^2} + \frac{II + D}{r^3} + \frac{I}{r^4} + O\left(r^{-5}\right), \tag{1.9}$$

as the affine parameter $r \to \infty$ in a null direction when one observes a gravitational field further and further away from the finite source. Here N, III, II, D and I refer to tensorial quantities of respective Petrov types denoted by the symbols. Therefore the general far field of gravitational radiations is of Petrov type N, which consists of a single gravitational wave propagating in the direction of the unique PND.

Additionally, we can say more about shearfree null congruences as to their relations with algebraical special spacetimes:

Theorem 1.1. (Goldberg-Sachs [30]) Given that a spacetime satisfies the Einstein equations $\mathrm{Ric}(\mathbf{g}) = \Lambda\mathbf{g}$, then the following conditions are equivalent to each other:

(i) The spacetime admits a congruence of shearfree null geodesics tangent to a vector field k.

(ii) The spacetime is algebraically special, i.e., $\Psi_0 = \Psi_1 = 0$, with a multiple PND tangent to k.

1.3 Killing Symmetries

A *symmetry* or *isometry* is a transformation under which the form of the metric is invariant. More specifically for infinitesimal transformations generated by a vector field X, it requires that the Lie derivative of the metric tensor $\mathbf{g}$ in the direction of X must vanish, i.e.,

$$\mathcal{L}_X \mathbf{g} = 0, \tag{1.10}$$

which is known as the Killing equation. Accordingly, a vector field X satisfying the Killing equation is called a Killing vector. For a given spacetime, the existence and number of Killing vectors, among other physical properties and interpretations, do not rely on a particular coordinate representation. In addition, it is well-known that in Lorentzian spacetimes, the maximum number of independent symmetries is ten (including four translations, three rotations and three boosts), in which case the spacetime has constant scalar curvature (Minkowski spacetime, (Anti-)de Sitter spacetime). Most exact solutions, however, admit significantly lower number of symmetries.

To study a new spacetime, one of the first steps for physical interpretation is to identify its symmetries. In fact, as we will demonstrate later, in order to simplify the field equations sufficiently for exact solutions to be obtained, one almost always needs to assume the existence of certain symmetries in the beginning. Specifically, if the metric components are all independent of one special coordinate (for our metric, the coordinate u, see Section 4.1), a Killing vector can be immediately identified as aligning in the direction of this special coordinate. However, other symmetries may be much more difficult to find, due to the non-triviality of solving the Killing equation without missing any useful solution.

Chapter 2

CR Structures and Reduced Einstein Equations

2.1 CR Structures

CR (Cauchy-Riemann or complex-real) structures were first introduced into mathematics by Poincaré and extensively studied by E. Cartan [19, 20]. They later appear as the geometric structures of algebraically special spacetimes that admit a shearfree null congruence. Good sources of background on these two geometric concepts may be found, for instance, in the thesis of Nurowski [31], and also in the very detailed discussion of their use for Einstein spaces in his joint article with Hill and Lewandowski [17]. Generally speaking, CR geometry is an *invariant* way of characterizing algebraically special spacetimes and reformulating associated field equations. In these sections, we will describe how CR structures (especially, strictly pseudoconvex ones) play such a role in relativity.

A CR manifold is a 3-dimensional real manifold M equipped with an equivalence class of pairs of 1-forms λ (real) and μ (complex) such that

$$\lambda \wedge \mu \wedge \bar{\mu} \neq 0.$$

Another pair (λ', μ') is considered equivalent to (λ, μ), and therefore simply another representative of the same class $[(\lambda, \mu)]$, iff there exist functions $f \neq 0$ (real) and $h \neq 0$, g (complex) on M such that

$$\lambda' = f\lambda, \qquad \mu' = h\mu + g\lambda, \qquad \bar{\mu}' = \bar{h}\bar{\mu} + \bar{g}\lambda. \tag{2.1}$$

As an alternative definition, for a non-vanishing complex vector field ∂ satisfying

$$\partial \lrcorner \bar{\mu} = \partial \lrcorner \lambda = 0,$$

the equivalence relation (2.1) allows precisely the following transformation:

$$\partial \to \frac{1}{h}\partial,$$

within the same CR structure.

Given a 3-dimensional CR manifold M, it is important to consider whether or not M can be locally embedded as a hypersurface in $\mathbb{C}^2$. The question is related to the following first-order linear PDE known as the tangential CR equation:

$$\bar{\partial}\zeta = 0, \tag{2.2}$$

or equivalently,

$$\mathrm{d}\zeta \wedge \lambda \wedge \mu = 0, \tag{2.3}$$

for a complex-valued function ζ. The solution ζ is called a *CR function*. If the equation (2.2) locally admits two CR functions ζ and η that are functionally independent, namely,

$$\mathrm{d}\zeta \wedge \mathrm{d}\eta \neq 0, \tag{2.4}$$

then one can construct a local embedding of M into $\mathbb{C}^2$ as

$$M \ni (x_1, x_2, x_3) \to (\zeta(x_i), \eta(x_i)) \in \mathbb{C}^2$$

where x_i are coordinates on M. In fact, the converse of the argument is also true [31]. Hence a CR manifold is locally embeddable iff the equation (2.2) locally admits two functionally independent CR functions. Nevertheless, a generic situation is that (2.2) may have no local solutions other than trivial constants, even if the vector field $\bar{\partial}$ is of differentiability class C^∞ [32, 33].

From now on, we only consider CR structures that are *strictly pseudoconvex* (non-degenerate), which are defined by the condition

$$\lambda \wedge \mathrm{d}\lambda \neq 0, \tag{2.5}$$

or equivalently, that the vector fields ∂, $\bar{\partial}$ and $[\partial, \bar{\partial}]$ are linearly independent at each point of M. Note that this definition is independent of the choice of representatives from (2.1). In Section 2.5, we will see that the condition (2.5) implies the shearfree null congruence being *twisting* in a spacetime.

2.2 Cartan Invariants

Given two CR structures with a great degree of freedom in choosing vastly different representatives, one may ask if there can be a set of procedures to conveniently decide whether or not they are in fact the same CR structure, merely represented differently. The method was originally created by E. Cartan [19, 20, 21] to show equivalency of CR structures, without the need of actually determining an explicit transformation between representatives and coordinates on two manifolds. Instead, one calculates the values of a complete set of invariant quantities, called the *Cartan invariants*, for a CR structure, which must be the same for all equivalent such structures. Here we describe how they are defined.

Assuming that (2.5) holds, one can always make a suitable choice of 1-forms (λ', μ') from the class $[(\lambda, \mu)]$ satisfying

$$\lambda' \wedge \mathrm{d}\lambda' = \mathrm{i}\lambda' \wedge \mu' \wedge \bar{\mu}' \neq 0. \tag{2.6}$$

This condition, which can be assumed without loss of generality, will restrict the transformation (2.1) to the following ($h \neq 0$, g complex and arbitrary):

$$\Omega = h\bar{h}\lambda', \qquad \Omega_1 = h(\mu' + g\lambda'), \tag{2.7}$$

such that the new pair (Ω, Ω_1) are still in the class $[(\lambda, \mu)]$ and satisfy the condition (2.6). From Ω and Ω_1, we can construct 1-forms Ω_2, Ω_3 (complex), Ω_4 (real) and define a complex-valued function $\mathcal{R}$ through the following equations [19]:

$$\begin{aligned} \mathrm{d}\Omega &= \mathrm{i}\Omega_1 \wedge \bar{\Omega}_1 + \left(\Omega_2 + \bar{\Omega}_2\right) \wedge \Omega, \\ \mathrm{d}\Omega_1 &= \Omega_2 \wedge \Omega_1 + \Omega_3 \wedge \Omega, \\ \mathrm{d}\Omega_2 &= 2\mathrm{i}\Omega_1 \wedge \bar{\Omega}_3 + \mathrm{i}\bar{\Omega}_1 \wedge \Omega_3 + \Omega_4 \wedge \Omega, \\ \mathrm{d}\Omega_3 &= \Omega_4 \wedge \Omega_1 + \Omega_3 \wedge \bar{\Omega}_2 + \mathcal{R}\bar{\Omega}_1 \wedge \Omega. \end{aligned} \tag{2.8}$$

Note that they all depend on the functions h and g. In particular, one can show that the vanishing or not of $\mathcal{R}$ is *invariant* under the transformation (2.7). For $\mathcal{R} = 0$, there exists a unique CR structure which can always be represented by the canonical form [19]

$$\Omega = \mathrm{d}u - \frac{\mathrm{i}}{2}\bar{\zeta}\mathrm{d}\zeta + \frac{\mathrm{i}}{2}\zeta\mathrm{d}\bar{\zeta}, \qquad \Omega_1 = \mathrm{d}\zeta \tag{2.9}$$

in a suitable coordinate chart $(u, \zeta, \bar{\zeta})$. This is in fact the local CR structure of a hyperquadric or a sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ [21] (see p. 150). We will encounter this CR structure (also known to physicists as related to the Robinson congruence [34]) in the Hauser solution (see Section 2.7) of the twisting type N problem. For $\mathcal{R} \neq 0$, by the transformation (2.7), we can always pick some h and g to achieve the special "gauge"

$$\mathcal{R} = 1. \tag{2.10}$$

Eventually, the choice of the functions h (up to a sign) and g (or, Ω and $\pm\Omega_1$) will become unique if one imposes, in addition to (2.10),

$$\mathrm{d}\Omega = \mathrm{i}\Omega_1 \wedge \bar{\Omega}_1.$$

Hence from these special h (with the sign fixed) and g, one can uniquely determine the 1-forms Ω_2, Ω_3 and Ω_4, which, together with Ω and Ω_1, are called the *Cartan invariant forms*. Finally, note that $(\Omega, \Omega_1, \bar{\Omega}_1)$ forms a basis. Thus by expanding these invariant forms as

$$\begin{aligned}
\Omega_2 &= \alpha_I \Omega_1 - \bar{\alpha}_I \bar{\Omega}_1 + \mathrm{i}\beta_I \Omega, \\
\Omega_3 &= \mathrm{i}\gamma_I \Omega_1 + \theta_I \bar{\Omega}_1 + \eta_I \Omega, \\
\Omega_4 &= -\frac{\mathrm{i}}{2}\bar{\eta}_I \Omega_1 + \frac{\mathrm{i}}{2}\eta_I \bar{\Omega}_1 + \zeta_I \Omega,
\end{aligned}$$

we define the six *Cartain invariants* (invariant functions) denoted respectively by

$$\begin{aligned}
&\alpha_I, \theta_I, \eta_I \text{ (complex)}, \\
&\beta_I, \gamma_I, \zeta_I \text{ (real)}.
\end{aligned}$$

Under additional assumptions on the 1-forms (λ', μ') in (2.7):

$$\begin{aligned}
&\mathrm{d}\mu' = 0, \qquad \mathrm{d}\bar{\mu}' = 0, \\
&\mathrm{d}\lambda' = \mathrm{i}\mu' \wedge \bar{\mu}' + (c\mu' + \bar{c}\bar{\mu}') \wedge \lambda',
\end{aligned} \tag{2.11}$$

with a complex function c on M, Cartan found explicit expressions for the Cartan invariant forms [19] (or [21], see pp. 123-127), based on which we have managed to calculate the Cartan invariants as those listed in Appendix B. The above assumptions will also be used in the CR formulation of the field equations in Section 2.5.

2.3 Symmetries

A symmetry of a CR structure is a diffeomorphism that preserves the relation (2.1). Specifically, a real vector field X on M is an infinitesimal symmetry iff

$$\mathcal{L}_X\lambda = a\lambda, \qquad \mathcal{L}_X\mu = b\mu + g\lambda$$

or equivalently,

$$[X, \partial] = -b\partial, \tag{2.12}$$

where a is a real function and b, g are complex functions.

The classification of infinitesimal symmetries of CR structures has been resolved in [35]. Here we quote two theorems from it regarding the canonical forms of CR structures with one or two infinitesimal symmetries.

Theorem 2.1. If a CR structure admits one infinitesimal symmetry then it is equivalent to the following CR structure defined by

$$\mu = \mathrm{d}x + \mathrm{i}\mathrm{d}y, \qquad \lambda = \mathrm{d}u + f(x,y)\mathrm{d}x, \qquad \partial_y f \neq 0, \tag{2.13}$$

in a real coordinate chart (u, x, y), with some real function $f(x, y)$. The associated symmetry is given by

$$X = \partial_u.$$

Theorem 2.2. If a CR structure admits two infinitesimal symmetries then it is equivalent to the following CR structure defined by

$$\mu = \mathrm{d}x + \mathrm{i}\mathrm{d}y, \qquad \lambda = \mathrm{e}^{-\varepsilon x}\mathrm{d}u + f(y)\mathrm{d}x, \qquad \partial_y f \neq 0, \qquad \varepsilon = 0 \text{ or } 1, \tag{2.14}$$

in a real coordinate chart (u, x, y), with some real function $f(y)$. The associated symmetries are given by

$$X_1 = \partial_u, \qquad X_2 = \varepsilon u\partial_u + \partial_x, \tag{2.15}$$

such that

$$[X_1, X_2] = \varepsilon X_1.$$

These two types of CR structures will be very useful for our later discussions. In some sense, the first theorem provides a motivation to the key assumption of the u-independence that we make in Section 4.1. Our new class of type N metrics have CR structures of the type described in the second theorem. Moreover, in Chapter 8, starting with the symmetries of (2.15), we will be able to make a quick guess on the Killing vectors of our new type N metrics, and verify that they indeed lead to true Killing vectors (an inheritability of symmetries). For further applications of symmetries in other circumstances, we refer to [36].

2.4 Lifting CR Manifolds to Spacetimes

CR structures are naturally related to spacetimes admitting congruences of null geodesics without shear. Such spacetimes are automatically algebraically special by the Goldberg-Sachs theorem (Theorem 1.1), and have been studied by physicists since the late 1950s with intentions of characterizing gravitational radiations.

Given a 3-dimensional CR manifold M with a representative (λ, μ) of its CR structure (strictly pseudoconvex or not), we consider, on the Cartesian product $\mathcal{M} = M \times \mathbb{R}$, an entire class of metrics[1] of the following:

$$\mathbf{g} = 2P^2[\mu\bar{\mu} + \lambda(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda)], \tag{2.16}$$

where r is a coordinate along $\mathbb{R}$ and $P \neq 0$, H (real) and W (complex) are arbitrary functions on $\mathcal{M}$. Note that the form of (2.16) is invariant under the change of representatives (λ', μ') through (2.1), and thus is called the class of metrics *adapted*

[1] Here we keep using the same letters for pullbacks of λ and μ from M to $\mathcal{M}$.

to the CR structure $[(\lambda, \mu)]$. In addition, the spacetime $\mathcal{M}$ possesses a congruence of shearfree null geodesics along the direction of the vector field $k = P^{-1}\partial_r$ on $\mathcal{M}$, since one has

$$k \lrcorner \mu = k \lrcorner \lambda = 0, \qquad \mathbf{g}(k, k) = 0$$

and the condition (1.6) is satisfied. Here the congruence being shearfree implies that the same CR structure in the 3-parameter leaf space transverse to k at each fixed value of r is preserved along the congruence and can always be identified with M [17].

The above procedure of *lifting* a CR structure to a Lorentizan spacetime also has the following converse.

Theorem 2.3. ([17], see its *Theorem 1.2* and references therein) Let $(\mathcal{M}, g)$ be a 4-dimensional manifold equipped with a Lorentzian metric and foliated by a 3-parameter congruence of shearfree null geodesics (twisting or not). Then $\mathcal{M}$ is locally a Cartesian product $\mathcal{M} = M \times \mathbb{R}$. The CR structure $[(\lambda, \mu)]$ on M is uniquely determined by $(\mathcal{M}, g)$ and the shearfree null congruence on $\mathcal{M}$.

In Section 4.4 and Chapter 7, we will rely heavily on this theorem to show that the class of type N metrics we find indeed contains new metrics with distinct CR structures, the fact of which otherwise would be much more difficult to prove.

2.5 Reduction of the Einstein Equations

Now combining the metric already defined in Section 1.2,

$$\mathbf{g} = 2\left(\theta^1\theta^2 + \theta^3\theta^4\right), \tag{2.17}$$

with the lifting (2.16), we pick the following null tetrad [17]:

$$
\begin{aligned}
&\theta^1 = P\,\mu, \qquad \theta^2 = P\,\bar{\mu}, \\
&\theta^3 = P\,\lambda, \qquad \theta^4 = P\left(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda\right),
\end{aligned}
\tag{2.18}
$$

such that r is an affine parameter along the shearfree null congruence and the associated null vector field/PND $k = P^{-1}\partial_r$ (recall $\theta^3 = \mathbf{g}(k,\cdot)$). Also, we assume that the 1-forms (λ, μ) satisfy the same condition (2.11) used by Cartan,

$$
\begin{aligned}
&\lambda \wedge \mu \wedge \bar{\mu} \neq 0, \qquad \mu = \mathrm{d}\zeta, \qquad \bar{\mu} = \mathrm{d}\bar{\zeta}, \\
&\mathrm{d}\lambda = \mathrm{i}\mu \wedge \bar{\mu} + (c\mu + \bar{c}\bar{\mu}) \wedge \lambda,
\end{aligned}
\tag{2.19}
$$

where c is a complex-valued function on the CR manifold M. Here ζ is chosen to be first non-constant CR function ($\bar{\partial}\zeta = 0$) the existence of which is guaranteed by the Einstein equations $R_{22} = R_{24} = R_{44} = 0$ [17] (see also [25] p. 417).

Referring back to (2.18), we can see that the condition $\theta^3 \wedge \mathrm{d}\theta^3 \neq 0$ for the congruence to be twisting is satisfied since $\lambda \wedge \mathrm{d}\lambda \neq 0$ (cf. (2.5)) due to the last equations of (2.19). In fact, the null congruence being twisting is equivalent to the CR manifold being locally strictly pseudoconvex [17]. Furthermore, using the closure of the same equation, one determines an important reality condition on the derivatives of c:

$$
\partial\bar{c} = \bar{\partial}c,
$$

provided that a dual basis of vector fields is introduced, which, however, is not a commutative basis:

$$
\begin{aligned}
&\left(\partial_0, \partial, \bar{\partial}\right) \quad \text{dual to} \quad (\lambda, \mu, \bar{\mu}), \\
&\left[\partial, \bar{\partial}\,\right] = -\mathrm{i}\partial_0, \qquad [\partial_0, \partial] = c\partial_0, \qquad \left[\partial_0, \bar{\partial}\,\right] = \bar{c}\partial_0.
\end{aligned}
\tag{2.20}
$$

At this point one has sufficient information to write down explicitly the Einstein equations (1.7), as well as the Weyl scalars (by Theorem 1.1, $\Psi_0 = \Psi_1 = 0$ automatically), all but Ψ_4 required to vanish for type N. We quote from [17] and [15] which show that the results are the following:

$$P = \frac{p}{\cos(\frac{r}{2})}, \qquad (\Leftarrow R_{44} = 0) \tag{2.21}$$

$$W = \mathrm{i}\, a\,(1 + \mathrm{e}^{-\mathrm{i}r}), \qquad (\Leftarrow R_{2412} + R_{2434} = 0,\ R_{24} = R_{22} = 0) \tag{2.22}$$

$$H = q\,\mathrm{e}^{\mathrm{i}r} + \bar{q}\,\mathrm{e}^{-\mathrm{i}r} + h, \qquad (\Leftarrow R_{12} + R_{34} = 2\Lambda,\ R_{13} = 0,\ \Psi_2 = 0) \tag{2.23}$$

where the functions a, q (complex) and h, p (real), all independent of r, satisfy

$$a = c + 2\partial \log p, \tag{2.24}$$

$$q = \frac{2}{3}\Lambda p^2 + \frac{2\partial p\, \bar{\partial} p - p\,\left(\partial\bar{\partial} p + \bar{\partial}\partial p\right)}{2p^2} - \frac{\mathrm{i}}{2}\,\partial_0 \log p - \bar{\partial} c, \tag{2.25}$$

$$h = 2\Lambda p^2 + \frac{2\partial p\, \bar{\partial} p - p\,\left(\partial\bar{\partial} p + \bar{\partial}\partial p\right)}{p^2} - 2\bar{\partial} c. \tag{2.26}$$

Given all the above, the functions a, c, h, p and q define a twisting type N Einstein space, of the form given in (2.17-2.19) iff the unknown functions c and p satisfy the following system of PDEs on M:

$$\partial\bar{c} = \bar{\partial} c \tag{2.27}$$

$$\left[\partial\bar{\partial} + \bar{\partial}\partial + \bar{c}\partial + c\bar{\partial} + \tfrac{1}{2}c\bar{c} + \tfrac{3}{4}\left(\partial\bar{c} + \bar{\partial} c\right)\right] p = \tfrac{2}{3}\Lambda p^3, \qquad (\Leftarrow R_{12} = R_{34},\ \Psi_2 = 0) \tag{2.28}$$

$$R_{33} = 0,$$

$$\Psi_3 = 0, \tag{2.29}$$

as well as one inequality

$$\Psi_4 \neq 0, \tag{2.30}$$

in order that the spacetime should not be conformally flat. In terms of those variables already defined, the Ricci tensor component R_{33} and the Weyl scalars Ψ_3 and Ψ_4

[17, 5] take on quite complicated-looking expressions

$$\begin{aligned} R_{33} = \Bigg[& \frac{8}{p^4}\,(\partial + 2c)\left(p^2 \partial \bar{I}\right) - 8\Lambda \Big(\tfrac{4}{3}\Lambda p^2 + 6\left(\bar{c}\partial + c\bar{\partial}\right)\log p \\ & + 12\partial \log p\, \bar{\partial} \log p + 3c\bar{c} - \bar{\partial} c - 2\mathrm{i}\partial_0 \log p \Big) \Bigg] \cos^4\left(\frac{r}{2}\right), \end{aligned} \tag{2.31}$$

$$\Psi_3 = \left[\frac{2\mathrm{i}}{p^2}\,\partial \bar{I} - 4\mathrm{i}\Lambda\left(2\bar{\partial}\log p + \bar{c}\right)\right] \mathrm{e}^{\mathrm{i}r/2}\cos^3\left(\frac{r}{2}\right), \tag{2.32}$$

$$\Psi_4 = \left\{\frac{2\mathrm{i}}{p^2}\,\partial_0 \bar{I} + \frac{4}{3}\Lambda\left[\left(\bar{\partial} + \bar{c}\right)\left(2\bar{\partial}\log p + \bar{c}\right) + 2\left(2\bar{\partial}\log p + \bar{c}\right)^2\right]\right\} \mathrm{e}^{-\mathrm{i}r/2}\cos^3\left(\frac{r}{2}\right), \tag{2.33}$$

where the function I is defined by

$$I = \partial\left(\partial \log p + c\right) + \left(\partial \log p + c\right)^2, \tag{2.34}$$

and this calculated Ψ_4 for $\Lambda \neq 0$ has been *simplified* with the use of $\Psi_3 = 0$. Despite the frightening appearance of R_{33}, the equations (2.28) and $\Psi_3 = 0$ together do imply the requirement $R_{33} = 0$. This tells us that within the established formalism the twisting type N solutions to the Einstein equations automatically satisfy the condition for an Einstein space, i.e., vacuum with or without a cosmological constant. For $\Lambda = 0$, the statement is obviously true with $\partial \bar{I} = 0$ (see also [25] p. 451) and was used in [17] to prove the CR embeddability of twisting type N vacuums, without cosmological constant. For $\Lambda \neq 0$, one uses (2.28) to substitute the term $\frac{4}{3}\Lambda p^2$ in R_{33} and notices that the resulting expression is a linear combination of $\partial \Psi_3$ and Ψ_3. The equation $R_{33} = 0$ is therefore superfluous for the type N problem, which facilitates our calculation greatly.

For the actual solving of equations, it is important to understand the meaning of the operator ∂ by introducing a real coordinate system (x, y, u) on M such that we have

$$\begin{aligned} \zeta &= x + \mathrm{i}y, & \partial_\zeta &= \tfrac{1}{2}\left(\partial_x - \mathrm{i}\partial_y\right), & \quad \lambda = \frac{\mathrm{d}u + L\mathrm{d}\zeta + \bar{L}\mathrm{d}\bar{\zeta}}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})}, \\ \partial &= \partial_\zeta - L\partial_u, & \partial_0 &= \mathrm{i}(\bar{\partial}L - \partial\bar{L})\partial_u, \end{aligned} \tag{2.35}$$

with $L = L(\zeta, \bar{\zeta}, u)$ a complex-valued function [37] subject to

$$\bar{\partial} L - \partial \bar{L} \neq 0, \tag{2.36}$$

as required by the strict pseudo-convexity ($[\partial, \bar{\partial}] \neq 0$). In addition, the function L relates to the function c in the following way

$$c = -\partial \ln(\bar{\partial} L - \partial \bar{L}) - \partial_u L. \tag{2.37}$$

Hence generally, the system (2.27-2.29) are in fact PDEs for the unknown functions L, $\bar{L}$, and p. Note that the CR structure, which can be determined by L alone, is not given beforehand for the field equations (2.27-2.29), and thus must be resolved simultaneously with the unknown function p.

2.6 Comparison with Non-CR Formulations

With in hand the twisting type N metric form (2.17-2.37) formulated according to CR geometry, it is important to know how it is different from other formalisms that have been extensively used long before this new one was proposed. We here quote (with a slight modification to the tetrad) from [25] (see p. 439-451) a most common one of those pre-existing formalisms proposed by Kerr, Debney and Schild [38, 39, 40] without including the cosmological constant Λ. The extension with $\Lambda \neq 0$ can be found in [36]. For simplicity, we only consider here $\Lambda = 0$. Also we follow closely the notation of [25] with sub- or superscript s added to avoid confusion.

Theorem 2.4. A type N spacetime admits a geodesic, shearfree and twisting null congruence and satisfies the Einstein equation $\mathrm{Ric}(\mathbf{g}) = 0$, iff the metric can be written as

$$\begin{aligned} &\mathbf{g} = 2(\omega^1\omega^2 + \omega^3\omega^4), \qquad \omega^1 = -\frac{\mathrm{d}\zeta}{P_s \bar{\rho}_s} = \bar{\omega}^2, \\ &\omega^3 = \mathrm{d}u + L\mathrm{d}\zeta + \bar{L}\mathrm{d}\bar{\zeta}, \qquad \omega^4 = \mathrm{d}r_s + W_s\mathrm{d}\zeta + \bar{W}_s\mathrm{d}\bar{\zeta} + H_s\omega^3, \end{aligned} \tag{2.38}$$

with metric components subject to

$$\rho_s^{-1} = r_s - \mathrm{i}\Sigma_s, \qquad \frac{2\mathrm{i}\Sigma_s}{P_s^2} = \bar{\partial}L - \partial\bar{L} \neq 0, \tag{2.39}$$

$$W_s = -\left(\rho_s^{-1}\partial_u L + \mathrm{i}\partial\Sigma_s\right), \qquad \partial = \partial_\zeta - L\partial_u, \tag{2.40}$$

$$H_s = -P_s^2 \mathrm{Re}\left[\partial\left(\bar{\partial}\log P_s - \partial_u\bar{L}\right)\right] - r_s\partial_u \log P_s, \tag{2.41}$$

such that the unknown functions $L = L(\zeta,\bar{\zeta},u)$ (complex) and $P_s = P_s(\zeta,\bar{\zeta},u)$ (real) satisfy

$$\mathrm{Im}\ \partial\partial\bar{\partial}\bar{\partial}V_s = 0, \qquad P_s = \partial_u V_s, \tag{2.42}$$

$$\partial I_s = 0, \tag{2.43}$$

$$\Psi_4^s = P_s^2\rho_s\partial_u I_s \neq 0, \tag{2.44}$$

where the function I_s is defined by

$$I_s = \bar{\partial}\left(\bar{\partial}\log P_s - \partial_u\bar{L}\right) + \left(\bar{\partial}\log P_s - \partial_u\bar{L}\right)^2 = P_s^{-1}\partial_u\bar{\partial}\bar{\partial}V_s. \tag{2.45}$$

In this metric form, the coordinates $(\zeta,\bar{\zeta},u)$ and the function L have been chosen identically with those introduced in (2.35); hence, unlike other quantities, each is not given a sub- or superscript s. In addition, the real coordinate r_s, like its counterpart r in (2.18), is also an affine parameter along the null congruence.

Even without taking a hard look, one can readily see some resemblance between the metrics (2.38-2.45) and (2.17-2.37), for instance, in the expressions of ω^3 and λ both in terms of L. Taking $\Lambda = 0$ in (2.17-2.37), we can show that the two metrics are equivalent to each other by the following transformation:

$$P_s = \frac{2p}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})}, \tag{2.46}$$

$$r_s = \frac{2p^2}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})}\tan\left(\frac{r}{2}\right), \qquad |r| < \pi, \tag{2.47}$$

with the inverse

$$p = \frac{\mathrm{i}}{2}(\bar{\partial}L - \partial\bar{L})P_s, \tag{2.48}$$

$$r = 2\arctan\left(\frac{2}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})P_s^2}\, r_s\right). \tag{2.49}$$

To verify this equivalency, first one substitutes (2.46) into the definition (2.45) of the function I_s, and compares the resulting expression with the function I given by (2.34) with (2.37) plugged in, and thereby obtains an equality

$$I_s = \bar{I},$$

with both sides in terms of p and L. Therefore the conditions $\partial I_s = 0$ and $\partial_u I_s \neq 0$ is equivalent to $\partial\bar{I} = 0$ and $\partial_0\bar{I} \neq 0$ (or, respectively, $\Psi_3 = 0$ and $\Psi_4 \neq 0$ with $\Lambda = 0$). By a similar argument, one can show that the equation (2.42) can be transformed to (2.28) with $\Lambda = 0$ (preferably with the help of Maple), despite their drastically different appearances. For the tetrad, a tedious but straightforward calculation confirms that the metric components of (2.38) indeed match those of (2.18) through (2.46) and (2.47). In particular, we have found

$$W_s + \partial r_s = \frac{p^2}{\cos^2(\frac{r}{2})} \cdot \frac{W}{\mathrm{i}(\bar{\partial}L - \partial\bar{L})},$$

$$H_s + \partial_u r_s = -\frac{p^2}{\cos^2(\frac{r}{2})} \cdot \frac{H}{(\bar{\partial}L - \partial\bar{L})^2},$$

with (2.46) and (2.47) applied to the right hand sides. Lastly, one can reverse the whole process by using the inverse transformation (2.48) and (2.49), hence proving the equivalency of the two metric forms.

One usefulness of such a comparison is to acquire the coordinate freedom and transformation properties of the new metric form (2.17-2.37) and the associated field equations (2.27-2.29), the knowledge of which may help us to understand better their invariant features. First, we start with the type N metric (2.38-2.45) which is known to admit the following coordinate transformation [25] (see p. 442):

$$\zeta' = f(\zeta), \qquad u' = F(\zeta, \bar{\zeta}, u), \qquad \partial_u F \neq 0, \tag{2.50}$$

with f holomorphic and F a real-valued function. Correspondingly, the coordinate r_s and the functions L and P_s must change according to the following:

$$r'_s = \frac{r_s}{\partial_u F}, \tag{2.51}$$

$$\partial' = \frac{1}{f'}\partial, \qquad \partial'_u = \frac{1}{\partial_u F}\partial_u, \tag{2.52}$$

$$L' = -\frac{1}{f'}(\partial_\zeta F - L\partial_u F) = -\frac{1}{f'}\partial F, \qquad \bar\partial' L' - \partial'\bar L' = \frac{\partial_u F}{f'\bar f'}(\bar\partial L - \partial \bar L), \tag{2.53}$$

$$P'_s = \frac{|f'|}{\partial_u F}P_s, \qquad V'_s = |f'|V_s. \tag{2.54}$$

under which, as one may expect, the forms of the field equations (2.42) and (2.43) are invariant for the new P'_s and L'. Quite importantly, the coordinate freedom (2.50) always allows one to pick a function F to obtain the special gauge $P_s = 1$, $V_s = u$ [38], under which the original field equations can be much simplified. Related to this idea, the transformations (2.46) and (2.47) themselves can be simply viewed as another special gauge.

Now we proceed to see what the coordinate freedom (2.50) brings about for the type N metric (2.17-2.37) in the CR formalism. To begin with, we comment that the new coordinate $\zeta' = f(\zeta)$, as a CR function ($\bar\partial f(\zeta) = f'\bar\partial\zeta = 0$), is not functionally independent of ζ (cf. (2.4)). Hence no knowledge of a second CR function that is functionally independent to ζ is involved here, which also means that one only needs to consider a restricted form of the transformation (2.1) (also cf. (2.7)):

$$\mu' = h\mu, \qquad \lambda' = h\bar h\lambda,$$

with $h = f'(\zeta)$ in this particular case. Since the new basis $(\partial'_0, \partial', \bar\partial')$ must be dual to $(\lambda', \mu', \bar\mu')$, it immediately yields

$$\partial' = \frac{1}{f'}\partial, \qquad \partial'_0 = \frac{1}{f'\bar f'}\partial_0. \tag{2.55}$$

which, by a quick check, are consistent with (2.52) and (2.53). However, unlike those two relations, the deduction of (2.55) requires no explicit presence of the function L or its transformation (though (2.53) is still valid in the CR formalism since $(\zeta, \bar{\zeta}, u)$ and L are identical in both metrics). This is in fact true for all our following derivations of the transformation laws. To see how the function c transforms, we insist that the same commutation relations (2.20) be observed for the new basis, i.e.,

$$\left[\partial', \bar{\partial}'\right] = -\mathrm{i}\partial'_0, \qquad [\partial'_0, \partial'] = c'\partial'_0, \qquad \left[\partial'_0, \bar{\partial}'\right] = \bar{c}'\partial'_0,$$

from which we obtain

$$c' = \frac{1}{f'}c + \frac{f''}{(f')^2}. \tag{2.56}$$

Moreover, omitting details, we point out that the function p relates to p' through

$$p' = \frac{1}{|f'|}p. \tag{2.57}$$

As expected, under the above transformation laws, the field equations for the new p' and c' can be shown taking on the same form of (2.27-2.29) with $(\partial_0, \partial, \bar{\partial})$ simply replaced by $(\partial'_0, \partial', \bar{\partial}')$.

It is worthwhile to comment that the adoption of the "coordinate-free" basis $(\partial_0, \partial, \bar{\partial})$ effectively hides away all presence of the function L, which accounts for the noticeable feature that neither the function $F(\zeta, \bar{\zeta}, u)$ nor its derivatives appear in (2.55-2.57). In fact, there is even no need for a coordinate u being chosen in order to carry out all the derivations above, as long as the basis $(\partial_0, \partial, \bar{\partial})$ is defined from its dual $(\lambda, \mu, \bar{\mu})$, which by themselves are more intrinsic geometric objects than their coordinate representations.

Finally, the whole set of transformation laws are completed by a remarkable invariance of the coordinate r, i.e.,

$$r' = r, \tag{2.58}$$

which is quite contrary to its counterpart (2.51). For a double-check, one may easily verify that (2.57) and (2.58) are consistent with (2.54) and (2.51) via the transformation (2.46) and (2.47) and the relation (2.53) (with a bit more effort to see the consistency of (2.56) and (2.53) through (2.37)). Furthermore, we have

$$\frac{r_s}{\Sigma_s} = \frac{r_s'}{\Sigma_s{}'} = -\tan\left(\frac{r}{2}\right), \qquad |r| < \pi, \tag{2.59}$$

in which $r_s\Sigma_s^{-1}$ is known to be a gauge invariant [40]. Therefore we have proved the following theorem.

Theorem 2.5. In the type N metric (2.17-2.37), the affine parameter r along the null congruence is invariant under the coordinate transformation $\zeta' = f(\zeta)$, $u' = F(\zeta, \bar{\zeta}, u)$ with $\partial_u F \neq 0$.

The theorem still holds as long as the Einstein equations (1.7) are satisfied without further requirements on Petrov types. Since all explicit *periodic* dependence on r are already solved for the metric (see (2.21-2.23)), it implies a *circle bundle* structure $\mathbb{S}^1 \to \mathcal{M} \to M$ in the spacetime, based on which Hill and Nurowski conducted their periodic universe argument [41] relating Penrose's idea on a "pre-big-bang era".

Remark. [42] The overall factor $1/\cos^2(\frac{r}{2})$ (cf. (2.21)) of the metric (2.17-2.37) is in fact associated with the *Penrose conformal factor.* From Penrose's approach to asymptotically simple/flat spacetimes, the metric (2.17-2.37) constitutes a *conformal compactification* of the metric (2.38-2.45), with the conformal boundary attained at $r = \pm\pi$.

To conclude, we point out that the metric (2.17-2.37), like its counterpart, also has a gauge freedom. For an example, if $\partial_0 p \neq 0$, we can always choose a local coordinate u such that

$$p = u \implies \partial p = -L,$$

thereby simplifying the field equations, just as the gauge $P_s = 1$, $V_s = u$ [38] does, as mentioned before.

2.7 The Hauser Solution

For the twisting type N metric (2.38-2.45) with $\Lambda = 0$, only one class of exact solutions, the Hauser solution [6, 7], has been found so far, which is given by [25] (see p. 451)

$$L = 2\mathrm{i}\left(\zeta + \bar{\zeta}\right), \qquad P_s = \left(\zeta + \bar{\zeta}\right)^{3/2} f(w), \qquad w = \frac{u}{\left(\zeta + \bar{\zeta}\right)^2}, \tag{2.60}$$

where $f(w)$ is a solution of the hypergeometric differential equation

$$16(1 + w^2)f'' + 3f = 0. \tag{2.61}$$

Nevertheless, there have been arguments regarding that this solution is not asymptotically flat and hence does not describe gravitational radiations from a finite source [43]. From a geometric point of view, a distinctive feature of the Hauser solution is that its underlying CR structure is that of a hyperquadric, since

$$c = 0$$

as calculated from (2.60) and $c = 0$ implies $\mathcal{R} = 0$ in (2.8) [19]. Furthermore, to see the Hauser solution in the CR formalism (2.17-2.37), we note that the canonical form (2.9) for a hyperquadric provides a rather simple choice for L, which, together with $c = 0$, can significantly simplify the field equations (2.27-2.29) to a set of two PDEs for a single unknown p. By studying the classical symmetries (see Chapter 3) of the resulting PDEs, we have managed to find an equivalent form of (2.60):

$$\begin{aligned} &L = -\frac{\mathrm{i}}{2}\bar{\zeta}, \qquad c = 0, \qquad \Lambda = 0, \\ &p = \left(\frac{\zeta + \bar{\zeta}}{2}\right)^{3/2} f(w), \qquad w = \frac{4u + \mathrm{i}(\zeta^2 - \bar{\zeta}^2)}{(\zeta + \bar{\zeta})^2}, \end{aligned} \tag{2.62}$$

where $f(w)$ satisfies the same ODE (2.61) as before.

Chapter 3

Classical Symmetries of PDEs

3.1 Background

The Einstein equations are a system of highly nonlinear PDEs and thus very difficult to solve in general. A most important technique for finding their special exact solutions is to study the symmetry properties of those PDEs. The classical symmetry theory for PDEs was established by Sophus Lie more than 100 years ago, based on which he introduced the fundamental notions of Lie groups and Lie algebras. In the 1960s, symmetry theory entered upon a new era, starting with the discovery of *completely integrable* systems (KdV equation, nonlinear Schrödinger equations, etc.) and the development of the inverse scattering method, and thereby generalizing Lie's original idea on classical point symmetries to the concept of *higher symmetries*.

Generally speaking, a symmetry of a system of differential equations is a (continuous or discrete) transformation of its "solution manifold" into itself, i.e., an automorphism that takes one solution to another. For the classical symmetry, this solution manifold is determined by the initial equations alone, while for the higher symmetry, all differential consequences of the equations at hand, called the infinite prolongation, shall also be considered. Nonetheless, both types of symmetries share the same computing scheme. Once calculated, a knowledge of symmetries of PDEs can be used to reduce the order of the equations and/or the number of variables, hence, with a better chance, leading to exact solutions. As a most astonishing feature, all these symmetry methods are quite universal and, in principle, can be applied to any types of differential equations, though the calculation may present a significant, even formidable challenge.

All symmetries considered in the following are continuous and local, i.e., infinitesimal symmetries. Thus we can talk about their Lie algebras instead of the actual transformation groups. Our goal here is to lay out all ground work before we embark on calculating the classical symmetries of a special case of the field equations (2.27-2.29). All materials here are adapted and reorganized from [44].

3.2 Jet Manifolds

Consider a generic system of r (nonlinear) differential equations of order k with n independent variables $\mathbf{x} = (x_1, \dots, x_n)$ and m dependent variables (unknowns) $\mathbf{u}(\mathbf{x}) = (u^1, \dots, u^m)$,

$$\begin{cases} F_1(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0, \\ \cdots \\ F_r(\mathbf{x}, \mathbf{u}, \mathbf{p}) = 0, \end{cases} \tag{3.1}$$

where F_l's are smooth functions and $\mathbf{p}$ denotes the set of all partial derivatives

$$p^j_\sigma = \frac{\partial^{|\sigma|} u^j}{\partial x^\sigma} = \frac{\partial^{|\sigma|} u^j}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}. \tag{3.2}$$

with a multi-index $\sigma = (i_1, \dots, i_n)$ and $|\sigma| = i_1 + \cdots + i_n \leq k$.

The basic idea of geometric studies of differential equations lies in treating the variables

$$x_1, \ldots, x_n, u^1, \ldots, u^m, p^j_\sigma, \qquad |\sigma| \leq k,$$

as the coordinates of the so-called *jet space* $J^k(n,m)$. Therefore the equations (3.1) determine a surface $\mathcal{E}$ of codimension r in $J^k(n,m)$. This surface $\mathcal{E} \subset J^k(n,m)$, also known as the *solution manifold*, is the geometric object for which we define symmetry transformations. To acknowledge the fact that p^j_σ's correspond to partial derivatives, we introduce the total derivative operator (or a *vector field* on $J^k(n,m)$ if truncated accordingly)

$$D_i = \frac{\partial}{\partial x_i} + \sum_{|\sigma|=0}^{\infty} \sum_{j=1}^{m} p^j_{\sigma+1_i} \frac{\partial}{\partial p^j_\sigma}, \qquad p^j_{(0,\ldots,0)} = u^j, \qquad i = 1, \ldots, n,$$

with respect to x_i, where $1_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th place and hence

$$p^j_{\sigma+1_i} = \frac{\partial}{\partial x_i} \left(\frac{\partial^{|\sigma|} u^j}{\partial x^\sigma} \right). \tag{3.3}$$

To see that this operator contains information on differential relations between variables, we consider an n-dimensional surface in $J^0(n,m)$,

$$\begin{cases} u^1 = f^1(x_1, \ldots, x_n), \\ \cdots \\ u^m = f^m(x_1, \ldots, x_n), \end{cases} \tag{3.4}$$

determined by some smooth vector function $(f^1, \ldots, f^m)$. Then by repetitively applying the total derivative operators on both sides, we can lift (3.4) up to an n-dimensional surface in $J^k(n,m)$ given by

$$p^j_\sigma = D_\sigma(u^j) = \frac{\partial^{|\sigma|} f^j}{\partial x^\sigma}(x_1, \ldots, x_n), \qquad j = 1, \ldots, m, \qquad |\sigma| \leq k, \tag{3.5}$$

which is consistent with (3.2). If this surface happens to lie in the solution manifold $\mathcal{E}$, we conclude that (3.4) is a solution to the system (3.1).

Another important use of total derivative operators is to calculate the differential consequences of the initial equations (3.1), for instance,

$$0 = D_i(F_l(\mathbf{x}, \mathbf{u}, \mathbf{p})) = \frac{\partial F_l}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial u^j}{\partial x_i} \frac{\partial F_l}{\partial u^j} + \cdots , \tag{3.6}$$

justifying the name “total derivative”.

3.3 Defining Equations

From a geometric point of view, classical symmetries are diffeomorphisms of the solution manifold $\mathcal{E} \subset J^k(n, m)$, preserving the differential relations between variables as encoded in the (truncated) total derivative operators.

In an infinitesimal form, consider a Lie group of point transformations

$$\begin{aligned} \tilde{x}_i &= x_i + \varepsilon a_i(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \qquad i = 1, \ldots, n, \\ \tilde{u}^j &= u^j + \varepsilon b^j(\mathbf{x}, \mathbf{u}) + O(\varepsilon^2), \qquad j = 1, \ldots, m, \end{aligned} \tag{3.7}$$

where ε is a group parameter. They correspond to a Lie algebra of infinitesimal operators

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} b^j \frac{\partial}{\partial u^j},$$

which is also a vector field on $J^0(n, m)$. In a similar manner as we lift the surface (3.4) to a surface in $J^k(n, m)$, there is a unique way to prolong the vector field X to a vector field on $J^k(n, m)$ such that all the differential relations mentioned before are respected. Such a kth *lifting* of X, called a Lie field, is given by

$$X^{(k)} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + \sum_{|\sigma| \le k} \sum_{j=1}^{m} b^j_\sigma \frac{\partial}{\partial p^j_\sigma}, \qquad b^j_{(0,\ldots,0)} = b^j,$$

where the coefficients b^j_σ are computed from the recursion relation

$$b^j_{\sigma+1_i} = D_i(b^j_\sigma) - \sum_{s=1}^{n} p^j_{\sigma+1_l} D_i(a_s), \qquad 0 \le |\sigma| \le k - 1,$$

and hence are all uniquely determined by a_i and b^j. The point transformation (3.7) with its Lie field $X^{(k)}$ will become a classical point symmetry of the system (3.1) iff shifts along the trajectories generated by $X^{(k)}$ leave invariant the solution manifold $\mathcal{E}$, i.e., that $X^{(k)}$ is *tangent* to $\mathcal{E}$:

$$X^{(k)}(F_l)\big|_{\mathcal{E}} = 0, \qquad l = 1, \ldots, r,$$

with $\mathcal{E} = \{(F_1, \ldots, F_r) = 0\}$, or equivalently, by removing the part of $X^{(k)}$ that is already tangent to $\mathcal{E}$ due to (3.6),

$$\left(X^{(k)} - \sum_{i=1}^{n} a_i D_i\right) (F_l)|_{\mathcal{E}} = \sum_{j=1}^{m} \left(b^j - \sum_{i=1}^{n} a_i p_i^j\right) \frac{\partial F_l}{\partial u^j}\bigg|_{\mathcal{E}} + \cdots = 0. \tag{3.8}$$

In practice, the above defining equation for classical symmetries can be more conveniently written as

$$\sum_{j,\sigma} \frac{\partial F_l}{\partial p_\sigma^j} D_\sigma(\varphi^j)\Bigg|_{\mathcal{E}} = 0, \qquad l = 1, \ldots, r. \tag{3.9}$$

where the vector function $(\varphi^1, ..., \varphi^m)$ with components given by

$$\varphi^j = b^j - \sum_{i=1}^{n} a_i p_i^j \tag{3.10}$$

is called the *generating section* of the Lie field. From (3.9), an over-determined system of linear PDEs for $a_i(\mathbf{x}, \mathbf{u})$ and $b^j(\mathbf{x}, \mathbf{u})$ can be obtained, which may contain hundreds or thousands of equations, depending on the complexity of (3.1). The reduction and solving of these determining PDEs often require extensive algebraic manipulations, with the possibility that non-trivial solutions may not even exist.

As an extra comment on (3.9), the higher symmetries, in fact, share the same defining equation except that φ^j may have a dependence on variables p_σ^j from higher jet spaces and that the solution manifold $\mathcal{E}$ shall be extended to its infinite prolongation $\mathcal{E}^\infty = \{D_\sigma(F_l) = 0\} \subset J^\infty(n, m)$, i.e., all differential consequences of the equations (3.1).

To conclude, we present a quick example of the Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{3.11}$$

The generating functions for its classical symmetries are listed as follows

$$xu_x + 3tu_t + 2u, \qquad 6tu_x + 1, \qquad u_x, \qquad u_t. \tag{3.12}$$

with infinitesimal generators respectively given by

$$\begin{aligned} 2u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x} - 3t\frac{\partial}{\partial t}, &\qquad \text{scale symmetry,} \\ \frac{\partial}{\partial u} - 6t\frac{\partial}{\partial x}, &\qquad \text{Galilean symmetry,} \\ \frac{\partial}{\partial x}, &\qquad \text{translation along } x, \\ \frac{\partial}{\partial t}, &\qquad \text{translation along } t. \end{aligned}$$

3.4 Invariant Solutions

Invariant solutions are special solutions that are invariant under certain symmetry transformations. Specifically, they are the *fixed points* of the "flow" generated by the Lie field (3.8):

$$X^{(k)} - \sum_{i=1}^{n} a_i D_i = \sum_{j=1}^{m} \varphi^j \frac{\partial}{\partial u^j} + \cdots = \sum_{j,\sigma} D_\sigma(\varphi^j)\frac{\partial}{\partial p^j_\sigma}.$$

Hence if the generating section φ^j has been calculated, the invariant solutions are subject to the over-determined, but compatible system

$$\begin{cases} (\varphi^1, \ldots, \varphi^m) = 0, \\ F_1 = 0, \\ \cdots \\ F_r = 0, \end{cases} \tag{3.13}$$

that makes the above Lie field vanish. Note that each equation $\varphi^j = 0$ is simply a first-order linear PDEs for the unknown u^j alone. Thus they usually can be solved first by the method of characteristics to obtain an ansatz of solutions for other equations.

Chapter 4

ODEs for Twisting Type N Vacuums

4.1 Killing Vector in the u-Direction

Now we come back to the field equations. One major difficulty of fully solving the system (2.27-2.29) is that unlike ordinary coordinate differentiations, the selected (dual) basis for the tangent space is not commutative, and, even worse, the operator ∂ itself involves the unknown function c (or L, cf. (2.37)). When this dependence on the coordinates $(\zeta, \bar{\zeta}, u)$ is written out explicitly with the functions L, $\bar{L}$ and p, the original PDEs will become formidably lengthy. Instead of facing this entire conundrum, we have decided to circumvent it, at least in this work, by looking at the special case that the unknowns p and c have no u-dependence, i.e., $\partial_0 p = 0 = \partial_0 c$. Geometrically speaking, we insist that the spacetime admits a Killing vector in the u-direction. Such an assumption simplifies the problem greatly in that one can treat the operator ∂ the same as ∂_ζ, when acting on either p or c. This is a generalization of the assumption made by Nurowski [15], where it was simplified to just dependence on y, i.e, two Killing vectors assumed.

Theorem 4.1. (*CR embeddability* [35]) A CR structure (2.19) with $c = c(\zeta, \bar{\zeta})$ is CR embeddable. From this particular form of c, a u-independent form of the function L can be constructed as

$$L(\zeta, \bar{\zeta}) = -\tfrac{\mathrm{i}}{2} \int \alpha(\zeta, \bar{\zeta})\, \mathrm{d}\bar{\zeta} \tag{4.1}$$

with a real-valued function $\alpha \neq 0$ satisfying

$$\partial_\zeta \alpha = -c\,\alpha, \qquad \partial_{\bar{\zeta}} \alpha = -\bar{c}\,\alpha. \tag{4.2}$$

Associated to this L, the tangential CR equation $\bar{\partial}\eta = 0$ yields a second CR function:

$$\eta = u + \tfrac{\mathrm{i}}{2} \iint \alpha(\zeta, \bar{\zeta})\, \mathrm{d}\zeta \mathrm{d}\bar{\zeta}. \tag{4.3}$$

Proof. Because of the restraint $\partial_\zeta \bar{c} = \partial_{\bar{\zeta}} c$, the system (4.2) is compatible and has a real solution $\alpha \neq 0$. Hence one can directly check that (4.1) satisfies (2.37) and that (4.3) satisfies the equation $\bar{\partial}\eta = (\partial_{\bar{\zeta}} - \bar{L}\partial_u)\eta = 0$. Clearly, the CR functions η and ζ are functionally independent, i.e., $\mathrm{d}\zeta \wedge \mathrm{d}\eta \neq 0$. Therefore we acquire a second CR function. $\square$

For a given function $c = c(\zeta, \bar{\zeta})$, the equation (2.37), viewed as a PDE for L, may give rise to multiple choices of the function L, hence various λ's. However, such an ambiguity only constitutes different representatives of the same CR structure. To see this, one may look into the six Cartan invariants (details in Section 4.4) and notice that they are all uniquely determined by the function $c = c(\zeta, \bar{\zeta})$ (see, e.g., (4.21) and Appendix B), given that the function r defined in (4.21) does not vanish. For CR structures with $r = 0$ ($r \propto \mathcal{R}$ in (2.8) [19]), they are all locally equivalent to a 3-dimensional hyperquadric inside $\mathbb{C}^2$. An alternative proof would be to show that there always exists a coordinate transformation $u \to \tilde{u}(\zeta, \bar{\zeta}, u)$ ($\partial\tilde{u}/\partial u \neq 0$) that takes a function $L = L(\zeta, \bar{\zeta}, u)$ satisfying (2.37) to the u-independent form

(4.1). This can be confirmed by checking the compatibility of PDEs regarding such an existence. In conclusion, the CR structure on M is *uniquely* determined once a function $c = c(\zeta, \bar{\zeta})$ is given.

The converse of the last statement above is, however, not true. In fact various choices of the function c may correspond to the same CR structure. We will see examples of this in Section 4.4. Relevant to this issue, our assumption of the function c being u-independent is thus not a CR invariant property. A function $c = c(\zeta, \bar{\zeta})$ may acquire u-dependence through the transformation (2.1) that takes one representative (λ, μ) of the CR structure to another.

Now we apply the assumption and the following notations

$$\partial f \to \partial_\zeta f = f_1,\ \bar{\partial} f \to \partial_{\bar{\zeta}} f = f_2,\ f = f_0,\ (f = p,\ c \text{ and } \bar{c} \text{ only})$$

and then rewrite the system (2.27-2.29) as

$$\bar{c}_1 = c_2, \tag{4.4}$$

$$2p_{12} + \bar{c}_0 p_1 + c_0 p_2 + \tfrac{1}{2} c_0 \bar{c}_0 p_0 + \tfrac{3}{4}(\bar{c}_1 + c_2) p_0 = \tfrac{2}{3}\Lambda p_0^3, \tag{4.5}$$

$$p_0 p_{122} - p_1 p_{22} + 2\bar{c}_0 p_0 p_{12} - 2\bar{c}_0 p_1 p_2 + 2\bar{c}_1 p_0 p_2 + (\bar{c}_{12} + 2\bar{c}_0 \bar{c}_1) p_0^2 = 2\Lambda(2p_2 + \bar{c}_0 p_0) p_0^3, \tag{4.6}$$

where the last equation arises from $\Psi_3 = 0$. These are the PDEs we aim to solve. Moreover, the Weyl scalar Ψ_4 reads

$$\Psi_4 = \frac{4}{3}\Lambda \left[2p_0 p_{22} + 6p_2^2 + 10\bar{c}_0 p_0 p_2 + (\bar{c}_2 + 3\bar{c}_0^2) p_0^2\right] \frac{\mathrm{e}^{-\mathrm{i}r/2}}{p_0^2} \cos^3\left(\frac{r}{2}\right). \tag{4.7}$$

4.2 Infinite-Dimensional Classical Symmetries

We follow the standard procedure described in Chapter 3 (see also [44]) to calculate the classical symmetries of the system (4.4-4.6). Since (4.6) is generally complex, we have to include its complex conjugate as well in the calculation. Moreover, we treat (4.4) as a constraint and encode it and its differential consequences directly into the choice of *intrinsic coordinates* so that this equation no longer needs further attention. This gives us three PDEs for three dependent variables p, c and $\bar{c}$ which depend on two independent variables ζ and $\bar{\zeta}$. The intrinsic coordinates within the first four jet spaces that are relevant to the calculation are chosen as follows

$$p_0, c_0, \bar{c}_0,$$

$$p_1, p_2, c_1, \bar{c}_1, \bar{c}_2,$$

$$p_{11}, p_{22}, c_{11}, \bar{c}_{22},$$

$$p_{111}, p_{222}, c_{111}, \bar{c}_{222}.$$

The rest of the jet variables, such as p_{12}, p_{122}, $\bar{c}_{12} (= c_{22})$ etc., can be expressed in terms of the intrinsic coordinates through the PDEs and their differential consequences.

With a considerable amount of manual work on the algebraic computer program Maple, we have managed to find the classical symmetries with the generating section given by

$$\begin{aligned}
\Psi &= -\tfrac{1}{2}(\partial_\zeta A + \partial_{\bar{\zeta}} \bar{A}) p_0 - A p_1 - \bar{A} p_2, \\
\Theta &= \partial_\zeta^2 A - (\partial_\zeta A) c_0 - A c_1 - \bar{A} \bar{c}_1, \\
\bar{\Theta} &= \partial_{\bar{\zeta}}^2 \bar{A} - (\partial_{\bar{\zeta}} \bar{A}) \bar{c}_0 - A \bar{c}_1 - \bar{A} \bar{c}_2,
\end{aligned}$$

where $A = A(\zeta)$ is an arbitrary function of ζ that is sufficiently differentiable. The Lie bracket of two symmetries with, respectively, $A_1(\zeta)$ and $A_2(\zeta)$ yields a third symmetry with a new $A_3(\zeta)$ given by

$$A_3 = [A_1, A_2] := A_1 \partial_\zeta A_2 - A_2 \partial_\zeta A_1.$$

Therefore, we indeed obtain an infinite-dimensional set of classical symmetries for the system (4.4-4.6). In particular, they reduce to translational symmetries for nonzero constant A, and scaling symmetries for $A \propto \zeta$.

4.3 Invariant Solutions and Reductions to ODEs

Setting the generating section $(\Psi, \Theta, \bar{\Theta})$ to zero (cf. Section 3.4), i.e.,

$$
\begin{aligned}
0 &= -\tfrac{1}{2}(\partial_\zeta A + \partial_{\bar\zeta}\bar A)p - A\partial_\zeta p - \bar A\partial_{\bar\zeta} p,\\
0 &= \partial_\zeta^2 A - (\partial_\zeta A)c - A\partial_\zeta c - \bar A\partial_{\bar\zeta} c,\\
0 &= \partial_{\bar\zeta}^2 \bar A - (\partial_{\bar\zeta} \bar A)\bar c - A\partial_\zeta \bar c - \bar A\partial_{\bar\zeta} \bar c,
\end{aligned}
$$

we aim to solve these *linear* first-order PDEs for p, c and $\bar c$, so as to acquire an ansatz for the field equations. According to the method of characteristics, for the equation for $p(\zeta, \bar\zeta)$, we know that its characteristic curves must satisfy

$$
\frac{\mathrm{d}\bar\zeta}{\bar A} = \frac{\mathrm{d}\zeta}{A} = \frac{\mathrm{d}p}{-\frac{1}{2}(\partial_\zeta A + \partial_{\bar\zeta}\bar A)p}.
$$

The first equality can be integrated as

$$
\int \frac{1}{\bar A}\mathrm{d}\bar\zeta - \int \frac{1}{A}\mathrm{d}\zeta = C, \tag{4.8}
$$

where C is an integration constant. For the second equality, one can rewrite it as

$$
\frac{\mathrm{d}p}{p} = -\frac{\partial_\zeta A + \partial_{\bar\zeta}\bar A}{2A}\,\mathrm{d}\zeta = -\frac{\partial_\zeta A}{2A}\,\mathrm{d}\zeta - \frac{\partial_{\bar\zeta}\bar A}{2\bar A}\,\mathrm{d}\bar\zeta,
$$

where the first equality is used to substitute one $\mathrm{d}\zeta$ in order to make a perfect derivative. Clearly, one can immediately integrate the resulting equation as follows:

$$
\mathrm{d}(\log p) = \mathrm{d}\left(-\tfrac{1}{2}\log(A\bar A)\right) \Longrightarrow p = \frac{F(C)}{\sqrt{A\bar A}},
$$

where the arbitrary function $F(C)$ acts as an integration constant. Then the final general solution p is obtained by substituting C with (4.8) in the above expression.

Following the same procedure, one can solve the equations for c and $\bar c$ as well. Then taking into account the constraint $\partial_\zeta \bar c = \partial_{\bar\zeta} c$ and p being real-valued, we are able to obtain a remarkable ansatz for the field equations:

$$
p(\zeta,\bar\zeta) = \frac{F_1(z)}{\sqrt{A\bar A}}, \quad c(\zeta,\bar\zeta) = \frac{\partial_\zeta A + \mathrm{i}F_2(z) + C_1}{A}, \quad \bar c(\zeta,\bar\zeta) = \frac{\partial_{\bar\zeta}\bar A - \mathrm{i}F_2(z) + C_1}{\bar A} \tag{4.9}
$$

with a new real argument (taking $C = -\mathrm{i}z$ in (4.8))

$$z = -\mathrm{i}\left(\int \frac{1}{A}\mathrm{d}\zeta - \int \frac{1}{\bar{A}}\mathrm{d}\bar{\zeta}\right) = \mathrm{Im}\int \frac{2}{A}\mathrm{d}\zeta. \tag{4.10}$$

Here the constant C_1 and the undetermined functions $F_{1,2}(z)$ are all real-valued. One may easily verify these expressions by direct calculation.

Substituting the ansatz into (4.5) and (4.6) and noticing that all dependence on $A, \bar{A} \neq 0$, except those in the argument z, can be factored out, we have a neat reduction from the PDEs to a system of two ODEs for F_1 and F_2 only:

$$\begin{aligned} 0 &= -F_1'' + F_2 F_1' + \tfrac{1}{3}\Lambda F_1^3 - \tfrac{1}{4}(F_2^2 - 3F_2' + C_1^2)F_1, \\ 0 &= -H' + 2(F_2 + \mathrm{i}C_1)H, \end{aligned}$$

where in the the second ODE (derived from (4.6)), the function $H(z)$ is defined by

$$H = F_1'' F_1 - (F_1')^2 - \Lambda F_1^4 - F_2' F_1^2.$$

The above set of ODEs contains *two* separate cases for solutions. If $C_1 = 0$, the system reduces to

$$\begin{aligned} 0 &= -F_1'' + F_2 F_1' + \tfrac{1}{3}\Lambda F_1^3 - \tfrac{1}{4}(F_2^2 - 3F_2')F_1, \\ 0 &= -H' + 2F_2 H, \end{aligned} \tag{4.11}$$

which allows solutions with $H \neq 0$, in addition to the obvious case $H = 0$. But generally for $C_1 \neq 0$, since C_1, $F_{1,2}$ and H are all real-valued, the vanishing of the imaginary part of the complex ODE for H requires $H = 0$, i.e., that we have

$$\begin{aligned} 0 &= -F_1'' + F_2 F_1' + \tfrac{1}{3}\Lambda F_1^3 - \tfrac{1}{4}(F_2^2 - 3F_2' + C_1^2)F_1, \\ 0 &= -F_1'' F_1 + (F_1')^2 + \Lambda F_1^4 + F_2' F_1^2. \end{aligned} \tag{4.12}$$

In the following discussion, we will focus on this second case (4.12) regardless of C_1 being zero or not. An example of solutions with $C_1 = 0$ and $H \neq 0$ is given in Appendix D, which is shown to have the *hyperquadric* CR structure.

For the system (4.12), the satisfaction of the second ODE is given by introducing a single new, real-valued function $J = J(z)$ such that

$$F_1 = \pm\sqrt{J'}, \qquad F_2 = \frac{J''}{2J'} - \Lambda J. \tag{4.13}$$

Then the first ODE simply becomes

$$J''' = \frac{(J'')^2}{2J'} - 2\Lambda J J'' - \frac{10}{3}\Lambda(J')^2 - 2(\Lambda^2 J^2 + C_1^2)J'. \tag{4.14}$$

Since this ODE does not have the argument z appearing explicitly, we can lower the order of the ODE through the standard transformation

$$J' = P(J) \Longrightarrow J'' = PP' \Longrightarrow J''' = P(PP')'$$

and obtain an even simpler equation of the second-order

$$P'' = -\frac{(P' + 2\Lambda J)^2}{2P} - \frac{2C_1^2}{P} - \frac{10}{3}\Lambda. \tag{4.15}$$

A solution $P = P(J)$ to (4.15) can give rise to a solution $J = J(z)$ to (4.14) at least locally by inverting

$$z + C_0 = \int \frac{1}{P(J)} \mathrm{d}J \tag{4.16}$$

with C_0 constant. This solution will be physical if it also makes $F_{1,2}(z)$ real-valued via (4.13), which requires that locally

$$P(J) > 0,\ J' > 0 \text{ and } J \text{ real-valued.} \tag{4.17}$$

Therefore we are only interested in solutions for $J(z)$ that are monotonically increasing, or equivalently, positive $P(J)$.

We can also consider the special case of (4.15) with $C_1 = 0$ and $\Lambda \neq 0$, i.e.,

$$P'' = -\frac{(P' + 2\Lambda J)^2}{2P} - \frac{10}{3}\Lambda. \tag{4.18}$$

By introducing the following integral transformation

$$J = \frac{1}{\Lambda} \exp\left(\int f(t)\ \mathrm{d}t\right), \qquad P(J) = \frac{t}{\Lambda} \exp\left(2 \int f(t)\ \mathrm{d}t\right),$$

of which the inverse has the form

$$t = \frac{P}{\Lambda J^2}, \qquad f(t) = \frac{\Lambda J^2}{J P' - 2P},$$

we can further reduce (4.18) to an Abel ODE of the first kind [45], as already noted in Section 1.1:

$$f' = \frac{4}{t}\left(t + \frac{3}{2}\right)\left(t + \frac{1}{3}\right) f^3 + \frac{5}{t}\left(t + \frac{2}{5}\right) f^2 + \frac{1}{2t}\ f. \tag{4.19}$$

Once the general solution $f = f(t, C_2)$ is acquired with a constant C_2, we can find the general solution $P(J)$ of (4.18) by solving the following ODE

$$f\left(\frac{P}{\Lambda J^2}, C_2\right) = \frac{\Lambda J^2}{J P' - 2P},$$

of which the solution is given by

$$P(J) = Z(J) J^2, \quad \text{with} \ \ 0 = -\ln J + \int^{Z/\Lambda} f(t, C_2)\ \mathrm{d}t + C_3. \tag{4.20}$$

Simple as both (4.15) and (4.19) may appear, so far we have had no luck finding their explicit general solutions. For more comments on (4.19) and Abel ODEs in general, see Appendix A.

4.4 CR Equivalency as Classical Symmetry

To identify new twisting type N Einstein spaces obtained from (4.14), we will use Theorem 2.3 as a natural way to classify metrics equipped with CR structures.

By definition, a type N spacetime at each point has a unique PND pointing in the direction of the null congruence. In the case of vacuums (with or without Λ), this

PND must be geodesic and shearfree [30]. Thus for every twisting type N Einstein space, the shearfree null congruence is unique. Hence by Theorem 2.3, to confirm a new twisting type N vacuum metric, it is sufficient to show that its CR structure is distinct from the one of known metrics. This can be routinely done by computing the six Cartan invariants (see Section 2.2), which are denoted respectively by

$$\alpha_I, \theta_I, \eta_I \text{ (complex)},$$
$$\beta_I, \gamma_I, \zeta_I \text{ (real)}.$$

Cartan showed that two local CR structures are equivalent iff their six CR invariants (defined when $r \neq 0$ in (4.21)) are identical, except possibly for a sign difference in both α_I and η_I [19]. With the assumption of u-independence, we can write down, for instance, the simplest invariant computed from the 1-forms defined in (2.19):

$$\begin{aligned} &\alpha_I(\zeta,\bar\zeta) = -\frac{5\bar r \partial_\zeta r + r\partial_\zeta \bar r + 8cr\bar r}{8\varepsilon\sqrt{\bar r}\cdot\sqrt[8]{(r\bar r)^7}}, \qquad \varepsilon = \pm 1, \\ &r = \tfrac{1}{6}\left(\partial_{\bar\zeta}\bar l + 2\bar c\bar l\right), \qquad l = -\partial_\zeta\partial_{\bar\zeta}c - c\partial_{\bar\zeta}c. \end{aligned} \tag{4.21}$$

Here the function r ($r \propto \mathcal{R}$ in (2.8) [19]), following the notation of Cartan, is not to be confused with the coordinate r along the null congruence. For our calculated β_I, γ_I, θ_I and η_I, see Appendix B. Note that this α_I only relies on $c(\zeta,\bar\zeta)$, $\bar c$ and their derivatives, which is also the case for all the other Cartan invariants. We first point out a remarkable feature of these invariants computed from the ansatz (4.9,4.10).

Theorem 4.2. Given the ansatz (4.9,4.10), all the following quantities are independent of $A(\zeta)$ and $\bar A(\bar\zeta)$ except those in the argument z:

$$\alpha_I^2, \eta_I^2, \alpha_I\bar\eta_I, \beta_I, \gamma_I, \theta_I, \zeta_I.$$

In another word, they are all functions of z only, e.g., $\beta_I(\zeta,\bar\zeta) = \beta_I(z)$.

Proof. Except for a lengthy but straightforward symbolic computation with Maple, we are, at the moment, still not aware of any other more insightful way of proving this result. Here we only emphasize that the law $\sqrt{v}\sqrt{w} = \sqrt{vw}$ is in general *not* true in the complex domain; failing to notice this may cause an erroneous conclusion. □

Remark. For a fixed z, the presence of the functions A and $\bar{A}$ in α_I and η_I themselves only affects their signs. More specifically, the only dependence on A and $\bar{A}$ takes the following forms:

$$\alpha_I \propto \frac{1}{A(\zeta)}\sqrt{\frac{A^2(\zeta)}{F(z)}}, \qquad \eta_I \propto \frac{1}{\bar{A}(\bar{\zeta})}\sqrt{\frac{\bar{A}^2(\bar{\zeta})}{\bar{F}(z)}},$$

$$F(z) = -F_2''' + (F_2')^2 + 3F_2F_2'' - 2F_2^2F_2' + 2C_1^2F_2' + \mathrm{i}C_1(3F_2'' - 4F_2F_2').$$

Hence the product $\alpha_I\bar{\eta}_I$ is a function of z only. According to Cartan [19], this sign situation is accounted for by a local CR diffeomorphism and therefore does not generate a new CR structure. Hence, we have proved the following theorem.

Theorem 4.3. Locally, the CR structure (2.19) (as an equivalence class) determined by the function c given in (4.9,4.10) is independent of the choice of the function $A(\zeta) \neq 0$, once the form of $F_2(z)$ is fixed.

Altogether, the theorem tells us that locally the freedom of choosing various $A(\zeta) \neq 0$ does not affect the CR structure of a type N metric of our concern. Hence, for the simplicity of representing new metrics distinguished by CR structure, we can just set $A(\zeta) = \bar{A}(\bar{\zeta}) = 2$ (see Section 9.1). In hindsight, the classical symmetries we have obtained are nothing more than a particular manifestation of the underlying CR equivalency. We believe this connection between the two may as well suggest a more general concern if one aims to find, through the (classical or higher) symmetries, additional exact solutions to the Einstein equations formulated with CR structures.

We will see later examples of solutions that have *constant* CR invariants, and remarkably, one of them is the solution of Leroy-Nurowski. Nonetheless, this feature is generally not true for other solutions.

Chapter 5

Examples of Exact Solutions

5.1 Conformally Flat Solutions

Before we try to solve (4.14) for type N solutions, it is useful to find out in advance those conformally flat solutions satisfying $\Psi_4 = 0$ which are automatically contained in the general solution of (4.14). We insert the ansatz (4.9,4.10) into the expression for Ψ_4 given by (4.7), and re-normalize Ψ_4 to pull out just a simple complex-valued function of z:

$$\begin{aligned} K(z) &:= -\frac{3\bar{A}^2 F_1^2 \mathrm{e}^{\mathrm{i}r/2}}{4\Lambda \cos^3\left(\frac{r}{2}\right)} \Psi_4 \\ &= 2F_1 F_1'' + 6(F_1')^2 - 10(F_2 + \mathrm{i}C_1)F_1 F_1' + (-F_2' + 3F_2^2 + 6\mathrm{i}C_1 F_2 - 3C_1^2)F_1^2. \end{aligned} \tag{5.1}$$

We now apply (4.13) and use (4.14) to substitute for J''', which gives us

$$K = \left[\Lambda J J'' - \tfrac{2}{3}\Lambda (J')^2 + 2(\Lambda^2 J^2 - 2C_1^2)J'\right] + \mathrm{i}\left[-2C_1\left(J'' + 3\Lambda J J'\right)\right], \tag{5.2}$$

or in terms of $P(J)$,

$$K = P\left[\Lambda J P' - \tfrac{2}{3}\Lambda P + 2\Lambda^2 J^2 - 4C_1^2\right] + \mathrm{i}\left[-2C_1 P\left(P' + 3\Lambda J\right)\right],$$

where we have put the real and imaginary parts in separate brackets. Replacing J'' with the help of (5.2) being zero, we can rewrite the equation (4.14) as

$$0 = \tfrac{1}{3}\Lambda K(J')^2 - (2\Lambda KJ + K')(\Lambda J - 2\mathrm{i}C_1)J' + \tfrac{1}{2}K^2$$

which clearly has $K = 0$, i.e, all conformally flat solutions, as some of its solutions.

If $P(J)$ is not restricted to the real domain, then solving the first-order ODE $K = 0$ for $P(J)$ leads to the following general solution

$$P(J) = -\frac{3}{2}\Lambda\left(J^2 + \frac{4C_1^2}{\Lambda^2}\right) + C_2\left(J \pm \frac{2\mathrm{i}C_1}{\Lambda}\right)^{2/3}. \tag{5.3}$$

with a complex constant C_2.

If, instead, we restrict $P(J)$ to be real, a simultaneous vanishing of the real and imaginary parts of K respectively yields the following set of two equations, provided $P \neq 0$,

$$0 = C_1(P' + 3\Lambda J),$$
$$P' = \frac{2P}{3J} - 2\Lambda J + \frac{4C_1^2}{\Lambda J},$$

both of which are consistent with (4.15). There are now two cases for solutions.

The case $C_1 \neq 0$ requires that both ODEs be satisfied, so that we have a unique solution

$$P(J) = -\frac{3}{2}\Lambda J^2 - \frac{6C_1^2}{\Lambda} \tag{5.4}$$

which, by solving $J' = P(J)$, gives rise to

$$J = \frac{2C_1}{\Lambda\tan(3C_1(z + C_0))}. \tag{5.5}$$

In the limit $C_1 \to 0$, the above solution becomes even simpler[1]:

$$J = \frac{2}{3\Lambda(z + C_0)}. \tag{5.6}$$

[1] Both (5.5) and (5.6) would be of particular importance for perturbation theory on type N solutions near conformally flat ones.

From (5.5), we have

$$F_1 = \pm\frac{\sqrt{6}C_1}{s\sin(3C_1(z+C_0))}, \qquad F_2 = -\frac{5C_1}{\tan(3C_1(z+C_0))} \tag{5.7}$$

with negative-valued $\Lambda = -s^2$. Note that it is only at this stage that the reality condition on $F_{1,2}$, i.e., $J' > 0$, requires $\Lambda < 0$, i.e, a negative cosmological constant. An important remark that can be made is that the extended form of the Leroy-Nurowski solution (see the next section) resembles this solution greatly, with simply differences in the numerical coefficients.

For the other case when $C_1 = 0$, we have

$$P(J) = -\tfrac{3}{2}\Lambda J^2 + C_2 J^{2/3}. \tag{5.8}$$

with a real constant C_2. From (4.16), the solution $J(z)$ is determined by

$$\int \frac{1}{-\frac{3}{2}\Lambda J^2 + C_2 J^{2/3}}\mathrm{d}J = z + C_0. \tag{5.9}$$

Since $J' > 0$, we cannot have both $\Lambda > 0$ and $C_2 \leq 0$. Hence, we can discuss three other sign possibilities, the details of which are put in Appendix C.

We note that the special solution (5.6) corresponding to $C_1 = C_2 = 0$ serves as the single "point" where these two families of conformally flat solutions are joined up.

Modulo possible sign differences in α_I and η_I caused by square roots as already discussed, the Cartan invariants for both (5.5) and (5.6), as calculated via (4.21) and the equations for the other invariants, as presented in Appendix B, are given by

$$\begin{aligned} &\alpha_I = -\frac{4\mathrm{i}}{\varepsilon}\sqrt[4]{\frac{2}{5}}, \qquad \beta_I = \frac{41}{2\sqrt{10}}, \qquad \gamma_I = \frac{29}{2\sqrt{10}}, \\ &\theta_I = 3\mathrm{i}\sqrt{\frac{2}{5}}, \qquad \eta_I = -\frac{\mathrm{i}}{\varepsilon}\cdot\frac{2^{19/4}}{5^{3/4}}, \qquad \zeta_I = -\frac{327}{40}, \qquad \varepsilon = \pm 1. \end{aligned} \tag{5.10}$$

Remarkably, they are all constant and do not depend on C_1. Nonetheless, this is not the case for the other conformally flat solutions obtained from (5.9) with $C_2 \neq 0$

of which the Cartan invariants are generally functions of z and C_2. For instance, simplified by (5.8) and $J' = P(J)$, the first Cartan invariant satisfies

$$\alpha_I^2(z, C_2) = -16\sqrt{\frac{2}{5}}\left(\frac{3\Lambda J^{4/3} + 2C_2}{3\Lambda J^{4/3} - 2C_2}\right)^2, \tag{5.11}$$

where $J = J(z)$ belongs to one of the three cases described in Appendix C.

Two conformally flat Einstein spaces may have non-equivalent CR structures. This does not conflict with Theorem 2.3 because in a conformally flat spacetime, one is free to make different choices from among the multiple shearfree null congruences and therefore may have non-equivalent CR structures attached to them.

5.2 An Extended Form of the Leroy-Nurowski Solution

Now we can reveal, to a fuller extent, the exact twisting type N solution first discovered by Leroy, and re-derived by Nurowski within the framework of CR geometry, upon the latter of which our current work is mainly based. We hope that our derivation of this solution will make the process behind the previous discoveries appear clearer.

Given Nurowski's form of the solution (see [15] or (5.19)) and recasting it into the form of the ansatz (4.9,4.10) and (4.13), we find the following special solution to (4.15)

$$P(J) = -\frac{1}{3}\Lambda J^2 - \frac{3C_1^2}{4\Lambda} \tag{5.12}$$

which gives rise to a solution to (4.14):

$$J = \frac{3C_1}{2\Lambda\tan(\frac{1}{2}C_1(z + C_0))}. \tag{5.13}$$

In the limit $C_1 \to 0$, the above expression becomes even simpler:

$$J = \frac{3}{\Lambda(z + C_0)}, \tag{5.14}$$

which is quite similar to that of (5.6). Back to the case with $C_1 \neq 0$, using (4.13), we have

$$F_1 = \pm \frac{\sqrt{3}C_1}{2s\sin(\frac{1}{2}C_1(z + C_0))}, \qquad F_2 = -\frac{2C_1}{\tan(\frac{1}{2}C_1(z + C_0))} \tag{5.15}$$

with a negative $\Lambda = -s^2$. Note that it is only at this stage that the reality condition on F_1 requires $\Lambda < 0$. In the end, our extended version of the Leroy-Nurowski solution takes the form

$$p(\zeta, \bar{\zeta}) = \pm \frac{\mathrm{i}\sqrt{3}C_1}{2s\sinh\left(\frac{\mathrm{i}}{2}C_1(z + C_0)\right)\sqrt{A\bar{A}}}, \tag{5.16}$$

$$c(\zeta, \bar{\zeta}) = \frac{1}{A}\left[\partial_\zeta A + \frac{2C_1}{\tanh\left(\frac{\mathrm{i}}{2}C_1(z + C_0)\right)} + C_1\right], \tag{5.17}$$

$$z = -\mathrm{i}\left(\int \frac{1}{A}\mathrm{d}\zeta - \int \frac{1}{\bar{A}}\mathrm{d}\bar{\zeta}\right). \tag{5.18}$$

The flexibility of choosing the function $A(\zeta)$ and real constant $C_{0,1}$ may perhaps facilitate a possible future application of the solution. From this extended version, one can obtain the original form of Nurowski [15] by setting

$$A(\zeta) = C_1\zeta, \qquad \bar{A}(\bar{\zeta}) = C_1\bar{\zeta}, \qquad C_0 = 0,$$

and consequently,

$$p(\zeta, \bar{\zeta}) = \pm \frac{\mathrm{i}\sqrt{3}}{s(\zeta - \bar{\zeta})}, \; c(\zeta, \bar{\zeta}) = \frac{4}{\zeta - \bar{\zeta}}, \; \Psi_4 = \frac{14s^2}{3y^2}\mathrm{e}^{-\mathrm{i}r/2}\cos^3\left(\frac{r}{2}\right). \tag{5.19}$$

Note that all C_1's are canceled out in the above expressions. Hence another way of obtaining (5.19) is by taking the limit $C_1 \to 0$ in (5.16) and (5.17) (cf. (5.14)) and setting $C_0 = 0$ and $A(\zeta) = 2$.

Modulo possible sign differences in α_I and η_I caused by square roots as already discussed, the Cartan invariants calculated from (5.13) and (5.14) are both given by

$$\begin{aligned} &\alpha_I = \frac{1}{\varepsilon}\sqrt{\frac{1}{2}\sqrt{\frac{3}{5}}}, \qquad \beta_I = -\frac{1}{2}\sqrt{\frac{3}{5}}, \qquad \gamma_I = \frac{1}{2}\sqrt{\frac{3}{5}}, \\ &\theta_I = \mathrm{i}\sqrt{\frac{3}{5}}, \qquad \eta_I = -\frac{1}{\varepsilon}\cdot\frac{2^{3/2}}{3^{1/4}\cdot 5^{3/4}}, \qquad \zeta_I = -\frac{1}{20}, \qquad \varepsilon = \pm 1. \end{aligned} \tag{5.20}$$

Like (5.10), they are all constant and do not depend on C_1.

Chapter 6

Painlevé Analysis

6.1 Weak Painlevé Tests

Now we have three ODEs, the equations (4.14) for $J(z)$, (4.15) for $P(J)$, and (4.19) of the Abel type at hand that may be explored for new twisting type N solutions. A particular, probably useful way to decide which one of these equations has a better chance for finding a solution is given by the (weak) Painlevé test [46, 47, 48]. This test reveals the nature of the movable singularities (poles, branch points, ...) of the general solution of a nonlinear ODE. Failing the test means the occurrence of certain undesirable movable singularities, e.g., infinitely branched singularities, that relate to non-integrability [48] or even chaoticity, although it may still be possible to find special solutions. Associated to the (weak) Painlevé test is the global property called the (weak) Painlevé property. An ODE possesses the Painlevé property if the general solution can be made single-valued (e.g., all movable singularities are poles). Examples of such are all linear ODEs, the elliptic equation and, most noteworthy, the six Painlevé equations. However, the weak Painlevé property only requires that the general solution be at most finitely branched around any movable singularity.

The tests themselves are by design sets of necessary conditions respectively for these properties.

In this section, we will show that none of the three ODEs pass the Painlevé test, and that (4.14) also fails the weak Painlevé test while the other two pass. To begin, we detail the test procedures on (4.15). Then we briefly comment on the Abel equation (4.19) and simply point out where the tests fail for (4.14) without dwelling on details.

The equation (4.15) surely does not have the Painlevé property since the coefficient of the $(P')^2$ term clearly violates the necessary conditions for the Painlevé property [46] (see p. 127). This is also confirmed by the test conclusion that (4.15) has movable algebraic singularities.

Step 1 (*Dominant behaviours*). Assume the leading behaviour of a solution $P(J)$ to be

$$P \sim u_0\chi^m, \qquad \chi = J - J_0, \qquad u_0 \neq 0, \qquad m \neq 0,$$

with m not a positive integer. Substitute this form into (4.15) and select out all possible lowest order terms as listed below

$$\tfrac{3}{2}u_0^2 m \left(m - \tfrac{2}{3}\right)\chi^{2m-2}, \qquad 2\Lambda u_0 J_0 m\chi^{m-1}, \qquad 2(\Lambda^2 J_0^2 + C_1^2).$$

Since $m \neq 1$, we only have two possibilities. For $m < 1$, χ^{2m-2} is the lowest order term and the vanishing of its coefficient requires

$$m = \frac{2}{3}$$

given $u_0, m \neq 0$. For $m > 1$, the constant $2(\Lambda^2 J_0^2 + C_1^2)$ is the lowest order term, which does not vanish in general, hence not interesting for the purpose. To summarize, we obtain $m = \frac{2}{3}$ with arbitrary $u_0 \neq 0$, i.e, we find that

$$P \sim u_0(J - J_0)^{2/3}$$

is the only detected dominant behaviour.

Step 2 (*Resonance conditions* [46] (see p. 87)). Having found the dominant behaviour, now we consider the possibility to extend it to a Puiseux series expansion

$$P = \sum_{j=0}^{\infty} u_j (J - J_0)^{(j+2)/3}.$$

This requires the determination of the locations $(j+2)/3$, called *Fuchs indices* or *resonances*, where arbitrary coefficients may enter the Puiseux series. Consider the dominant terms

$$\hat{E}(J, P) = PP'' + \tfrac{1}{2}(P')^2$$

of (4.15) that contribute to the leading behaviour $\chi^{2m-2} = \chi^{-2/3}$. Then compute the derivative

$$\lim_{\epsilon \to 0} \frac{\hat{E}(J, P + \epsilon V) - \hat{E}(J, P)}{\epsilon} = (P\partial_J^2 + P'\partial_J + P'')V.$$

The Fuchs indices satisfy the so-called *indicial equation*

$$\lim_{\chi \to 0} \chi^{-j-(2m-2)}(P\partial_J^2 + P'\partial_J + P'')\chi^{j+m} = u_0(j+1)j = 0.$$

Hence we obtain a fractional resonance at $(j+2)/3 = \frac{2}{3}$ with $j = 0$.

Step 3 (*Compatibility conditions*). At $j = 0$, we know, from the the first step, that $u_0 (\neq 0)$ is indeed an arbitrary coefficient. This completes the test. In conclusion, (4.15) passes the weak Painlevé test.

Remark. Note that no *pole* is detected from the test above. The ODE for P^3 still involves a Puisuex series instead of a Laurent series since the cubing does not eliminate all third roots of χ. According to [48], the presence of movable algebraic singularities is not incompatible with integrability.

The very design of the weak Painlevé test limits its usage only as necessary conditions for the weak Painlevé property. The test can neither detect movable (branched) essential singularities themselves nor exclude an accumulation of algebraic singularities forming a movable essential one that may be severely branched. These possibilities make a rigorous proof of the weak Painlevé property not at all a trivial one, which by itself may deserve a specialized article to discuss. See examples in [49, 50, 51].

According to Painlevé [52, 53], the only movable singularities of solutions to the first-order ODE $y' = F(x, y)$ where F is rational in y with coefficients that are algebraic functions of x, are poles and/or algebraic branch points. In addition, the only nonlinear ODE in this class that has the Painlevé property is the Riccati equation which (4.19) is certainly not. Hence the equation (4.19) automatically has the weak Painlevé property, but not the Painlevé property, and it is free from movable essential singularities.

The equation (4.14) admits two families of dominant behaviours (cf. (5.6) and (5.14)):

$$
\begin{aligned}
J &\sim \frac{2}{3\Lambda(z-z_0)}, \qquad \text{Fuchs indices} = -1, \frac{4}{3}, \frac{7}{3}; \\
J &\sim \frac{3}{\Lambda(z-z_0)}, \qquad \text{Fuchs indices} = -1, -\frac{1+\sqrt{57}}{2}, -\frac{1-\sqrt{57}}{2}.
\end{aligned}
$$

It fails the weak Painlevé test for having *irrational* resonances. This means that (4.14) has an infinitely branched movable singularity, which is a strong indicator for non-integrability [48].

Since our attempt of solving (4.19) has not been successful, we decided to focus on (4.15) and explore some of its features that may facilitate constructing new solutions.

6.2 Puiseux Series Solutions

As indicated by the weak Painlevé test, the ODE (4.15) for $P(J)$ possesses a formal Puiseux series solution

$$\begin{aligned}P=&\sum_{k=0}^{\infty}u_k(J-J_0)^{(k+2)/3}\\=&u_0(J-J_0)^{2/3}-3\Lambda J_0(J-J_0)-\frac{9(\Lambda^2J_0^2+4C_1^2)}{20u_0}(J-J_0)^{4/3}\\&-\frac{3\Lambda J_0(\Lambda^2J_0^2+4C_1^2)}{5u_0^2}(J-J_0)^{5/3}\\&-\left[\frac{3}{2}\Lambda+\frac{27(109\Lambda^2J_0^2+36C_1^2)(\Lambda^2J_0^2+4C_1^2)}{2800u_0^3}\right](J-J_0)^2+\cdots\end{aligned}\tag{6.1}$$

with two arbitrary complex constants $u_0\neq 0$ and J_0. In particular, this Puiseux series solution contains a special case for $J_0=\pm 2\mathrm{i}C_1/\Lambda$ ($\Lambda\neq 0$) such that

$$P=u_0\left(J\pm\frac{2\mathrm{i}C_1}{\Lambda}\right)^{2/3}-\frac{3}{2}\Lambda\left(J^2+\frac{4C_1^2}{\Lambda^2}\right).$$

This finite expression coincides with the known solution (5.3) (setting $u_0=C_2$).

Theorem 6.1. Given that $u_0\neq 0$ and $u_0,\ J_0\in\mathbb{C}$, the ODE (4.15) admits a formal Puiseux series solution (6.1) such that it converges in a neighborhood of J_0.

Proof. The idea of the proof, following many standard proofs of the Painlevé property, is to convert the Puiseux series into a power series solution of a regular initial value problem (e.g., [49, 51]). First we define

$$Z=P^{1/2}(P'+4\Lambda J).\tag{6.2}$$

Then differentiate it once with respect to J and substitute P'' using (4.15). Hence we obtain

$$Z'=\frac{2(\Lambda P-3\Lambda^2J^2-3C_1^2)}{3P^{1/2}}.\tag{6.3}$$

The system (6.2,6.3) is equivalent to the ODE (4.15). Now by introducing a new variable $U = P^{1/2}$, we can transform the system into

$$\begin{aligned} \frac{dJ}{dU} &= \frac{2U^2}{Z - 4\Lambda UJ}, \\ \frac{dZ}{dU} &= -\frac{4(3\Lambda^2 J^2 - \Lambda U^2 + 3C_1^2)U}{3(Z - 4\Lambda UJ)}. \end{aligned} \tag{6.4}$$

which has a unique power series solution about $U = 0$

$$\begin{aligned} J &= J_0 + \frac{2}{3Z_0}U^3 + \cdots, \\ Z &= Z_0 - \frac{2(\Lambda^2 J_0^2 + C_1^2)}{Z_0}U^2 + \cdots. \end{aligned} \tag{6.5}$$

By the Cauchy existence and uniqueness theorem, both series have non-vanishing radii of convergence. From the series (6.5), the corresponding solutions to (6.2,6.3) then take the form

$$\begin{aligned} P &= \left[\frac{3Z_0}{2}(J - J_0)\right]^{2/3} + \sum_{k=1}^{\infty} u_k (J - J_0)^{(k+2)/3}, \\ Z &= Z_0 + \sum_{k=0}^{\infty} v_k (J - J_0)^{(k+2)/3}. \end{aligned}$$

with $Z_0 \neq 0$. This completes the proof. □

The series (6.1) clearly contains type N solutions that are not equivalent to Leroy-Nurowski's since they all continuously deform to the conformally flat solution (5.8) in the limit $J_0 \to 0$, $C_1 \to 0$. We already know that the latter has a non-constant Cartan invariant α_I given by (5.11).

Chapter 7

Constructing New Solutions

7.1 An Example of Power Series Solutions

For simplicity, assume that $C_1 = 0$ in the ODE (4.14). Now consider the power series solution of (4.14) satisfying the regular initial conditions $J(0) = 0$, $J'(0) = u_0 > 0$ and $J''(0) = 0$. A simple calculation gives us the first few terms of this series

$$J(z) = \sum_{i=0}^{\infty} u_i z^{i+1} = u_0 z - \tfrac{5}{9}\Lambda u_0^2 z^3 + \tfrac{16}{45}\Lambda^2 u_0^3 z^5 + \cdots , \tag{7.1}$$

which is an odd function of z. Moreover, this series solution, convergent in a neighborhood of $z = 0$ according to the Cauchy existence and uniqueness theorem, is of type N with a non-vanishing Weyl scalar $\Psi_4 \propto K(z)$. Particularly,

$$K(0) = -\tfrac{2}{3}\Lambda u_0^2 \neq 0.$$

To see that the solution (7.1) is not equivalent to the Leroy-Nurowski solution, we calculate the first Cartan invariant $\alpha_I(z)$ via (4.21) which, in this case, is no longer a constant. In particular, this series solution has

$$\alpha_I(0) = 0,$$

with $\alpha_I(z)$ and also $K(z)$ continuous at $z = 0$, while the values given in (5.20) are always nonzero constants. This is sufficient to assert that the ODE (4.14) as well as its reductions (4.15) and (4.19) indeed contain new twisting type N solutions.

7.2 One-Parameter Deformation from a Conformally Flat Solution to the Leroy-Nurowski Solution

One feature that makes (4.15) preferable to the other two ODEs (4.14) and (4.19) is that the conformally flat solution (5.4) and the extended Leroy-Nurowski solution (5.12) are just simple quadratic functions, without poles in the complex plane, compared to their counterparts (5.5) and (5.13). Also note that these quadratic solutions with $C_1 = 0$ do not correspond to any solution of the Abel equation (4.19) since the form (4.20) with the non-constant function $Z(J)$, excludes all quadratic functions as solutions. These well-behaved quadratic solutions facilitate a study of the power series solutions near them, which complements the Puiseux series solutions presented in Section 6.2.

To simplify the notation, we apply the scaling transformation[1] $J = C_1 w/\Lambda$, $P(J) = C_1^2 g(w)/\Lambda$ with $\Lambda \neq 0$, $C_1 \neq 0$ such that (4.15) takes on the form already noted as (1.1) with $C = 1$:

$$g'' = -\frac{(g' + 2w)^2}{2g} - \frac{2}{g} - \frac{10}{3}. \tag{7.2}$$

We look for power series solutions for this equation corresponding to the regular

[1] Once having a solution $g(w)$, one may choose a sign for Λ in order to have $P(J) > 0$.

initial conditions $g(0) = u_0 \neq 0$, $g'(0) = 0$. The first few terms of this series read

$$\begin{aligned} g(w) =& \sum_{j=0}^{\infty} u_j w^j \\ =& u_0 - \frac{5\left(u_0 + \frac{3}{5}\right)}{3u_0} w^2 - \frac{2\left(u_0 + \frac{3}{4}\right)\left(u_0 + 6\right)}{27u_0^3} w^4 \\ &- \frac{76\left(u_0 + \frac{3}{4}\right)\left(u_0 + 6\right)\left(u_0 + \frac{33}{38}\right)}{1215u_0^5} w^6 + \cdots , \end{aligned} \tag{7.3}$$

where all odd order terms vanish. The remainder of the coefficients in the series can be determined by a recursion relation which is valid beginning with u_6:

$$0 = (2k+1)(k+1)u_0 u_{2k+2} + \left(2k + \tfrac{5}{3}\right) u_{2k} + \sum_{l=0}^{k-1}(k+l+1)(l+1)u_{2l+2}u_{2k-2l}, \; k \geq 2, \tag{7.4}$$

while u_2 and u_4 can be easily read off from (7.3). It is clear that this relation allows one to calculate the coefficients to whatever order desired. One can easily see that the coefficient of w^{2k}, namely u_{2k}, is a kth-order polynomial, $P_k(u_0)$, divided by $u_0^{2k-1} \neq 0$. Remarkably, this infinite series reduces to simple quadratic functions in two special cases. The reason for this is that for every value of $k \geq 2$, the polynomial $P_k(u_0)$ has the factors $\left(u_0 + \frac{3}{4}\right)(u_0+6)$, as can be seen in the few terms demonstrated in (7.3) above and can easily be shown by induction. Hence for $u_0 = -\frac{3}{4}$, we retrieve the Leroy-Nurowski solution (5.12), which in this notation is simply

$$g_{LN} = -\left(\tfrac{1}{3}w^2 + \tfrac{3}{4}\right). \tag{7.5}$$

As well, for $u_0 = -6$, we retrieve a conformally flat solution (5.4), which has the form

$$g_{CF} = -\left(\tfrac{3}{2}w^2 + 6\right). \tag{7.6}$$

For all other values of $u_0 \neq 0$, the formal series solution (7.3) may then be viewed as a generalization of these two known solutions, in terms of a power series with infinitely many terms. It is interesting that in every one of these polynomials, $P_k(u_0)$, all coefficients are negative, so that the only possible real roots would be negative. Our numerical calculations suggest that none are smaller than -6, and that there are no other roots common to all these different polynomials.

The series (7.3) does define, in the complex domain, a function holomorphic in some neighborhood of the origin as is shown by the following method of determining a nonzero radius of convergence for it.

Theorem 7.1. Given the series (7.3) with the recursion relation (7.4) and a fixed $u_0 \neq 0$, one has the following bound:

$$|u_{2j}| \leq \frac{CM^{2j}}{(2j)^2}, \; j = 2, 3, \cdots , \tag{7.7}$$

provided that one can pick two constants $C > 0$ and $M > 0$ such that they satisfy

$$\left| \frac{2\left(u_0 + \frac{3}{4}\right)(u_0 + 6)}{27u_0^3} \right| \leq \frac{CM^4}{16}, \tag{7.8}$$

$$\left(\frac{5}{3} + \frac{1}{|u_0|} \right) \frac{9}{4M^2} + \left(\frac{\pi^2}{12} - \frac{1}{4} \right) C \leq |u_0|. \tag{7.9}$$

Proof. The induction begins with

$$|u_4| \leq \frac{CM^4}{16}.$$

which holds by the assumption (7.8). Now assume that for $k \geq 2$ and $j = 2, \cdots , k$, the bound (7.7) is true. Then for $k \geq 3$ and $1 \leq l \leq k-2$, we can bound the product $u_{2l+2}u_{2k-2l}$ by

$$\begin{aligned} |u_{2l+2}u_{2k-2l}| &\leq \frac{C^2M^{2k+2}}{(2l+2)^2(2k-2l)^2} \\ &\leq 2\left[\frac{(2k-2l)^2 + (2l+2)^2}{(2k+2)^2} \right] \frac{C^2M^{2k+2}}{(2l+2)^2(2k-2l)^2} \\ &= 2\left[\frac{1}{(2l+2)^2} + \frac{1}{(2k-2l)^2} \right] \frac{C^2M^{2k+2}}{(2k+2)^2}. \end{aligned} \tag{7.10}$$

The second inequality above is due to $(a^2+b^2)/(a+b)^2 \geq \frac{1}{2}$. Rearranging (7.4) and using the triangular inequality together with (7.10), we obtain an upper bound for $|u_{2k+2}|$:

$$\begin{aligned}
|u_{2k+2}| \leq & \frac{\left(2k+\frac{5}{3}\right)|u_{2k}| + (2k^2+k+1)|u_2 u_{2k}|}{(2k+1)(k+1)|u_0|} + \frac{\sum_{l=1}^{k-2}(k+l+1)(l+1)|u_{2l+2}u_{2k-2l}|}{(2k+1)(k+1)|u_0|} \\
\leq & \frac{\left(2k+\frac{5}{3}\right) + (2k^2+k+1)|u_2|}{(2k+1)(k+1)|u_0|} \cdot \frac{CM^{2k}}{(2k)^2} \\
& + \frac{S(k)}{2(2k+1)(k+1)|u_0|} \cdot \frac{C^2 M^{2k+2}}{(2k+2)^2}, \quad k \geq 2,
\end{aligned} \tag{7.11}$$

where we define

$$S(k) = \sum_{l=1}^{k-2} \frac{(k+l+1)(l+1)}{(l+1)^2} + \sum_{l=1}^{k-2} \frac{(k+l+1)(l+1)}{(k-l)^2}, \; k \geq 3, \text{ and } S(2) = 0$$

We can evaluate the first summation above in terms of the digamma function

$$\sum_{l=1}^{k-2} \frac{(k+l+1)(l+1)}{(l+1)^2} = k\Psi(k) - (2-\gamma)k \leq k\Psi(k)$$

where γ is Euler's constant, which is approximately $0.57721\cdots$. The second summation has the following bound

$$\begin{aligned}
&\sum_{l=1}^{k-2} \frac{(k+l+1)(l+1)}{(k-l)^2} = \sum_{l=1}^{k-2} \frac{(2k-l)(k-l)}{(l+1)^2} \\
&\leq 2k^2 \sum_{l=1}^{k-2} \frac{1}{(l+1)^2} = \left(\frac{\pi^2}{3} - 2\right)k^2 - 2k^2\Psi(1,k) \leq \left(\frac{\pi^2}{3} - 2\right)k^2.
\end{aligned}$$

Note that the trigamma function $\Psi(1,k) \geq 0$ for all integers $k \geq 3$ and that $\Psi(1,k) \sim k^{-1}$ for $k \to +\infty$. Combining these two bounds, for $k \geq 3$, we obtain

$$\begin{aligned}
\frac{S(k)}{2(2k+1)(k+1)} &\leq \frac{\left(\pi^2/3 - 2\right)k^2 + k\Psi(k)}{2(2k+1)(k+1)} \\
&\leq \frac{\left(\pi^2/3 - 2\right)k^2 + k^2}{4k^2} = \frac{\pi^2}{12} - \frac{1}{4},
\end{aligned}$$

where we use the fact that $0 \le \Psi(k) \le k$ for all integers $k \ge 3$. In addition, the first term in (7.11) is bounded by

$$\begin{aligned}
\frac{\left(2k+\frac{5}{3}\right)+(2k^2+k+1)|u_2|}{(2k+1)(k+1)|u_0|}\cdot\frac{CM^{2k}}{(2k)^2} &\le \frac{\left(2k+\frac{5}{3}\right)+(2k^2+k+1)\left(\frac{5}{3}+\frac{1}{|u_0|}\right)}{(2k+1)(k+1)|u_0|}\cdot\frac{CM^{2k}}{(2k)^2}\\
&\le \frac{1}{|u_0|}\left(\frac{5}{3}+\frac{1}{|u_0|}\right)\frac{(k+1)^2}{k^2M^2}\cdot\frac{CM^{2k+2}}{(2k+2)^2}\\
&\le \frac{1}{|u_0|}\left(\frac{5}{3}+\frac{1}{|u_0|}\right)\frac{9}{4M^2}\cdot\frac{CM^{2k+2}}{(2k+2)^2},
\end{aligned}$$

where the last inequality becomes an equality for $k = 2$. Altogether, we obtain for $k \ge 2$

$$|u_{2k+2}| \le \frac{1}{|u_0|}\left[\left(\frac{5}{3}+\frac{1}{|u_0|}\right)\frac{9}{4M^2}+\delta_k^2\left(\frac{\pi^2}{12}-\frac{1}{4}\right)C\right]\frac{CM^{2k+2}}{(2k+2)^2} \le \frac{CM^{2k+2}}{(2k+2)^2},$$

given the assumption (7.9). Here δ_k^j is the Kronecker delta. This completes the induction. $\square$

The existence of such an upper bound (7.7) on u_{2j} guarantees a lower bound M^{-1} on the radius of convergence. For instance, if we take $u_0 = -2$, which lies nicely in the interval between $-\frac{3}{4}$ and -6, we can at least pick

$$C = \tfrac{1}{10}, \qquad M^{-1} = \tfrac{3}{5}$$

satisfying both (7.8) and (7.9). The bound (7.7) is by no means optimal at every $u_0 \neq 0$. In fact, our numerical integrations of (7.2) with u_0 sampled between -6 and $-\frac{3}{4}$ all indicate that in the *real* domain, the series solutions (7.3) with $-6 < u_0 < -\frac{3}{4}$ are all well sandwiched between the parabolic curves of (7.5) and (7.6), and therefore suggest an infinite radius of convergence on the real line. Moreover, by applying the transformation $w \to \frac{1}{w}$ to (7.2) and studying the formal (Puiseux) series expansion of the transformed ODE at the origin, we find the following asymptotic expansion[2]

[2] We also find another asymptotic expansion that has the first two leading terms identical to (7.5), but also involves fractional powers of w in a complicated way, hence not presented here.

of (7.2) as $w \to \infty$ (cf. (7.6)):

$$g \sim -\tfrac{3}{2}w^2 - 6 + u_{4/3}w^{2/3} + O(w^{-1/3}),$$

where $u_{4/3}$ is an arbitrary constant. This asymptotic behaviour at infinity, consistent with our numerical calculations, again suggests that we may significantly extend the radius of convergence for (7.3) at least in the real domain.

An additional comment is that the Cartan invariant α_I, computed from (7.3) with $-6 < u_0 < -\frac{3}{4}$, is generally not constant, contrary to the special cases for those values of u_0 at the two endpoints of the interval of values for u_0 being considered. To see this, we can use the following series expansion of α_I at $w = 0$ (ignoring the overall sign difference):

$$\begin{aligned}
\alpha_I[g(w)] &= -\frac{4 \cdot 2^{1/4}3^{3/4}(7u_0^2 + 21u_0 + 9)\,|8u_0^2 + 24u_0 - 9|^{1/4}}{(-8u_0^2 - 24u_0 + 9)^{3/2}} \\
&+ \frac{8\mathrm{i} \cdot 2^{1/4}3^{3/4}(u_0 + 6)(4u_0 + 3)(44u_0^3 + 162u_0^2 - 27u_0 - 135)\,|8u_0^2 + 24u_0 - 9|^{1/4}}{3u_0(-8u_0^2 - 24u_0 + 9)^{5/2}}\, w \\
&+ \cdots .
\end{aligned}$$

where the coefficient of w clearly vanishes at $u_0 = -6, -\frac{3}{4}$ (likewise for coefficients of higher-degree terms), but is generally nonzero for $-6 < u_0 < -\frac{3}{4}$.

Chapter 8

Killing Vectors

8.1 The Leroy-Nurowski Solution

The original metric found by Leroy [8] was constructed by assuming the existence of a three-parameter group of Killing symmetries, instead of directly solving certain field equations for type N which is how Nurowski discovered his version of the same metric [15]. Here we quote from [25] (see p. 201) a form of the Leroy solution:

$$\begin{aligned}\mathbf{g} =& \frac{t^2+1}{2kx^2}\left(\mathrm{d}x^2+\mathrm{d}y^2\right) \\ &-\frac{2}{k}\left(x^2\mathrm{d}u-\frac{\mathrm{d}y}{3x}\right)\left[\mathrm{d}t+\frac{2}{x}\left(t\mathrm{d}x+\mathrm{d}y\right)+\frac{t^2-1}{2}\left(x^2\mathrm{d}u-\frac{\mathrm{d}y}{3x}\right)\right],\end{aligned} \tag{8.1}$$

with $\Lambda = -3k < 0$, which admits *three* Killing vectors

$$\partial_u, \qquad \partial_y, \qquad x\partial_x + y\partial_y - 2u\partial_u.$$

One can immediately recognize the first two Killing vectors since no metric components of (8.1) depend on the coordinate u or y.

By the following coordinate substitutions

$$x \longrightarrow y, \qquad y \longrightarrow x, \qquad u \longrightarrow -\frac{u}{3}, \qquad t \longrightarrow \tan\left(\frac{r}{2}\right),$$

the metric (8.1) can be transformed to the form found by Nurowski [15]

$$\begin{aligned}\mathbf{g} =& \frac{1}{s^2 y^2 \cos^2(\frac{r}{2})} \left[\frac{3}{2}(\mathrm{d}x^2 + \mathrm{d}y^2)\right. \\ &\left. + (\mathrm{d}x + y^3 \mathrm{d}u) \left(y\mathrm{d}r + \frac{y^3}{3}\cos r \mathrm{d}u + \left(2 + \frac{7}{3}\cos r\right)\mathrm{d}x + 2\sin r \mathrm{d}y \right)\right]\end{aligned} \tag{8.2}$$

with $\Lambda = -s^2 < 0$. Correspondingly, the three Killing vectors are given by

$$\partial_u, \qquad \partial_x, \qquad x\partial_x + y\partial_y - 2u\partial_u. \tag{8.3}$$

As one may have expected (from Theorem 2.3), these Killing symmetries, without dependence on the coordinate r, are in fact also the symmetries of the underlying CR structure (use (2.12) and $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y - y^{-3}\partial_u)$ to verify), the three-parameter group of which belongs to the Bianchi type VI_h [35]. Indeed, we have the following theorem stating this coincidence to be a general property.

Theorem 8.1. [54] For metrics in the class (2.16), the projection of a Killing vector onto the CR manifold is a symmetry of the CR structure.

8.2 General Solutions Determined by ODEs

Theorem 8.1 suggests that to seek for Killing vectors, one may as well start with symmetries of the CR structure, which by themselves are easier to find and have been well classified [35]. Specifically for the metrics (2.17-2.37) with the ansatz (4.9, 4.10) and generally determined by the ODEs (4.11) or (4.12) (or (4.14), (4.15)), one can, by setting $A(\zeta) = 2$ (in this gauge, $z = y$) and without loss of generality, choose the following representative for the underlying CR structure:

$$\mu = \mathrm{d}x + \mathrm{i}z, \qquad \lambda = \frac{\mathrm{e}^{C_1 x}\mathrm{d}u - 2\left[\int \exp\left(\int F_2 \mathrm{d}z\right)\mathrm{d}z\right]\mathrm{d}x}{\exp\left(\int F_2 \mathrm{d}z\right)}. \tag{8.4}$$

Then by Theorem 2.2 (due to (2.1), a factor in λ is irrelevant), we immediately acquire two symmetries

$$X_1 = \partial_u, \qquad X_2 = \partial_x - C_1 u \partial_u,$$

with the commutation relation

$$[X_1, X_2] = -C_1 X_1.$$

If we take these symmetries as our first guess, it is quite straightforward to verify that the vector fields $X_{1,2}$ (especially X_2) satisfy the condition (1.10) (use the explicit metric form $\mathbf{g}$ in Section 9.1) and hence constitute true Killing vectors. This interesting fact indicates a certain inheritability of symmetries from CR structures to spacetimes, though this is not always true. For example, the Hauser solution has only one Killing vector (and one homothetic vector) [55] despite that its hyperquadric CR structure has the maximal eight symmetries.

To identify more symmetries of (8.4), one needs to know more about the function F_2, which may require solving associated ODEs. Relevantly, it still remains an open problem regarding the maximum number (≥ 3) of Killing vectors that a twisting type N vacuum with $\Lambda \neq 0$ (not necessarily in the ansatz (4.9)) may have. For $\Lambda = 0$, we indeed know that this maximum number is two [56, 57].

Chapter 9

Conclusions and Outlook

9.1 Conclusions

We have begun with the advantage of prior work done on the use of (3-dimensional) CR manifolds to look for solutions of the Einstein field equations that correspond to algebraically special Einstein spaces with twisting PNDs. A general solution of those reduced field equations for the two functions of three variables would generate all twisting solutions of Petrov type N. Of course we did not achieve this; however, after the assumption of a single Killing vector in a particular direction, our ansatz of invariant solutions obtained from the infinite-dimensional classical symmetries of the field equations, allowed us to obtain a single ODE[1], the solutions of which would generate a family of twisting spacetimes of type N admitting at least two Killing vectors. That ODE is either a rather simple, third-order nonlinear equation for $J = J(z)$ in which the independent variable z does not appear or, equivalently, an even simpler, second-order nonlinear equation for $g = g(w)$, where w is a dimensionless

[1]The ODE system (4.11) with $C_1 = 0$ and $H \neq 0$ is also very important, but we still know very little about it. See Appendix D for an example of solutions that is unique to (4.11) and not found in other ODEs.

re-scaling of J and g is a re-scaling of J', which includes a nonzero value for Λ, the cosmological constant. Within the same ansatz, we have also investigated all the cases of solutions corresponding to conformally flat spacetimes to which type N solutions may degenerate, helping us look for non-trivial cases.

We have studied this second-order equation at some length. In particular it contains one parameter C_1, which may always be re-scaled to the value $+1$ unless it happens to be zero. In the case that it is zero, the equation can be reduced still further to a first-order equation of Abel type. Following standard approaches to Abel equations, we were unable to determine any method that we thought would generate reasonable explicit type N solutions, although this is still an ongoing project of considerable interest. However, when C_1 is not zero we have considered various sorts of solutions which it might have. We have shown that it does have solutions which are holomorphic, in the complex plane in a neighborhood of the origin, and have found an asymptotic behavior near the (real) infinity. In particular we have picked out especially those solutions which are even functions of w and looked at power-series solutions about the origin, both analytically and numerically via Maple programs. We have determined a moderately-simple recursion relation for the coefficients of the powers of w^2 in the series solutions, which determines the coefficient u_{2k+2}, of w^{2k+2} ($k \geq 2$) in terms of all the previous coefficients, looking at all of them as determined by the value of $g(0) = u_0$. This series terminates quickly for just two particular values of u_0, in the form $-a(u_0)w^2 + u_0$, with a constant, different for the two values of u_0. The value $u_0 = -\frac{3}{4}$ generates the previously-known Leroy-Nurowski solution, while the other one $u_0 = -6$ is unfortunately simply a conformally flat solution. To ensure that these series solutions are distinct from the Leroy-Nurowski solution, we have used the work of Cartan on the question of the equivalence of two CR manifolds, which requires the equality of the set of six Cartan invariants. We have found that any value of u_0 between these two special values generates Cartan invariants that are quite different from those at the endpoints of this interval, and therefore distinct from those of the Leroy-Nurowski solution.

The solutions characterized by values of u_0 between -6 and $-\frac{3}{4}$ have an asymptotic behavior, via a Puiseux series around the (real) infinity, that has the same form $-a(-6)w^2 - 6$ as the conformally flat solution aforementioned, but also lower-order terms involving third-roots of w, which undoubtedly generate algebraic singularities there. Numerical integrations via Maple agree with this behavior, showing negative values of $g(w)$ as needed and very simple structure for all real values of w. The same numerical integrations do show singularities in the solutions for $u_0 > -\frac{3}{4}$. As well, numerical calculations of the coefficients u_{2k+2}, for several values of $u_0 \in \left(-6, -\frac{3}{4}\right)$ (e.g., $u_0 = -\frac{301}{400}$) show that starting at a large enough k, they alternate in sign while their absolute values are monotonically decreasing at rapid rates. We therefore postulate that these solutions are everywhere non-singular and well-behaved on the real axis, and believe that they might define new well-behaved, transcendental functions with algebraic singularities off the real w-axis. The proof of such a conjecture is still being pursued; nonetheless, we feel that the numerical calculations justify the belief that this is a sufficiently interesting result as to merit the attention of a wider audience.

To conclude the discussion, we present here our new class of metrics which, without loss of generality, may be considered by setting $A(\zeta) = 2$ in the ansatz (4.9). Although we present it here with the new real coordinate z introduced in (4.10), with this choice of $A(\zeta)$ it is the same as the usual coordinate y used in (2.35). As well, our studies with the equation for $P(J)$, equivalently $g(w)$, allow us to replace z by its form in terms of J as determining the imaginary part of $\mathrm{d}\zeta$, via $\mathrm{d}z = \mathrm{d}J/P(J)$, namely,

$$\zeta = x + \mathrm{i}z = x + \mathrm{i}z(J), \qquad \mathrm{d}\zeta = \mathrm{d}x + \mathrm{i}\mathrm{d}z = \mathrm{d}x + \frac{\mathrm{i}}{P}\mathrm{d}J.$$

For simplicity of presentation, we show both forms below, with coordinates $\{x, z, u, r\}$

and $\{x, J, u, r\}$ respectively:

$$\begin{aligned}
\mathbf{g} &= \frac{J'}{2\cos^2(\frac{r}{2})}\left[\mathrm{d}\zeta\mathrm{d}\bar{\zeta} + \lambda\left(\mathrm{d}r + W\mathrm{d}\zeta + \bar{W}\mathrm{d}\bar{\zeta} + H\lambda\right)\right] \\
&= \frac{P}{2\cos^2(\frac{r}{2})}\left[\mathrm{d}\zeta\mathrm{d}\bar{\zeta} + \lambda\left(\mathrm{d}r + W\mathrm{d}\zeta + \bar{W}\mathrm{d}\bar{\zeta} + H\lambda\right)\right]
\end{aligned}$$

with real-valued $J = J(z)$, $J' \equiv \mathrm{d}J/\mathrm{d}z = P(J) > 0$ and $P' \equiv \mathrm{d}P/\mathrm{d}J$ such that

$$\begin{aligned}
W &= \frac{1}{2}\left(\frac{J''}{2J'} + \Lambda J + \mathrm{i}C_1\right)(\mathrm{e}^{-\mathrm{i}r} + 1) = \tfrac{1}{2}\left(\tfrac{1}{2}P' + \Lambda J + \mathrm{i}C_1\right)(\mathrm{e}^{-\mathrm{i}r} + 1), \\
H &= -\tfrac{1}{6}\Lambda J'\cos(r) = -\tfrac{1}{6}\Lambda P\cos(r),
\end{aligned}$$

where C_1 is an arbitrary real parameter. The function L as in $\partial = \partial_\zeta - L\partial_u$ can be chosen so as to be real-valued:

$$L = -\mathrm{e}^{-C_1 x}\int \exp\left(\int F_2\mathrm{d}z\right)\mathrm{d}z = -\mathrm{e}^{-C_1 x}\int \frac{1}{P}\exp\left(\int \frac{F_2}{P}\mathrm{d}J\right)\mathrm{d}J,$$

such that from (2.35),

$$\begin{aligned}
\lambda &= \frac{\mathrm{e}^{C_1 x}\mathrm{d}u - 2\left[\int \exp\left(\int F_2\mathrm{d}z\right)\mathrm{d}z\right]\mathrm{d}x}{\exp\left(\int F_2\mathrm{d}z\right)} \\
&= \frac{\mathrm{e}^{C_1 x}\mathrm{d}u - 2\left[\int P^{-1}\exp\left(\int F_2 P^{-1}\mathrm{d}J\right)\mathrm{d}J\right]\mathrm{d}x}{\exp\left(\int F_2 P^{-1}\mathrm{d}J\right)},
\end{aligned}$$

where F_2 is given by

$$F_2 = \frac{J''}{2J'} - \Lambda J = \tfrac{1}{2}P' - \Lambda J.$$

Meanwhile, the functions $J(z)$ and $P(J)$ respectively satisfy

$$\begin{aligned}
J''' &= \frac{(J'')^2}{2J'} - 2\Lambda J J'' - \frac{10}{3}\Lambda(J')^2 - 2(\Lambda^2 J^2 + C_1^2)J', \\
P'' &= -\frac{(P' + 2\Lambda J)^2}{2P} - \frac{2C_1^2}{P} - \frac{10}{3}\Lambda.
\end{aligned}$$

In particular, the original type N metric by Nurowski [15] corresponds to the case $C_1 = 0$, $J = \frac{3}{\Lambda z}$ and a proper choice of the integration constants in λ, for which the expression (5.2), i.e., the Weyl scalar Ψ_4, does not vanish.

We can here note the philosophy that certain ODEs themselves may serve the purpose of defining *new* transcendental functions; for instance, we recall the Painlevé functions and the associated ODEs. Hence our situation with new type N solutions being determined by a second-order *nonlinear* ODE is presumably not too different from that of the Hauser solution (in terms of hypergeometric functions [6]) which is determined by a second-order *linear* ODE, although it is true that there has already been much more extensive studies made on the properties of hypergeometric functions than have been made for newer functions defined by solutions of nonlinear ODEs that may not even have the Painlevé property.

9.2 Outlook

We believe that in some sense, this work in fact raises more questions than it answers, and therefore seems likely to generate future research on the subject. Here we list a few of them in addition to those we already mentioned in previous sections. First, an obvious direction is to construct explicit general solutions to the second-order ODE (1.1) and the Abel equation (1.2). This may involve proving or disproving the irreducibility (in terms of all known functions) and transcendence of the general solutions, hence to confirm whether or not these ODEs themselves may define new transcendental functions. Second, we still know very little about the formal Puiseux series solution (6.1) with respect to the extent of its domain and other global properties. This class of solutions is important because they allow a positive cosmological constant, which is more relevant to our current universe. Third, a complete classification of the solutions determined by the ODEs (4.11) or (4.12) (or (4.14), (4.15)) is surely of certain interest. It can help one to decide which parameters are really necessary in the solutions. Above all, the true value of an exact solution to the Einstein equations cannot be fully appreciated without appropriate physical interpretations regarding its asymptotic behaviors, symmetries, singularities, sources, extensions, completeness, topology and stability, most of which we have barely touched upon in this work. Moveover, since our new solutions describe gravitational waves that only exist for a nonzero cosmological constant as the background/source (they become flat when $\Lambda = 0$), we anticipate their possible future applications in cosmology (see, e.g., [41]).

Appendices

Appendix A

The Abel ODE

The equation (4.19) actually does have the following special solution

$$f_{CF} = -\frac{3}{4t+6}. \tag{A.1}$$

However, it can be shown to correspond to a conformally flat solution (5.8), and hence is not interesting.

Unfortunately, we have had no luck so far in finding the general solution to (4.19) or any other special solution other than (A.1). Since constructing the general solution to the generic Abel ODE has remained an open problem for decades, the general strategy of integration nowadays mainly lies in recognizing, within a suitable class of transformations, the ODE in question as equivalent to a previously solved equation. Such a procedure has been programmed into the current state-of-the-art Maple code `dsolve` (or `abelsol`) [22, 23], which presumably covers all/most of the integrable classes presented in Kamke's book [58] and various other references (e.g., [59]). However, this code, as tested by us, does not recognize (4.19) as a known solved type, e.g., the AIR class. Other attempts by us, such as the symmetry method, on finding special solutions all have failed or just led to (A.1).

So far we have not been able to find a similar reduction for the ODE (4.15) with $C_1 \neq 0$, nor can we negate the possibility that (4.15) with $C_1 \neq 0$ may contain different type N solutions other than the case with $C_1 = 0$. In fact, the Cartan invariants calculated with (4.14) generally do have a dependence on the constant C_1 even though this is not the case for all the conformally flat solutions and the Leroy-Nurowski solution (see (5.10) and (5.20)).

Appendix B

Cartan Invariants

Given $c = c(\zeta, \bar{\zeta})$ and

$$r = \tfrac{1}{6}\left(\partial_{\bar{\zeta}}\bar{l} + 2\bar{c}\bar{l}\right), \qquad l = -\partial_{\zeta}\partial_{\bar{\zeta}}c - c\partial_{\bar{\zeta}}c, \qquad \varepsilon = \pm 1$$

taken from (4.21) where α_I is presented, the next four Cartan invariants, when $r \neq 0$, read

$$\begin{aligned}
\beta_I(\zeta,\bar{\zeta}) =& \frac{1}{32(r\bar{r})^{9/4}}\Big[3\bar{r}^2\partial_{\bar{\zeta}}r\,\partial_{\zeta}r + 3r^2\partial_{\zeta}\bar{r}\,\partial_{\bar{\zeta}}\bar{r} - r\bar{r}\Big(\partial_{\zeta}\bar{r}\,\partial_{\bar{\zeta}}r \\
&+ 7\partial_{\zeta}r\,\partial_{\bar{\zeta}}\bar{r} + 16\bar{c}\bar{r}\partial_{\zeta}r + 16cr\partial_{\bar{\zeta}}\bar{r} - 8r\bar{r}\partial_{\bar{\zeta}}c + 16c\bar{c}r\bar{r}\Big)\Big],\\
\gamma_I(\zeta,\bar{\zeta}) =& \frac{-1}{32(r\bar{r})^{9/4}}\Big[7\bar{r}^2\partial_{\bar{\zeta}}r\,\partial_{\zeta}r + 7r^2\partial_{\bar{\zeta}}\bar{r}\,\partial_{\zeta}\bar{r} - r\bar{r}\Big(8r\partial_{\zeta}\partial_{\bar{\zeta}}\bar{r} + 8\bar{r}\partial_{\zeta}\partial_{\bar{\zeta}}r \\
&+ \partial_{\zeta}\bar{r}\,\partial_{\bar{\zeta}}r + \partial_{\zeta}r\,\partial_{\bar{\zeta}}\bar{r} + 4c\bar{r}\partial_{\bar{\zeta}}r + 4\bar{c}r\partial_{\zeta}\bar{r} + 4cr\partial_{\bar{\zeta}}\bar{r} + 4\bar{c}\bar{r}\partial_{\zeta}r \\
&+ 24r\bar{r}\partial_{\bar{\zeta}}c + 16c\bar{c}r\bar{r}\Big)\Big],\\
\theta_I(\zeta,\bar{\zeta}) =& \frac{-\mathrm{i}}{16r(r\bar{r})^{7/4}}\Big[5\bar{r}^2(\partial_{\bar{\zeta}}r)^2 + 5r^2(\partial_{\bar{\zeta}}\bar{r})^2 - r\bar{r}\Big(4r\partial_{\bar{\zeta}}^2\bar{r} + 4\bar{r}\partial_{\bar{\zeta}}^2 r \\
&- 2\partial_{\bar{\zeta}}r\partial_{\bar{\zeta}}\bar{r} - 4\bar{c}\bar{r}\partial_{\bar{\zeta}}r - 4\bar{c}r\partial_{\bar{\zeta}}\bar{r} + 16r\bar{r}\partial_{\bar{\zeta}}\bar{c}\Big)\Big],
\end{aligned}$$

$$\begin{aligned}
\eta_I(\zeta,\bar\zeta) =& \frac{-1}{192\varepsilon r^{1/2}(r\bar r)^{25/8}}\Big[-3r^3\partial_\zeta\bar r\,(\partial_{\bar\zeta}\bar r)^2 - 3\bar r^3\partial_\zeta r\,(\partial_{\bar\zeta} r)^2 \\
&+ r\bar r\Big(-15r\partial_\zeta r\,(\partial_{\bar\zeta}\bar r)^2 - 48\bar c r^2\bar r\partial_\zeta\partial_{\bar\zeta}\bar r + 12r\bar r\partial_\zeta r\,\partial^2_{\bar\zeta}\bar r + 48r^2\bar r\partial_{\bar\zeta}\bar c\,\partial_\zeta\bar r \\
&- 24r^2\bar r\partial_{\bar\zeta}c\,\partial_{\bar\zeta}\bar r - 15\bar r\partial_\zeta\bar r\,(\partial_{\bar\zeta} r)^2 - 24r\bar r^2\partial_{\bar\zeta}c\,\partial_{\bar\zeta} r - 60c\bar r^2(\partial_{\bar\zeta} r)^2 \\
&+ 12r^2\partial_\zeta\bar r\,\partial^2_{\bar\zeta}\bar r + 48cr^2\bar r\partial^2_{\bar\zeta}\bar r + 12\bar r^2\partial_\zeta r\,\partial^2_{\bar\zeta} r - 48\bar c r\bar r^2\partial_\zeta\partial_{\bar\zeta} r \\
&- 64\bar c r^2\bar r^2\partial_{\bar\zeta}c - 60cr^2(\partial_{\bar\zeta}\bar r)^2 + 12r\bar r\partial_\zeta\bar r\,\partial^2_{\bar\zeta} r + 48cr\bar r^2\partial^2_{\bar\zeta} r \\
&- 12r\bar r\partial_{\bar\zeta}\bar r\,\partial_\zeta\partial_{\bar\zeta} r + 192cr^2\bar r^2\partial_{\bar\zeta}\bar c + 48r\bar r^2\partial_{\bar\zeta}\bar c\,\partial_\zeta r - 12\bar r^2\partial_{\bar\zeta} r\,\partial_\zeta\partial_{\bar\zeta} r \\
&- 12r\bar r\partial_{\bar\zeta} r\,\partial_\zeta\partial_{\bar\zeta}\bar r - 12r^2\partial_{\bar\zeta}\bar r\,\partial_\zeta\partial_{\bar\zeta}\bar r - 12\bar c r\bar r\partial_\zeta\bar r\,\partial_{\bar\zeta} r + 36\bar c r^2\partial_\zeta\bar r\,\partial_{\bar\zeta}\bar r \\
&+ 6r\partial_{\bar\zeta} r\,\partial_\zeta\bar r\,\partial_{\bar\zeta}\bar r + 6\bar r\partial_{\bar\zeta} r\,\partial_\zeta r\,\partial_{\bar\zeta}\bar r - 12\bar c r\bar r\partial_\zeta r\,\partial_{\bar\zeta}\bar r + 36\bar c\bar r^2\partial_\zeta r\,\partial_{\bar\zeta} r \\
&- 48c\bar c r\bar r^2\partial_{\bar\zeta} r - 24cr\bar r\partial_{\bar\zeta} r\,\partial_{\bar\zeta}\bar r - 48c\bar c r^2\bar r\partial_{\bar\zeta}\bar r + 32r^2\bar r^2\partial^2_{\bar\zeta} c\Big)\Big].
\end{aligned}$$

Due to the formidable length of ζ_I as also calculated with Maple (cf. Section 2.2) for our studies, we will not present it here. All Cartan invariants are uniquely determined by the function $c = c(\zeta,\bar\zeta)$.

Appendix C

Conformally Flat Solutions

For a further integration of (5.9), we have the following three separate cases.

Case 1: $\Lambda < 0,\ C_2 > 0$. We always have $J' \geq 0$. Then the solution is determined by

$$\ln \frac{G^2 + \sqrt{2}GM + M^2}{G^2 - \sqrt{2}GM + M^2} + 2\arctan\left(\frac{\sqrt{2}GM}{M^2 - G^2}\right) = -2\sqrt{2}M^3(z + C_0),$$
$$M = \pm\left(-\frac{2C_2\Lambda^{1/3}}{3}\right)^{1/4}, \qquad G = (\Lambda J)^{1/3}.$$

In the real domain, the inverse function $J = J(z)$ is well defined over $z + C_0 \in \left(-\frac{\pi}{\sqrt{2}|M^3|}, \frac{\pi}{\sqrt{2}|M^3|}\right)$ instead of the entire real line, and has singularities at $z + C_0 = \pm\frac{\pi}{\sqrt{2}|M^3|}$.

Case 2: $\Lambda < 0,\ C_2 < 0$. We need $|J| \geq (2C_2/3\Lambda)^{3/4}$ for $J' \geq 0$. The solution is determined by

$$\ln\left|\frac{M+G}{M-G}\right| + 2\arctan\left(\frac{G}{M}\right) = 2M^3(z + C_0),$$
$$M = \pm\left(\frac{2C_2\Lambda^{1/3}}{3}\right)^{1/4}, \qquad G = (\Lambda J)^{1/3}, \qquad |J| \geq \left(\frac{2C_2}{3\Lambda}\right)^{3/4}.$$

In the real domain, the inverse function $J = J(z)$ is well defined over $z + C_0 \in \left(-\infty, -\frac{\pi}{2|M^3|}\right) \cup \left(\frac{\pi}{2|M^3|}, +\infty\right)$, and has singularities at $z + C_0 = \pm\frac{\pi}{2|M^3|}$.

Case 3: $\Lambda > 0,\ C_2 > 0$. We need $|J| \leq (2C_2/3\Lambda)^{3/4}$ for $J' \geq 0$. The solution is determined by

$$\ln\left|\frac{M+G}{M-G}\right| + 2\arctan\left(\frac{G}{M}\right) = 2M^3(z + C_0),$$
$$M = \pm\left(\frac{2C_2\Lambda^{1/3}}{3}\right)^{1/4}, \qquad G = (\Lambda J)^{1/3}, \qquad |J| \leq \left(\frac{2C_2}{3\Lambda}\right)^{3/4}.$$

In the real domain, the inverse function $J = J(z)$ from the above is well defined over the entire real line.

Appendix D

Solutions with the Hyperquadric CR Structure

As already mentioned in Section 2.7, the CR structure of the Hauser solution is of a hyperquadric. Given the prominence of a hyperquadric as the most symmetric of all CR structures, it is quite natural to look for solutions of (4.11) or (4.12) having such a property, i.e., those also satisfying the condition

$$r = 0$$

with r given in (4.21) ($r \propto \mathcal{R}$ in (2.8) [19]). Again by the classical symmetry method, we are able to identify at least one such solution obtained from the system (4.11):

$$F_1 = \pm\frac{\sqrt{6}}{2s(z+C_0)}, \qquad F_2 = -\frac{2}{z+C_0}, \qquad H = \frac{3}{4s^2(z+C_0)^4},$$

with a negative $\Lambda = -s^2$ and the function $H \neq 0$. To verify that r vanishes, one can use the following expression of r in terms of F_2:

$$r = \frac{1}{6A\bar{A}^3}\left[-F_2''' + 3F_2F_2'' + (F_2')^2 - 2F_2^2F_2'\right],$$

which is obtained from plugging (4.9) into (4.21). Unfortunately, the above solution also makes Ψ_4 vanish, hence only a conformally flat one. Note that none of the conformally flat solutions of (4.12) we found in Section 5.1 allows $r = 0$ identically (cf. (5.11)), which indicates that the system (4.11) with $H \neq 0$ contains solutions that are unique to itself and therefore deserves special studies.

It is worthwhile to mention that the function c with F_2 given above, for its dependence on both ζ and $\bar{\zeta}$, cannot be transformed to $c = 0$ by the relation (2.56), which is different from that of the Hauser solution (2.62). Therefore we know that the condition $r = 0$ is indeed more general than $c = 0$ for finding more hyperquadric solutions.

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