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Arithmetic jet spaces and modular forms

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Dedication

To my parents, Arun Prasad Saha and Manisha Saha, whose constant support and encouragement was my source of inspiration. Also to my brother Atanu Saha and his wife Madhumita Saha and their beautiful litte daughter Anushka for all the joy.

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Abstract

In this thesis, we would look into the theory of arithmetic jet spaces and its application in modular forms. The arithmetic jet spaces can be thought of as an analogue of jet spaces in differential algebra. In the case of arithmetic jet spaces, a derivation is replaced by p-derivation δ. This theory was initiated by A. Buium in [7]. The results in the first chapter are concerning the connection between arithmetic jet spaces and Witt vectors. Let $R = \widehat{\mathbb{Z}_p^{ur}}$ be the p-adic completion of the maximal unramified extension of $\mathbb{Z}_p$. If A is an R-algebra and we denote $J^n A$ its n-th jet ring. Firstly, we show the adjunction property which says that the arithmetic jet functor from rings to rings is the left adjoint of the Witt vector functor. This property was also shown by Borger in [3]. However, we give an explicit proof of this fact and the highlight of this proof is the construction of a ring homomorphism $\mathcal{P} : A \to \mathbb{W}_n(J^n A)$ which is the analogue to the exponential map $\exp : A \to A[t]/(t^{n+1})$ given by $\exp(a) = \sum_{i=0}^{n} \frac{\partial^i a}{i!} t^i$. If we denote by $\mathbb{D}_n(B) := B[t]/(t^{n+1})$ then we show that there is a family of ring homomorphisms indexed by $\alpha \in B^{n+1}$, $\Psi_\alpha : \mathbb{D}_1 \circ \mathbb{W}_n(B) \to \mathbb{W}_n \circ \mathbb{D}_1(B)$ for any

ring B and n. This gives yields the relation between a usual derivation ∂ and a p-derivation δ given by $\partial\delta x = p\delta\partial x + (\partial x)^p - x^{p-1}\partial x$. This interaction is used to analyse the ring homomorphisms $\eta : TJ^nA \to J^nTA$ where T associates the tangent ring to the ring A.

In the second chapter of the thesis, we apply the theory of arithmetic jet spaces to modular forms. Let M denote the ring of modular forms over an affine open embedding $X \subset X_1(N)$ where $X_1(N)$ is the modular curve that parametrises elliptic curves and level N structures on it. Let M^∞ be the direct limit of the jet rings of M which we call the ring of δ-modular forms. Then from the universality property of jet spaces, there are ring homomorphism $E^n : M^n \to R((q))\hat{}[q', ..., q^{(n)}]\hat{}$ which are prolongation of the given Fourier expansion map $E : M \to R((q))$. Hence E^n is the δ-Fourier expansion of M^∞. Denote by $\mathbb{S}^\infty = \lim_n Im(E^n)$. If $\overline{\mathbb{S}^\infty}$ denote the reduction mod p of $\mathbb{S}^\infty$ then, one of our main results says that $\overline{\mathbb{S}^\infty}$ can be realised as an Artin-Schrier extension over $\overline{S^\infty}$ where S is the coordinate ring of X. If we set all the indeterminates $q' = .. = q^{(n)} = 0$ then we obtain a ring homomorphism $M^\infty \to \mathcal{W}$ where $\mathcal{W}$ is the ring of generalised p-adic modular forms. Our next result shows that the image of the above homomorphism is p-adically dense in $\mathcal{W}$. We also classify the kernel of this homomorphism which is the p-adic closure of the δ-ideal $(f^\partial - 1, f^1, \delta(f^\partial - 1), \delta f^1, ...,)$ where f^∂ and f^1 are δ-modular forms with weights. This should be viewed as δ-analogue of the Theorem of Swinnerton-Dyer and Serre where the Fourier expansion over $\mathbb{F}_p$ of the modular forms has the kernel $(E_{p-1} - 1)$, E_{p-1} is the Hasse invariant.

In the third chapter, we take the step to understand the 'δ-Fourier expansion principle' and the action of the Hecke operators on the Fourier expansion of differential modular forms. We work on $k[[q]][q']$ which is the reduction mod p of $R[[q]]\hat{}[q']\hat{}$, the "holomorphic subspace" of $R((q))\hat{}[q']\hat{}$. The definition of the Hecke operators away from the prime p extends naturally from the classical definition of Hecke operators.

At the prime p, we define $T_\kappa(p)$ on a "δ-symmetric subspace" of δ-modular forms using the definition of A. Buium introduced in [11]. Our main result states that there is a one-to-one correspondence between the classical cusp forms which are eigenvectors of all Hecke operators with "primitive" δ-modular forms whose δ-Fourier series lies in $k[[q]][q']$ and are eigenvectors of all Hecke operators. This chapter should be viewed as the first attempt to understand the structure of eigenforms on the Fourier side of δ-modular forms.

Contents

Chapter 1

Introduction

The theory of arithmetic differential geometry [10] draws inspiration from the principles of differential algebra. In differential algebra [5] one enlarges the usual algebraic geometry by 'adding' differential equations to the algebraic equations. This approach has found several applications in diophantine geometry over function fields e.g. [6, 25]. In the very similar spirit the consideration of arithmetic jet spaces enlarge the regular algebraic geometry by considering "differential equations" which are satisfied by numbers. Of course the derivation in the usual sense will not work. But one looks for a suitable operator δ which can replace a derivation and yet retains a lot of its flavor.

Just like in the case of differential algebra, given a scheme X over $\mathbb{Z}$, we would like to view X defined over a ring equipped with a derivation. But $\mathbb{Z}$ has no nontrivial derivation to start with. Let us consider $R = \widehat{\mathbb{Z}_p^{ur}} \supset \mathbb{Z}$, the p-adic completion of the maximal unramified extension of $\mathbb{Z}_p$. Then R is endowed with a unique lift of frobenius ϕ acting as $\phi(\zeta_n) = \zeta_n^p$ on the roots of unity. Set

$$\delta x = \frac{\phi(x) - x^p}{p};$$

then the map $x \to (x, \delta x)$ is a ring homomorphism between $R \to \mathbb{W}_1(R)$, where

$\mathbb{W}_1(R)$ is the Witt ring of R of length 2. The foundation of the theory of arithmetic jet spaces is to view δ as a "derivation" of a number with respect to the prime p. Recall from differential algebra, that a derivation $\partial : \mathcal{F} \to \mathcal{F}$ is a ring homomorphism $x \to (x, \partial x)$ in $\mathcal{F} \to \mathcal{F}[\epsilon]/(\epsilon^2) =: \mathbb{D}_1(\mathcal{F})$ where $\mathbb{D}_1$ means the "dual numbers". In other words, in the arithmetic case, we are viewing $\mathbb{W}_1(\_)$ as the analogue of the ring of dual numbers $\mathbb{D}_1(\_)$.

Translating the ring axioms of $\mathbb{W}_1(\_)$, we find that a p-derivation $\delta : A \to B$ satisfies

$$\begin{aligned} \delta(x+y) &= \delta x + \delta y + C_p(x,y) \\ \delta(xy) &= x^p \delta y + y^p \delta x + p \delta x \delta y \end{aligned}$$

where $C_p(x,y) = \frac{x^p + y^p - (x+y)^p}{p}$. The subset of constants of δ is $R^\delta = \{0\} \cup \{\zeta_n \mid p \nmid n\}$. Note that R^δ is a multiplicatively closed set and is not preserved under addition and this is unlike the sub-ring of constants for the usual derivation.

Based on the above δ now viewed as a p-derivation, the arithmetic jet spaces $J^n X \to X \to \mathit{Spec}\ R$ are defined in [10] in a way similar to the definition of jet spaces $J_n X$ in differential algebra. The idea is to study $J^n X$ that would shed some additional light on X itself.

By a prolongation sequence B^* we mean a sequence of maps between rings B^n's

$$B^0 \xrightarrow[\delta]{u} B^1 \longrightarrow \cdots\cdot \qquad B^{n-1} \xrightarrow[\delta]{u} B^n \longrightarrow \cdots$$

where u is a ring homomorphism and δ is a p-derivation which satisfies

$$\begin{aligned} \delta(x+y) &= \delta x + \delta y + C_p(u(x), u(y)) \\ \delta(xy) &= u(y)^p \delta x + u(x)^p \delta y + p \delta x \delta y \end{aligned}$$

where C_p is defined as before.

Let $\mathbb{W}_n(B)$ be a p-typical Witt vector of length $n+1$. Recall that there are two homomorphisms, $R,\ F : \mathbb{W}_n(B) \to \mathbb{W}_{n-1}$ where, R is called the restriction and F is the Frobenius.

Let A and B be R-algebras where $g : R \to B$ is the given algebra map. Let $Hom_\delta(A, \mathbb{W}_n(B))$ be the set of all ring homomorphisms γ from A to $\mathbb{W}_n(B)$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & \mathbb{W}_n(B) \\ \uparrow & & \uparrow{\scriptstyle \mathbb{W}_n(g)} \\ R & \longrightarrow & \mathbb{W}_n(R) \end{array}$$

We will review the basic definitions and constructions of arithmetic jet spaces.

Let A be a finitely generated ring over R, where recall R is the completed, maximal, unramified extension of $\mathbb{Z}_p$. Then $A = R[\mathbf{x}]/(\mathbf{f})$ where $\mathbf{x}$ is a collection of variables and $\mathbf{f}$ represent a collection of multivariate polynomial in $\mathbf{x}$.

Then define the functor J^1A from rings to sets as

$$J^1A(B) = Hom_\delta(A, \mathbb{W}_1(B)).$$

Then it is to easy to see that the above functor is representable by the ring $R[\mathbf{x}, \mathbf{x}']/(\mathbf{f}, \delta\mathbf{f})$ where $\mathbf{x}'$ are new set of indeterminates. We again call this ring J^1A by a slight abuse of notation. By construction, there are two ring homomorphisms-

i) $u : A \to J^1A$ induced by identifying $\mathbf{x}$ in J^1A.

ii) $\phi : A \to J^1A$ defined by $\phi(a) = u(a)^p + p\delta a$ where δ is induced from $\delta\mathbf{x} = \mathbf{x}'$.

Now we will define J^n inductively for n, provided J^{n-1} is already defined. We also have the two canonical maps $(u, \delta) : J^{n-2}A \to J^{n-1}A$ where δ is the set theoretic map satisfying

$$\begin{aligned}\delta(x+y) &= \delta x + \delta y + C_p(u(x), u(y))\\ \delta(xy) &= u(x)^p \delta y + u(y)^p \delta x + p\delta x \delta y\end{aligned}$$

where $C_p(x,y) = \frac{x^p + y^p - (x+y)^p}{p}$.

A ring homomorphism $(f, \Delta) : J^{n-1}A \to \mathbb{W}_1(B)$ will be called to satisfy $(*)$ if in the following diagram

$$\begin{array}{ccc} J^{n-1}A & \xrightarrow[\Delta]{f} & B \\ {\scriptstyle u}\uparrow{\scriptstyle \delta} & & \\ J^{n-2}A & & \end{array}$$

we have $\Delta \circ u = f \circ \delta$.

Then J^nA is the functor from rings to sets defined as

$$J^nA(B) = \{(f, \Delta) \in Hom_\delta(J^{n-1}A, B) \mid (f, \Delta) \text{ satisfies } (*)\}$$

Then it is easy to see that the above functor is represented by $J^nA = R[\mathbf{x}]/(\mathbf{f}, ...\delta^{\mathbf{n}}\mathbf{f})$ with the two canonical maps $(u, \delta) : J^{n-1}A \to J^nA$ and $\phi(a) = u(a)^p + p\delta a$ is the lift of the Frobenius like before.

Hence we obtain a prolongation sequence of rings as follows

$$A \xrightarrow[\delta]{u} J^1A \longrightarrow \cdots \qquad J^{n-1}A \xrightarrow[\delta]{u} J^nA \longrightarrow \cdots$$

Remark 1.0.1. Let $X = Spec\ A$ be the affine scheme. Then we will denote $J^nX = Spf\ \widehat{J^nA}$, where $\widehat{J^nA}$ is the p-adic completion of J^nA. In other words, all our rings will be "non-completed" whereas our spaces are completed

In the first chapter of this thesis, we prove the adjunction theorem which is the following

Theorem 1.0.2. *Hom* $(J^nA, B) \simeq Hom_\delta(A, \mathbb{W}_n(B))$

This result is also independently shown by Borger [2]. However, we give an explicit proof of the above fact. Note that the above result resembles the adjunction property in the case of differential algebra if we replaced $\mathbb{W}_n$ by $\mathbb{D}_n$ where $\mathbb{D}_n(B) = B[\epsilon]/(\epsilon^{n+1})$.

The adjunction property is proved by constructing the universal map $\mathcal{P} : A \to \mathbb{W}_n(J^nA)$, $\mathcal{P}(a) = \{P_0(a), ..., P_n(a)\}$ that satisfies

$$P_k(a) = \sum_{i=0}^{k-1} \sum_{j=1}^{p^{k-1-i}} \frac{p^{j-1}}{p^{k-i-1}} \begin{pmatrix} p^{k-i-1} \\ j \end{pmatrix} P_i(a)^{p(p^{k-i-1}-j)}(\delta P_i(a))^j \tag{1.0.1}$$

for all $k \geq 1$ and $P_0(a) = a$. The map $\mathcal{P}$ is the analogue of the exponential map in differential algebra, $\exp : A \to \mathbb{D}_n(J^nA)$

$$\exp(a) = \sum_{i=0}^{n} \frac{\partial^i(a)}{i!} t^i \tag{1.0.2}$$

since both of them are the image of $\mathbb{1} \in Hom\ (J^nA, J^nA)$ in $Hom_\delta(A, \mathbb{W}_n(J^nA))$ under the isomorphism in Theorem 1.0.2. Hence the coordinates $P_i(a)$ could be viewed as i-th order jets over the 'mythical $\mathbb{F}_1$'.

Such a $\mathcal{P}$ opens up the possiblility to develop a theory analogous to deformation theory where $\mathcal{F}[[t]]$ would be replaced by $\mathbb{W}(R)$. Such a theory should be viewed as deformation theory over $\mathbb{F}_1$. This is indeed an on going research interest for the author.

Also note that the above adjunction property leads to the fact that the n-th jet space of $\mathbb{A}^1$ is $\mathbb{W}_n$. This is a reinterpretation of Witt vectors through arithmetic jet spaces where the complicated formulas defining the Witt vector could be hidden and $\mathbb{W}_n$ can be viewed as a universal object since the jet spaces are.

We show the following fundamental property of Witt vectors:

Theorem 1.0.3. *If B is p-adically complete then $\mathbb{W}_n(B)$ is also p-adically complete.*

Theorem 1.0.3 enables us to give the adjunction property in the category of p-adically complete formal schemes instead of affine ones.

$$J^n X(Spf\ B) \simeq \widehat{X}(Spf\ \mathbb{W}_n(B)) \tag{1.0.3}$$

where $\widehat{}$ denotes the p-adic completion. We seek such a formulation because the arithmetic jet spaces $J^n X$ are formal schemes completed at the prime p. This completion is necessary for a suitable behaviour with respect to localization.

We can summarise the analogy between the geometric and the arithmetic jet spaces by the following

	Geometric jetspace	Arithmetic jetspace
Symbol	$J_n A$	$J^n A$
Adjunction property	$Hom\ (J_n A, B) \simeq$ $Hom_\partial(A, \mathbb{D}_n(B))$	$Hom\ (J^n A, B) \simeq$ $Hom_\delta(A, \mathbb{W}_n(B))$
Exponential	$A \overset{\exp}{\to} J_n A[\epsilon]/(\epsilon^{n+1})$ $\exp(a) = \sum_{i=0}^n \frac{\partial^i a}{i!}\epsilon^i$	$A \overset{\mathcal{P}}{\to} \mathbb{W}_n(J^n A)$ $\mathcal{P}(a) = (P_0(a), ..., P_n(a))$

Even though the apparent structure of a p-derivation δ and a usual derivation ∂ are different, they do interact with each other in a canonical way. We will show that for each n there is a family of ring homomorphisms Ψ_α

Theorem 1.0.4. *$\Psi_\alpha : \mathbb{D}_1 \circ \mathbb{W}_n(B) \to \mathbb{W}_n \circ \mathbb{D}_1(B)$ for any ring B and $\alpha \in B^{n+1}$.*

If Ω denotes the sheaf of differentials of A over R then denote $TA = Symm\ \Omega$, the symmetric product of Ω. This maybe referred to as the tangent ring because it is the ring of functions on the Tangent space TX where $X = Spec\ A$. Then Theorem 1.0.4 implies that there is a family of ring homomorphisms $\eta : TJ^n A \to J^n TA$. In fact something more general in the sense of prolongation sequences is true and we obtain the following commutation relation between a derivation ∂ and a p-derivation δ

$$\delta\partial x = \phi(\partial x) - x^{p-1}\partial x \tag{1.0.4}$$

This relation is defined in [10] but we show here that this relation follows due to Ψ in Theorem 1.0.4

Before proceeding with explaining our next results let us recall a few basic facts of modular curves and forms. We refer to [17] for detailed discussion. Consider the modular curve $Y_1(N)$ defined over $\mathbb{Z}[1/N, \zeta_N]$ which is the representable object for the functor from rings over $\mathbb{Z}[1/N, \zeta_N]$ to sets defined as: given a $\mathbb{Z}[1/N, \zeta_N]$-ring B, we consider the isomorphism classes of pairs (E, ι) where E is an elliptic curve defined over B and $\iota : (\mathbb{Z}/N\mathbb{Z})_B \subset E$ is a level $\Gamma_1(N)$-structure.

Let $\mathcal{E} \to Y_1(N)$ be the universal elliptic curve and $e : Y_1(N) \to \mathcal{E}$ be the identity section. Denote by $L = e^*\Omega_{\mathcal{E}/Y_1(N)}$ where $\Omega_{\mathcal{E}/Y_1(N)}$ is the sheaf of relative 1-forms on $\mathcal{E}$. Let $X_1(N)$ denote the Deligne-Rapoport compactification of $Y_1(N)$ and take the natural extension of L to $X_1(N)$, and call it L again.

Let $X \subset X_1(N)$ be an open embedding (not necessarily a proper open subscheme). Consider the restriction of L on X and call it L again. Then over any $\mathbb{Z}[1/N, \zeta_N]$-algebra B, the modular forms of weight κ, denoted by $M_X(B, \kappa, N)$, identifies with the space of global sections $H^0(X_B, L_B^{\otimes\kappa})$, where X_B is obtained by base change and L_B denotes the sheaf obtained by pullback. Denote

$$M_X = \bigoplus_{\kappa} M_X(B, \kappa, N).$$

The cusp $P = \infty$ is a $\mathbb{Z}[1/N, \zeta_N]$ point on $X_1(N)$ and there is a natural Fourier expansion map $E : M_X \to R((q))$ associated to P. We will call such a tuple (X, L, P, E) as a *Fourier framed curve.*

Recall another definition of modular forms. For any $\mathbb{Z}[1/N, \zeta_N]$-algebra B, let

E/B denote an elliptic curve defined over B, $\omega \in H^0(E, \Omega_{E/B})$ a basis of the free B-module $H^0(E, \Omega_{E/B})$ and ι as above. By a modular form of weight κ we will understand a rule f that associates to any tuple $(E/B, \omega, \iota)$ an element $f(E/B, \omega, \iota) \in B$ which depends only on the isomorphism class of the tuple, commutes with base change and satisfies

$$f(E/B, \lambda\omega, \iota) = \lambda^{-\kappa} f(E/B, \omega, \iota) \tag{1.0.5}$$

for all $\lambda \in B^{\times}$. This definition identifies with the one given previously using the global sections of higher tensor powers of L [22].

Now for $p \nmid N$, we choose a homomorphism $\mathbb{Z}[1/N, \zeta_N] \to R$ and denote by $Y_1(N)_R, L_R, P_R$ the objects over R obtained by base change. The space of modular forms $M = \bigoplus_{\kappa} M(R, \kappa, N)$ is a remarkable space of functions and is one of the central object of study in number theory. For example, M contains the normalized Eisenstein forms E_4, E_6, E_{p-1} belonging to the spaces $M(R, 4, N), M(R, 6, N), M(R, p-1, N)$ respectively. Note that E_{p-1} is a characteristic 0 lift of the Hasse invariant, a quantity that measures super-singularity.

Let us consider the n-th jet of M, completed p-adically and call it M^n. Let

$$M^{\infty} = \lim_{\rightarrow} M^n.$$

We call M^{∞} as the ring of δ- *modular forms*. Clearly, $M \subset M^{\infty}$. But then the question is, are there interesting new examples in M^{∞} that shed a new light on M? Is there a nice theory of Fourier (Serre-Tate) expansion?

We will exhibit a few examples of new 'objects' that live in M^{∞} which have no apparent counterpart in the world of classical modular forms. But firstly we would like to define δ-modular forms of a given weight.

For any polynomial $w \in \mathbb{Z}[\phi]$, $w = \sum a_i \phi^i$ define an element

$\chi_w(t) \in R[t, t^{-1}, t', ..., t^{(n)}]$ by the formula

$$\chi_w(t) = t^w := \prod (\phi^i(t))^{a_i}$$

Such a χ_w is a multiplicative δ-character [9]. Also let B^* denote a prolongation sequence where B^ns are p-adically complete for all n. By a δ-modular form of order $\leq n$ and weight w we will understand a rule f that associates to any triple $(E/B^0, \omega, \iota, B^*)$ an element $f(E/B^0, \omega, \iota, B^*) \in B^n$, which depends on the isomorphism class of the triple only, commutes with base change and satisfies

$$f(E/B^0, \lambda\omega, \iota, B^*) = \chi_w(\lambda)^{-1} f(E/B^0, \omega, \iota, B^*) \tag{1.0.6}$$

The space of such δ-modular forms will be denoted by $M^n(w)$. A Fourier framed curve is called *ordinary* if there exists an element $f \in M^1(\phi - 1)$ which is invertible in the ring M^1, such that $E^1(f) = 1$.

We shall say $f \in M^\infty$ is *isogeny covariant* if for any triple $(E/S^0, \omega, i, S^*)$ and for any étale isogeny $\pi : E' \to E$ (of elliptic curves over S^0) we have

$$f(E'/S^0, \omega', i', S^*) = [\deg \pi]^{-deg\ w/2} f(E/S^0, \omega, i, S^*) \tag{1.0.7}$$

where $\omega' = \pi^*\omega$ and $deg\ w = \sum a_i$.

Let

$$\Psi := \frac{1}{p} \log \frac{q^\phi}{q^p} = \sum_{n \geq 1} (-1)^{n-1} n^{-1} p^{n-1} \left(\frac{q'}{q^p} \right)^n \in R((q))\hat{}[q']\hat{} \tag{1.0.8}$$

Proposition 1.0.5. [9] There exists a unique form $f^1 \in M^1(-1 - \phi)$ whose Fourier expansion is given by

$$E^1(f^1) = \Psi.$$

Notation. Given a ring B, we will denote its reduction mod p by $\overline{B}$.

Proposition 1.0.6. [1, 9, 10] Assume the reduction mod p of X, $\overline{X}$, is contained in the ordinary locus of the modular curve. Then there exists a unique form $f^\partial \in M^1(\phi - 1)$ which is invertible in the ring M^1 such that

$$E^1(f^\partial) = 1.$$

Furthermore its reduction mod p, $\overline{f^\partial} \in \overline{M^1(\phi - 1)}$ coincides with the image of the Hasse invariant $\overline{H} \in \overline{M^0(p-1)}$.

The δ-modular forms in Proposition 3.2.2 and 3.2.3 are isogeny covariant. The forms f^1 and f^∂ will play a central role in the second chapter of this thesis.

Since M comes with a Fourier expansion map $E : M \to R((q))$, by universality property of jet spaces as in Theorem 1.0.2, extends naturally to

$$E^n : M^n \to R((q))\hat{}[q',,,q^{(n)}]\hat{} =: S^n_{for}$$

where $q^{(i)}$'s are new indeterminates. However, unlike the classical modular forms, E^n is not injective. For example, $f^\partial - 1$ and its higher p-derivatives $\delta^i(f^\partial - 1)$ for all $i \leq n-1$ are in the kernel of E^n. Although, if we restrict to δ-modular forms of a fixed weight w, denoted $M^n(w)$, then E^n is injective.

Denote by $E^\infty : M^\infty \to S^\infty_{for}$ where $S^\infty_{for} = \lim_{\to} S^n_{for}$ and $S^n_{for} := R((q))\hat{}[q', ..., q^{(n)}]\hat{}$ the δ-Fourier expansion principle induced from E^n's discussed above. Then by Proposition 3.2.3, we observe that $(f^\partial - 1, \delta(f^\partial - 1), ...) \subset$ Ker E^∞. Set $\mathbb{S}^\infty = Im(E^\infty : M^\infty \to S^\infty_{for})$. Then we will show that $\overline{\mathbb{S}^\infty} \simeq \frac{\overline{M^\infty}}{(\overline{f^\partial - 1}, \overline{\delta(f^\partial - 1)}, ...)}$. But $\overline{\mathbb{S}^\infty}$ has more structure to it. It can be realised by a sequence of Artin-Schrier extensions over $\overline{S^\infty}$.

Definition 1.0.7. Let A be a k-algebra where k is a field. Let $A \subset B$ a ring extension, and Γ a profinite abelian group acting on B by A-automorphisms. We say that B is a Γ-*extension* of A if one can write A and B as filtered unions of finitely generated k-subalgebras, $A = \bigcup A_i$, $B = \bigcup B_i$, indexed by some partially ordered set, with $A_i \subset B_i$, and one can write Γ as an inverse limit of finite abelian groups, $\Gamma = \varprojlim \Gamma_i$, such that the Γ-action on B is induced by a system of compatible Γ_i-actions on B_i and

$$B_i^{\Gamma_i} = A_i$$

for all i. (Then, of course, we also have $B^\Gamma = A$.) If in addition one can choose the above data such that each A_i is smooth over k and each B_i is étale over A_i we say that B is an *ind-étale* Γ*-extension* of A.

Theorem 1.0.8. *Let $X = Spec\ S$ be an ordinary Fourier-framed curve. Then the ring $\overline{\mathbb{S}^\infty}$ is a quotient of an ind-etale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty}$.*

Let $\pi : S^\infty_{for} \to R((q))$ be the ring homomorphism obtained by setting $\pi(q^{(n)}) = 0$ for all $n \geq 1$. Then we will show that the image of π is p-adically dense in $\mathcal{W}$ where $\mathcal{W}$ is Katz's ring of generalised p-adic modular forms. Hence we have the following:

$$\begin{array}{ccc} M^\infty & \xrightarrow{E^\infty} & \mathbb{S}^\infty \subset S^\infty_{for} \\ \uparrow & & \uparrow \ \big\downarrow \pi \\ M & \xrightarrow[E]{} & \mathcal{W} \subset R((q))\widehat{\ } \end{array}$$

It is easy to see that setting $q^{(n)}$ to 0 is equivalent to setting $\phi^n(q) = q^{p^n}$. Hence combining with Proposition 3.2.2 we can see that f^1 is in the kernel of π. Our main result gives a complete characterisation of the kernel of π.

Theorem 1.0.9. *Assume $X = Spec\ S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. The following hold:*

1) The inclusion $\mathbb{S}^\infty \subset S^\infty_{for}$ has torsion free cokernel.

2) The kernel of $M^\infty \to S^\infty_{for}$ is the p-adic closure of the ideal generated by the elements

$$f^\partial - 1,\ \delta(f^\partial - 1),\ \delta^2(f^\partial - 1), \ldots$$

3) The kernel of $\mathbb{S}^\infty \to R((q))\hat{}$ *is the p-adic closure of the ideal generated by the images of the elements*

$$f^1,\ \delta f^1,\ \delta^2 f^1, ...$$

4) The kernel of $M^\infty \to R((q))\hat{}$ *is the p-adic closure of the ideal generated by the elements*

$$f^\partial - 1,\ f^1,\ \delta(f^\partial - 1),\ \delta f^1,\ \delta^2(f^\partial - 1),\ \delta^2 f^1, ...$$

Conclusion 1 in Theorem 1.0.9 should be viewed as a "strong" δ-expansion principle. Conclusions 2 and 4 should be viewed as δ-analogues of the Theorem of Swinnerton-Dyer and Serre according to which the kernel of the Fourier expansion map

$$\bigoplus_{\kappa \geq 0} M(\mathbb{F}_p, \kappa, N) \to \mathbb{F}_p[[q]]$$

is generated by $E_{p-1} - 1$;cf. [20],p. 459.

At the end of the second chapter, we show that there can not be a modular form ϵ such that $\epsilon^{p-1} = E_{p-1}$ i.e. the $(p-1)$-th root of E_{p-1} is not a modular form. We show this by using the irreducibility of the Igusa curve.

The third chapter is the joint work with A. Buium and the author in [14]. It takes the step to understand the δ-Fourier expansion principle' and the action of the Hecke operators on the Fourier expansion of differential modular forms. We work on $k[[q]][q']$ which is the reduction mod p of $R[[q]]\hat{}[q']\hat{}$, which is the "holomorphic subspace" of S^1_{for}.

For n coprime to p and $f \in R[[q]]\hat{}[q', ..., q^{(r)}]\hat{}$(or $k[[q]][q', ..., q^{(r)}]$) we define the *Hecke operator* $T_\kappa(n)$ for each integer κ as

$$T_\kappa(n) f := n^{\kappa-1} \sum_{A,B,D} D^{-\kappa} f(\zeta_D^B q^{A/D}, \delta(\zeta_D^B q^{A/D}), ..., \delta^r(\zeta_D^B q^{A/D})). \tag{1.0.9}$$

where A, B, D belong to the set

$$\{(A,B,D); A,B,D \in \mathbb{Z}_{\geq 0}, AD = n, (A,N) = 1, B < D\}$$

The above definition of $T_\kappa(n)$ is a natural extension from the classical definition of Hecke operators. However to find an analogue of the U operator in our case is a challenging question. We use the definition of A. Buium introduced in [11] as the analogue of the U operator. One draw back of this definition is that U is not defined on the whole of $\overline{S^1_{for}}$ but rather on a linear subspace called the δ-symmetric subspace. Set

$$A := R[[s_1, ..., s_p]][s'_1, ..., s'_p, ..., s_1^{(r)}, ..., s_p^{(r)}]\hat{},$$

$$B := R[[q_1, ..., q_p]][q'_1, ..., q'_p, ..., q_1^{(r)}, ..., q_p^{(r)}]\hat{}.$$

where $s_1, ..., s_p, s'_1, ..., s'_p, ...$ and $q_1, ..., q_p, q'_1, ..., q'_p, ...$ are indeterminates. If $S_1, .., S_p$ are the fundamental symmetric polynomials in $q_1, ..., q_p$ then the natural algebra map

$$A \to B, \quad s_j^{(i)} \mapsto \delta^i S_j,$$

is injective with torsion free cokernel [11].

An element $G \in B$ will be called *Taylor* $\delta-$*symmetric* if it is the image of some element $G_{(p)} \in A$ (which is then unique) under the above map. An element $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ will be called *Taylor* $\delta - p$*-symmetric* if

$$\Sigma_p f := \sum_{j=1}^{p} f(q_j, ..., q_j^{(r)}) \in B$$

is Taylor $\delta-$symmetric.

We define Uf where f Taylor $\delta - p$-symmetric

$$Uf := p^{-1}(\Sigma_p f)(0, ..., 0, q, ..., 0, ..., 0, q^{(r)})$$

which is an element in $p^{-1}R[[q]][q', ..., q^{(r)}]$. The restriction of U to $R[[q]]$ takes values in $R[[q]]$ and is equal to the classical Atkin's operator

$$U(\sum a_m q^m) = \sum a_{mp} q^m.$$

We also define the extension of the Frobenius operator V as

$$Vf = f(q^p, ..., \delta^r(q^p)) \in R[[q]][q', ..., q^{(r)}].$$

Hence for any $\kappa \in \mathbb{Z}$ and Taylor $\delta - p$-symmetric f we may define

$$pT_\kappa(p)f = pUf + p^\kappa Vf.$$

Note that the restriction of $pT_\kappa(p)$ to $R[[q]]$ is the classical Hecke operator $T_k(p)$ defined by

$$T_\kappa(p)(\sum a_m q^m) = \sum a_{pm} q^m + p^{\kappa-1} \sum a_m p^{pm}.$$

A series $\varphi \in k((q))$ will be called *primitive* if $U\varphi = 0$. A δ-series in $k((q))[q', ..., q^{(r)}]$ will be called *primitive* if its image in $k((q))$ under the specialization $q' = ... = q^{(r)} = 0$ is primitive. One can define Hecke operators $T_\kappa(n)$, $pT_\kappa(p)$ on $R((q))[q', ..., q^{(r)}]\hat{}$ (where $pT_\kappa(p)$ is only "partially defined" i.e. defined on an appropriate subspace); cf. Chapter 3 for all the relevant details. These operators induce operators $T_\kappa(n)$, "$pT_\kappa(p)$" on $k((q))[q', ..., q^{(r)}]$ (where "$pT_\kappa(p)$" is only "partially defined" i.e. defined on an appropriate subspace; the " " signs are meant to remind us that the operator $T_\kappa(p)$ itself is not defined mod p).

The following is our main result; it is a consequence of Theorems 4.5.6 and 4.5.7 in Chapter 3. Assume $\kappa \in \mathbb{Z}_{\geq 0}$.

Theorem 1.0.10. *There is a one-to-one correspondence between the following sets of objects:*

i) Series in $qk[[q]]$ which are eigenvectors of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p) = 1$, and which are Fourier expansions of classical modular forms over k of weight $\equiv \kappa + 2$ mod $p - 1$;

ii) Primitive δ-series in $k[[q]][q']$ which are eigenvectors of all Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p) = 1$, and which are δ-Fourier expansions of δ-modular forms of some order $r \geq 0$ and weight w with deg$(w) = \kappa$.

This correspondence preserves the respective eigenvalues.

Remark 1.0.11. 1) As Theorems 4.5.6 and 4.5.7 will show the correspondence in Theorem 1.0.10 is given, on a computational level, by an entirely explicit formula (but note that the proof that this formula establishes the desired correspondence is *not* merely computational.) The formula is as follows. If $\varphi = \sum_{m\geq 1} a_m q^m \in k[[q]]$ is a series as in i) of the Theorem then $a_1 \neq 0$ and the corresponding δ-series in ii) is given by

$$\varphi^{\sharp,2} := \sum_{(n,p)=1} \frac{a_n}{n} q^n - \frac{a_p}{a_1} \cdot \left(\sum_{m\geq 1} a_m q^{mp} \right) \frac{q'}{q^p} + e \cdot \left(\sum_{m\geq 1} a_m q^{mp^2} \right) \cdot \left(\frac{q'}{q^p} \right)^p$$

where e is 1 or 0 according as κ is 0 or > 0. (The upper index 2 in $\varphi^{\sharp,2}$ is meant to reflect the p^2 exponent in the right hand side of the above equality; later in the body of the thesis we will encounter a $\varphi^{\sharp,1}$ series as well. The $\sharp$ sign is meant to reflect the link between these objects and the objects $f^\sharp$ introduced in [11].)

2) Theorem 1.0.10 provides a complete description of primitive δ-series mod p of order 1 which are eigenvectors of all the Hecke operators and which are δ-Fourier expansions of δ-modular forms of arbitrary order. It would be desirable to have such a description in characteristic zero and/or for higher order δ-series. However note that all known examples (so far) of δ-modular forms of order ≥ 2 which are eigenvectors of all Hecke operators have the property that their δ-Fourier expansion reduced mod p has order 1; by the way some of these forms play a key role in [11, 12, 13]. So it is reasonable to ask if it is true that *any δ-modular form of order* ≥ 1 *which is an eigenvector of all the Hecke operators must have a δ-Fourier expansion whose reduction mod p has order* 1.

3) Note that in ii) of the above Theorem one can take the order to be $r = 1$ and the weight to be $w = \kappa$. Also note that the δ-modular forms in ii) above have, a priori, "singularities" at the cusps and at the supersingular points. Nevertheless, in the special case when the classical modular forms in i) above come from newforms

on $\Gamma_0(N)$ over $\mathbb{Z}$ of weight 2 one can choose the δ-modular forms in ii) of weight 0, order 2, and *without singularities at the cusps or at at the supersingular points*; this was done in [11] where the corresponding δ-modular forms were denoted (at least in the "non-CL" case) by $f^\sharp$. These $f^\sharp$s played, by the way, a key role in the proof of the main results in [13] about linear dependence relations among Heegner points. It would be interesting to find analogues of the forms $f^\sharp$ in higher weights.

4) One of the subtleties of the above theory is related to the fact that the operator "$pT_\kappa(p)$" is not everywhere defined as mentioned before. The failure of this operator to be everywhere defined is related to the failure of "the fundamental theorem of symmetric polynomials" in the context of δ-functions; cf. [11, 12]. The domain of definition of "$pT_\kappa(p)$" will be the space of all δ-series for which the analogue of "the fundamental theorem of symmetric polynomials" holds; these δ-series will be called *Taylor $\delta - p$-symmetric*. One of our main results will be a complete determination of the space of Taylor $\delta - p$-symmetric δ-series; cf. Theorems 4.3.1 and 4.3.2.

5) This chapter should be viewed as a first attempt to understand the structure of eigenforms on the Fourier side. It is an on going research project to push this further to characteristic 0 and for higher orders and to consequently develop a p-adic analysis à la Katz [19, 22].

Chapter 2

Interaction between Arithmetic and Geometric Jet Spaces

We will first show that if B is p-adically closed then so is the Witt ring $\mathbb{W}_n(B)$. Then in section 2.4 we show that $\mathbb{W}_n$ is the right adjoint of the jet space functor J^n. In section 2.6, we construct a family of canonical ring homomorphism Ψ from $\mathbb{D}_1 \circ \mathbb{W}_n(B) \to \mathbb{W}_n \circ \mathbb{D}_1(B)$. We apply this fact to prolong derivatives from the base to the entire of prolongation sequences. As a result we obtain the commutation relation between a derivation and a p-derivation. In section 2.8, we record an important property of "non-completed" jet ring as to how they behave with respect to taking fractions. In section 2.9, we have also recorded another geometric insight as to how the canonical lifts of points on the arithmetic jet space can be viewed as intersection of pull-back of subschemes.

2.1 Witt Vectors

Here we review the basic theory of Witt vectors. We refer to [21] for a detailed exposition. Let $\mathbb{N}$ be the set of positive integers. We call a set $S \subset \mathbb{N}$ a truncation set if $n \in S$ and d is a divisor in n then $d \in S$. Then the *big Witt ring* $\mathbb{W}_S(A)$ is the ring structure endowed on A^S such that the ghost map

$$w : \mathbb{W}_S(A) \to A^S$$

which takes $(a_n)_{n\in S}$ to $(w_n)_{n\in S}$ where

$$w_n = \sum_{d|n} d a_d^{\frac{n}{d}} \tag{2.1.1}$$

is a natural transformation of functors from the category of rings to itself. As it turns out [21], this ring structure is unique.

If $T \subset S$ are truncation sets, then the forgetful map

$$R_T^S : \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

is a natural ring homomorphism and is called the *restriction map*. For any $n \in \mathbb{N}$, we can define a new truncation set

$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

Then there exists a natural ring homomorphism $F_n : \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$, called the *Frobenius* such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{w} & A^S \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n^w} \\ \mathbb{W}_{S/n}(A) & \xrightarrow[w]{} & A^{S/n} \end{array}$$

where $(F_n^w((x_m)))_d = x_{nd}$.

Consider the truncation set $S = \{1, p, p^2, ...\} \subset \mathbb{N}$ containing all the powers of p. Then $\mathbb{W}(A) = \mathbb{W}_S(A)$ is called the ring of $p - typical\ Witt\ vectors$ and $\mathbb{W}_n(A) = \mathbb{W}_{\{1,p,...,p^n\}}(A)$ are called the p-typical Witt vectors of length $n+1$ in A.

Notation. For the rest of the section, we will denote the p-typical Witt vectors by $\mathbb{W}$. The restriction map $R^{\{1,...,p^n\}}_{\{1,...,p^{n-1}\}}$ and the frobenius map F_p would now be short-handed as R and F respectively. Also note that F_p^w is nothing but the left-shift operator of sequences.

2.2 p-adic completeness of the Witt rings

We would show that if B is a p-adically complete ring then $\mathbb{W}_n(B)$ is also p-adically complete. Consider the kernel of the reduction map $\mathbb{W}_n(B) \to \mathbb{W}_n(\frac{B}{p^k B})$ and call it I_k. An element in $p^\nu \mathbb{W}_n(B)$ is of the form $p^\nu x$ for some $x = \{x_0, ..., x_n\} \in \mathbb{W}_n(B)$.

$$p^\nu x = \{L_0(x), ..., L_n(x)\}$$

where L_i's satisfy

$$\sum_{i=0}^{k} p^i L_i^{p^{n-i}} = p^\nu (\sum_{i=0}^{k} p^i x_i^{p^{n-i}})$$

for all $k \geq 0$.

Lemma 2.2.1. *$lp^i - i \geq l$ for all $i \geq 1$ and $l \geq 1$*

Proof. We have

$$\begin{aligned} p^{i-1} + p^{i-2} + ... + 1 &\geq 1 + ... + 1 = i \\ \text{So } l(p-1)(p^{i-1+} ... + 1) &\geq i \text{ because both } l, p-1 \geq 1 \\ \text{Hence } l(p^i - 1) &\geq i \\ \text{Hence } lp^i - i &\geq l \end{aligned}$$

and this completes the proof. □

Lemma 2.2.2. *For* $0 \leq i \leq n-1,\ p^{n-i}(\nu - i) \geq \nu$

Proof. The result follows from the following inequality

$$\begin{aligned} p^{n-i} \geq \quad 1 \quad & \geq 1 - i/\nu \\ p^{n-i}\nu & \geq \nu - i \\ p^{n-i}(\nu - i) & \geq \nu \end{aligned}$$

and this completes the proof. □

Lemma 2.2.3. *If* $0 \leq m \leq \nu - n$ *then* $p^{\nu}\mathbb{W}_n(B) \subset I_m$.

Proof. Any element $y \in p^{\nu}\mathbb{W}_n(B)$ can be written as $y = \{L_0(x), ..., L_n(x)\}$ for some $x \in \mathbb{W}_n(B)$. We will prove this by induction on n. In the case when $n = 0$ the result is true. Let us assume that it is true for $n-1$. We know that $L_n = p^{\nu-n}\sum_{i=0}^{k} p^i x_i^{p^{n-i}} - \sum_{i=0}^{n-1} p^{i-n} L_i^{p^{n-i}}$. By the induction hypothesis, $v_p(L_i) \geq \nu - i$.

$$\begin{aligned} v_p(p^{i-n}L_i^{p^{n-i}}) & \geq i - n + p^{n-i}(\nu - i) \\ & \geq \nu - n + i, \text{ by lemma 2.2.2} \\ & \geq \nu - n \end{aligned}$$

We have shown that $v_p(L_n) \geq \nu - n$. Since all the components L_i have valuation greater than $\nu - n$ implies that the element $\{L_0, ..., L_n\} \in I_m$ and this completes the proof. □

Lemma 2.2.4. *If* $m > \nu$ *and* $m > n$ *then* $p^{\nu}\mathbb{W}_n(B) \supset I_m$

Proof. It is sufficient to show that for any $y_0, ..., y_m \in B$ there exists $x_0, ..., x_n \in B$ satisfying

$$\begin{aligned} L_0(x_0) &= p^m y_0 \\ L_1(x_0, x_1) &= p^m y_1 \\ &\dots \\ L_n(x_0, ..., x_n) &= p^m y_n \end{aligned}$$

We will prove this by induction. $L_0(x_0) = p^\nu x_0 = p^m y_0$ and hence $x_0 = p^{m-\nu}$ satisfies the equation. Also we have $v_p(x_0) \geq m - \nu$. Let us assume that there are solutions $x_0, ..., x_{k-1}$ satisfying $L_i = p^m y_i$ and also $v_p(x_i) \geq m - \nu$ for all $i \leq k-1$. Define

$$x_k = \sum_{i=0}^{k} \frac{L_i^{p^{k-i}}}{p^{k+\nu-i}} - \sum_{i=1}^{k} \frac{x_{k-i}^{p^i}}{p^i}$$

$v_p\left(\frac{L_i^{p^{k-i}}}{p^{k+\nu-i}}\right) \geq m(p^{k-i}) - (k+\nu-i)$ because each of the L_i have valuations greater than equal to m.

We claim that $mp^{k-i} - (k-i) \geq m$. If $i = 0$ then the both sides of the inequality are 0 and hence the inequality is true. When $i > 0$, since we have $m > n \geq k$ and $p^{k-i} \geq 1$, the above inequality is true again This shows that $v_p\left(\frac{L_i^{p^{k-i}}}{p^{k+\nu-i}}\right) \geq m - \nu$.

For the other terms in the sum $v_p\left(\frac{x_{k-i}^{p^i}}{p^i}\right) \geq p^i(m-\nu)$ assume the inductive hypothesis of the valuation of x_i's. Since $m - \nu \geq 1$, by Lemma 2.2.1, we have $v_p\left(\frac{x_{k-i}^{p^i}}{p^i}\right) \geq m - \nu$. We have shown that all the terms in the definition of x_k have valuation greater than equal to $m - \nu$, in particular, greater than equal to 0 and hence $x_k \in B$ and it easily follows that x_k is a solution for $L_k(x_0, ..., x_k) = p^m y_k$ and this completes the inductive step and hence the proof. □

Theorem 2.2.5. *If B is p-adically complete then $\mathbb{W}_n(B)$ is p-adically complete.*

Proof. Combining lemma 2.2.4 and lemma 2.2.3 we obtain

$$I_{\nu+1} \subset p^{\nu}\mathbb{W}_n(B) \subset I_{\nu-1}$$

for all $\nu > n$. Hence we obtain the following diagram

$$\begin{array}{ccccc} \mathbb{W}_n(B/p^{\nu+1}B) & \longrightarrow & \frac{\mathbb{W}_n(B)}{p^{\nu}\mathbb{W}_n(B)} & \longrightarrow & \mathbb{W}_n(B/p^{\nu-1}B) \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{W}_n(B/p^{\nu+2}B) & \longrightarrow & \frac{\mathbb{W}_n(B)}{p^{\nu+1}\mathbb{W}_n(B)} & \longrightarrow & \mathbb{W}_n(B/p^{\nu}B) \end{array}$$

where all the maps are surjections. Since the left and the right hand side have $\mathbb{W}_n(B)$ as limit, we get that $\mathbb{W}_n(B)$ is p-adically complete. $\square$

2.3 The Right Adjointness of the arithmetic jet functor

Let $\mathbb{W}_n(B)$ be a p-typical Witt vector of length $n+1$. Recall that there are two homomorphisms, $R,\ F : \mathbb{W}_n(B) \to \mathbb{W}_{n-1}$ where, R is called the restriction and F is the Frobenius. Also recall the ring $R = \mathbb{Z}_p^{ur}$.

Remark 2.3.1. Notation wise, it might be a little confusing since R represents the restriction map $R : \mathbb{W}_n(B) \to \mathbb{W}_{n-1}(B)$ and also $R = \mathbb{Z}_p^{ur}$. But the usage would be very clear from the context and we do no wish to change one of them as they both are very standard.

Since R has a unique lift of Frobenius, this induces a unique ring homomorphism $R \to \mathbb{W}_n(R)$. Let A and B be R-algebras. Let $Hom_{\delta}(A, \mathbb{W}_n(B))$ be the set of all ring homomorphisms $\gamma : A \to \mathbb{W}_n(B)$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\gamma} & \mathbb{W}_n(B) \\ \uparrow & & \uparrow \mathbb{W}_n(g) \\ R & \longrightarrow & \mathbb{W}_n(R) \end{array}$$

where $g : R \to B$ is the given algebra map of R.

We will review the basic definitions and constructions of arithmetic jet spaces. Let A be a finitely generated ring over R, where recall R is the completed, maximal, unramified extension of $\mathbb{Z}_p$. Then $A = R[\mathbf{x}]/(\mathbf{f})$ where $\mathbf{x}$ is a collection of variables and $\mathbf{f}$ represent a collection of multivariate polynomial in $\mathbf{x}$. Then define the functor J^1A from rings to sets as

$$J^1A(B) = Hom_\delta(A, \mathbb{W}_1(B)).$$

It is to easy to see that the above functor is representable by the ring $R[\mathbf{x}, \mathbf{x}']/(\mathbf{f}, \delta\mathbf{f})$ where $\mathbf{x}'$ are new set of indeterminates. We call this ring J^1A again by a slight abuse of notation. By construction, there are two ring homomorphisms-

i) $u : A \to J^1A$ induced by $u(\mathbf{x}) = \mathbf{x}$ where the right hand side is the image of $\mathbf{x}$ in J^1A.

ii) $\phi : A \to J^1A$ defined by $\phi(a) = u(a)^p + p\delta a$ where δ is induced from $\delta\mathbf{x} = \mathbf{x}'$.

Now we will define J^n inductively for n, provided J^{n-1} is already defined. We also have the two canonical maps $(u, \delta) : J^{n-2}A \to J^{n-1}A$ where δ is the set theoretic map satisfying

$$\begin{aligned}
\delta(x + y) &= \delta x + \delta y + C_p(x, y) \\
\delta(xy) &= x^p\delta y + y^p\delta x + p\delta x\delta y
\end{aligned}$$

where $C_p(x, y) = \frac{x^p + y^p - (x+y)^p}{p}$.

A ring homomorphism $(f, \Delta) : J^{n-1}A \to \mathbb{W}_1(B)$ will be said to satisfy $(*)$ if in the following diagram

$$\begin{array}{ccc}
J^{n-1}A & \xrightarrow[\Delta]{f} & B \\
{\scriptstyle u}\uparrow{\scriptstyle \delta} & & \\
J^{n-2}A & &
\end{array}$$

we have $\Delta \circ u = f \circ \delta$. Then $J^n A$ is the functor from rings to sets defined as

$$J^n A(B) = \{(f, \Delta) \in Hom_\delta(J^{n-1}A, B) \mid (f, \Delta) \text{ satisfies } (*)\}$$

It is easy to see that the above functor is represented by $J^n A = R[\mathbf{x}]/(\mathbf{f}, ...\delta^{\mathbf{n}}\mathbf{f})$ with the two canonical maps $(u, \delta) : J^{n-1}A \to J^n A$ and where $\phi(a) = u(a)^p + p\delta a$ is the lift of the Frobenius like before. Hence we obtain a sequence of rings as follows

$$A \underset{\delta}{\overset{u}{\longrightarrow}} J^1 A \longrightarrow \cdots \qquad J^{n-1}A \underset{\delta}{\overset{u}{\longrightarrow}} J^n A \longrightarrow \cdots$$

Definition 2.3.2. Let $X = Spec\ A$ be the affine scheme. Then we define $J^n X = Spf\ \widehat{J^n A}$, where $\widehat{J^n A}$ is the p-adic completion of $J^n A$. We will also denote $\mathfrak{J}^n X = Spec\ J^n A$.

We will show that $J^n(\_)$ is the left adjoint of the $\mathbb{W}_n(\_)$ functor, that is,

$$Hom\ (J^n A, \_) \simeq Hom_\delta(A, \mathbb{W}_n(\_)).$$

where A is finitely generated over R. The case when $n = 1$ is true by definition.

We will prove the adjointness of the two functors by induction. Consider the following statements whose conjunction we call $P(n-1)$:

1) For all $k \leq n-1$, $Hom\ (J^k A, B) \overset{\Psi_k}{\simeq} Hom_\delta(A, \mathbb{W}_k(B))$. Also define $\mathcal{P}_k := \Psi_k(\mathbb{1}_{J^k A})$ where $\mathbb{1}_{J^k A} \in Hom\ (J^k A, J^k A)$ is the identity element. Also, let $\Psi_k^{-1} : Hom_\delta(A, \mathbb{W}_k(B)) \to Hom\ (J^k A, B)$ be the inverse of Ψ_k, for all $k \leq n-1$.

2) If $Hom\ (J^k A, J^k A) \simeq Hom_\delta(A, \mathbb{W}_k(J^k A))$ then $\Psi_{k-1}^{-1}(R \circ \Psi_k(\mathbb{1}_{J^k A})) = u$ and $\Psi_{k-1}^{-1}(F \circ \Psi_k(\mathbb{1}_{J^k A})) = \phi$.

Under the induction hypothesis, let $\mathcal{P}_{n-1} = \Psi_{n-1}(\mathbb{1}_{J^{n-1}A})$. Then $\mathcal{P}_{n-1} : A \to \mathbb{W}_{n-1}(J^n A)$ is the universal family for the isomorphism in 1) above.

We will prove $P(n)$ in Theorem 2.3.24.

Let $R : \mathbb{W}_k(\_) \to \mathbb{W}_{k-1}(\_)$ denote the restriction map and $F : \mathbb{W}_k(\_) \to \mathbb{W}_{k-1}(\_)$ the Frobenius.

Given any ring C, let $w : \mathbb{W}_k(C) \to C^{k+1}$ denote the ghost map for all k. Then we can consider the following diagram

$$\begin{array}{ccc}
A & \xrightarrow{\mathcal{P}_{n-1}} & \mathbb{W}_{n-1}(J^{n-1}A) \\
 & & \Big\downarrow {\scriptstyle \mathbb{W}_{n-1}(u)\times\mathbb{W}_{n-1}(\phi)} \\
\mathbb{W}_n(J^nA) & \xrightarrow{R\times F} & \mathbb{W}_{n-1}(J^nA) \times \mathbb{W}_{n-1}(J^nA) \\
{\scriptstyle w}\Big\downarrow & & \Big\downarrow {\scriptstyle w\times w} \\
(J^nA)^n & \xrightarrow{R\times F} & (J^nA)^{n-1} \times (J^nA)^{n-1}
\end{array}$$

We will show that there exists a ring homomorphism $\mathcal{P}_n : A \to \mathbb{W}_n(J^nA)$ which makes the above diagram commutative in the following Proposition 2.3.6.

But before we show that for any finitely generated ring A in general, we will prove existence in the case when $A = R[\mathbf{x}]$ where $\mathbf{x}$ is a collection of indeterminate variables. Hence in this particular case, $J^nA = R[\mathbf{x}, ..., \mathbf{x^{(n)}}]$. Since R is torsion free, implies that J^nA is torsion free too.

Lemma 2.3.3. *$(R \times F) \circ w$ is a monomorphism.*

Proof. If $(x_0, ..., x_n) \in \mathbb{W}_n(J^nA)$ then $(R \times F)(x_0, ..., x_n) = (x_0, ..., x_{n-1}) \times (x_1, ..., x_n)$ which is injective. And $w : \mathbb{W}_n(J^nA) \to (J^nA)^n$ is injective too because J^nA is torsion free. Hence their composition is injective and we are done. □

Let $\mathcal{P}_{n-1} : A \to \mathbb{W}_{n-1}(J^{n-1}A)$ be given by $\mathcal{P}_{n-1}(a) = (P_0(a), ..., P_{n-1}(a))$, for all $a \in A$ where $P_0, ...P_{n-1} : A \to J^{n-1}(A)$ are set-theoretic maps. Define $P_n(a) \in J^nA \otimes \mathbb{Q}$,

$$P_n(a) = \sum_{i=0}^{n-1} \sum_{j=1}^{p^{n-1-i}} \frac{p^{j-1}}{p^{n-i-1}} \begin{pmatrix} p^{n-i-1} \\ j \end{pmatrix} P_i(a)^{p(p^{n-i-1}-j)} (\delta P_i(a))^j \tag{2.3.1}$$

The following Lemma proves that the coefficients in the above definition are integral.

Lemma 2.3.4. $P_n(a) \in J^n A$.

Proof. We need to check that $P_n(a)$ is p-integral. Hence it suffices to show that the valuation, $\nu_p\left(\frac{p^{j-1}}{p^{n-i-1}}\begin{pmatrix} p^{n-i-1} \\ j \end{pmatrix}\right) \geq 0$ for all i and j. We know that $\nu_p\begin{pmatrix} p^{n-i-1} \\ j \end{pmatrix} = n - i - 1 - \nu_p(j)$ and hence

$$\begin{aligned} \nu_p\left(\frac{p^{j-1}}{p^{n-i-1}}\begin{pmatrix} p^{n-i-1} \\ j \end{pmatrix}\right) &= n - i - 1 - \nu_p(j) + j - 1 - (n - i - 1) \\ &= j - \nu_p(j) - 1 \\ &\geq 0 \end{aligned}$$

and we are done. □

Proposition 2.3.5. $(R \times F) \circ w$ surjects onto the image of $(w \times w) \circ (\mathbb{W}_{n-1}(u) \times \mathbb{W}_{n-1}(\phi)) \circ \mathcal{P}_{n-1}$. In particular we have,

$$((R \times F) \circ w)(P_0(a), ..., P_n(a)) = (w \times w) \circ (\mathbb{W}_{n-1}(u) \times \mathbb{W}_{n-1}(\phi)) \circ \mathcal{P}_{n-1}(a)$$

for all $a \in A$.

Proof. Note that

$$\begin{aligned} (w \times w) \circ (\mathbb{W}_{n-1}(u) \times \mathbb{W}_{n-1}(\phi)) \circ \mathcal{P}_{n-1}(a) &= \left(\sum_{i=0}^{k} p^i P_i(a)^{p^{k-i}}\right)_{k=0}^{n-1} \times \\ &\quad \left(\sum_{i=0}^{k} p^i (P_i(a)^p + p\delta P_i(a))^{p^{k-1-i}}\right)_{k=0}^{n-1} \end{aligned}$$

On the other hand, we have

$$((R \times F) \circ w)(P_0(a), ..., P_n(a)) = \left(\sum_{i=0}^{k} p^i P_i(a)^{p^{k-i}}\right)_{k=0}^{n-1} \times \left(\sum_{i=0}^{k} p^i P_i(a)^{p^{k-i}}\right)_{k=1}^{n}$$

Hence by comparing the ghost components, it is sufficient to show that,

$$\sum_{i=0}^{n} p^i P_i(a)^{p^{n-i}} = \sum_{i=0}^{n-1} p^i (P_i(a)^p + p\delta P_i(a))^{p^{n-1-i}}$$

The following computation proves the above claim.

$$\begin{aligned}
\sum_{i=0}^{n} p^i P_i(a)^{p^{n-i}} &= \sum_{i=0}^{n-1} p^i P_i(a)^{p^{n-i}} \\
&\quad + p^n \sum_{i=0}^{n-1} \sum_{j=1}^{p^{n-1-i}} \frac{p^j - 1}{p^{n-i-1}} \binom{p^{n-i-1}}{j} P_i(a)^{p(p^{n-i-1}-j)} (\delta P_i(a))^j \\
&= \sum_{i=0}^{n-1} p^i P_i(a)^{p^{n-i}} \\
&\quad + \sum_{i=0}^{n-1} p^i p^{n-1-i} p \sum_{j=1}^{p^{n-1-i}} \frac{p^j - 1}{p^{n-i-1}} \binom{p^{n-i-1}}{j} P_i(a)^{p(p^{n-i-1}-j)} (\delta P_i(a))^j \\
&= \sum_{i=0}^{n-1} p^i (P_i(a)^p)^{(p^{n-i-1})} \\
&\quad + \sum_{i=0}^{n-1} p^i \sum_{j=1}^{p^{n-1-i}} \binom{p^{n-i-1}}{j} P_i(a)^{p(p^{n-i-1}-j)} (p\delta P_i(a))^j
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{n-1} p^i \Big((P_i(a)^p)^{(p^{n-i-1})} \\
&\quad + \sum_{j=1}^{p^{n-1-i}} \binom{p^{n-i-1}}{j} P_i(a)^{p(p^{n-i-1}-j)} (p\delta P_i(a))^j \Big) \\
&= \sum_{i=0}^{n-1} p^i (P_i(a)^p + p\delta P_i(a))^{p^{n-1-i}} \qquad \square
\end{aligned}$$

By Lemma 2.3.3, $(R \times F) \circ w$ is a monomorphism. And by Proposition 2.3.5, it makes sense to compose $(w \times w) \circ (\mathbb{W}_{n-1}(u) \times \mathbb{W}_{n-1}(\phi)) \circ \mathcal{P}_{n-1}(a)$ with the inverse of $(R \times F) \circ w$. Now define,

$$\mathcal{P}_n(a) = ((R \times F) \circ w)^{-1} \circ (w \times w) \circ (\mathbb{W}_{n-1}(R_L) \times \mathbb{W}_{n-1}(\phi_L)) \circ \mathcal{P}_{n-1}(a)$$

Since $\mathcal{P}_n$ is a composition of ring homomorphisms, we get that $\mathcal{P}_n$ is a ring homomorphism. Also in proposition 2.3.5 we have shown that, $\mathcal{P}_n(a) = (P_0(a), ..., P_n(a))$. Hence we have proved the following,

Proposition 2.3.6. If $A = R[\mathbf{x}]$ then there exists a ring homomorphism $\mathcal{P}_n : A \to \mathbb{W}_n(J^n A)$ making the following diagrams commutative.

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{P}_{n-1}} & \mathbb{W}_{n-1}(J^{n-1}A) \\ \downarrow{\scriptstyle \mathcal{P}_n} & & {\scriptstyle \mathbb{W}_{n-1}(u)}\downarrow \ \downarrow{\scriptstyle \mathbb{W}_{n-1}(\phi)} \\ \mathbb{W}_n(J^n A) & \underset{F}{\overset{R}{\rightrightarrows}} & \mathbb{W}_{n-1}(J^n A) \end{array}$$

□

Let $\phi : R \to R$ be the unique lift of the Frobenius. Suppose $E = \sum_I c_I x^I$ be a multivariate polynomial with coefficients in R and I running through an indexing set. Define $E^\phi = \sum_I \phi(c_I) x^I$. In other words, E^ϕ is obtained from E by twisting its coefficients by ϕ. Let us rewrite $P_n(a)$ as follows

$$P_k(a) \;=\; \delta P_{k-1}(a) + \sum_{i=0}^{k-2} \sum_{j=1}^{p^{k-1-i}} \frac{p^{j-1}}{p^{k-i-1}} \binom{p^{k-i-1}}{j} P_i(a)^{p(p^{k-i-1}-j)} (\delta P_i(a))^j$$

Lemma 2.3.7. $\phi_L(E(a_0, ..., a_k)) = E^\phi(E(a_0^p + p\delta a_0, ..., a_k^p + p\delta a_k))$

Proof. For any $y \in J^n A$, we know $\phi_L(y) = y^p + p\delta y$. Hence

$$\begin{aligned} \phi_L(E(a_0, ..., a_k)) &= \sum_I \phi_L(c_I)\phi_L(a_0, ..., a_k)^I \\ &= \sum_I \phi(c_I)\phi_L(a_0, ..., a_k)^I \\ &= E^\phi(E(a_0^p + p\delta a_0, ..., a_k^p + p\delta a_k)) \end{aligned}$$ □

Lemma 2.3.8. *There exists a polynomial E_k with coefficients in R such that, $\delta P_k(a) = E_k(P_0(a), ..., P_{k+1}(a))$ for all k.*

Proof. We will prove this by induction. For $k = 0$, $\delta P_0(a) = P_1(a)$ and hence the result holds true. Suppose the lemma is true for $k-1$, that is, $\delta P_i(a) = E_{i-1}(P_0(a), ..., P_i(a))$ for all $i \leq k-1$. Then we know that, $P_{k+1}(a) = \delta P_k(a) + H'(P_0(a), ..., P_k(a))$ for some polynomial H', which implies, $\delta P_k(a) = P_{k+1}(a) - H'(P_0(a), ..., P_k(a))$ and this completes the proof. □

Lemma 2.3.9. *Let G be a polynomial with coefficients in R. Then there exists an H such that, $\delta G(P_0(a), ..., P_k(a)) = H(P_0(a), ..., P_{k+1}(a))$*

Proof. We know that

$$\begin{aligned}\delta G(P_0(a), ..., P_k(a)) &= \frac{\phi_L(G(P_0(a),...,P_k(a)) - G(P_0(a),...,P_k(a))^p}{p} \\ &= \frac{G^{\phi}(P_0(a)^p + p\delta P_0(a),...,P_k(a)^p + p\delta P_k(a)) - G(P_0(a),...,P_k(a))^p}{p}\end{aligned}$$

, by lemma 2.3.7.

By lemma 2.3.8, for all i, there exists E_i with coefficients in R such that $\delta P_i(a) = E_i(P_0(a), ..., P_{i+1}(a))$. And hence substituting this in the above equation, we have proved the result. □

Lemma 2.3.10. *For each k, there exists a polynomial G_k with coefficients in R such that $P_k(a) = \delta^k a + G_k(P_0(a), ..., P_{k-1}(a))$*

Proof. We will proceed by induction on k. For $k = 1$, we know that $P_1(a) = \delta a$ and hence the result is true. Suppose it is true for $k-1$. Then there exists a polynomial G_{k-1} with coefficients in R such that $P_{k-1}(a) = \delta^{k-1}a + G_{k-1}(P_0(a), ..., P_{k-2}(a))$.

Then,

$$\begin{aligned}\delta P_{k-1}(a) &= \delta^k a + \delta G_{k-1}(P_0(a),...,P_{k-2}(a)) + \\ &\qquad C_p(\delta^{k-1}a, G_{k-1}(P_0(a),...,P_{k-2}(a))) \\ &= \delta^k a + \delta G_{k-1}(P_0(a),...,P_{k-2}(a)) + \\ &\quad C_p(P_{k-1}(a) - G_{k-1}(P_0(a),...,P_{k-2}(a), G_{k-1}(P_0(a),...,P_{k-2}(a))).\end{aligned}$$

By lemma 2.3.9,

$\delta G_{k-1}(P_0(a),...,P_{k-2}(a)) = H(P_0(a),...,P_{k-1}(a))$ for some polynomial H. Hence we get

$$\delta P_{k-1}(a) = \delta^k a + G'(P_0(a),...,P_{k-1}(a))$$

Hence using the formula of $P_k(a)$ we get,

$$\begin{aligned}P_k(a) &= \delta P_{k-1}(a) + H'(P_0(a),...,P_{k-1}(a)), \text{ for some polynomial } H'. \\ &= \delta^k a + G'(P_0(a),...,P_{k-1}(a)) + H'(P_0(a),...,P_{k-1}(a)).\end{aligned}$$

and this completes the proof. □

Proposition 2.3.11. $R[x,...,x^{(n)}] \simeq R[P_0(x),...,P_n(x)]$

Proof. We will prove this by induction on n. When $n = 1$, we know that $x = P_0(x)$ and hence the result holds true. Suppose true for $n-1$, that is $R[x,...,x^{(n-1)}] \simeq R[P_0(x),...,P_{n-1}(x)]$. Then by the formula of $P_n(x)$, we have $P_n(x) = Q(x,...,x^{(n)})$ where Q is a polynomial with coefficients in R.

By lemma 2.3.10, we know that $x^{(n)} = P_n(x) - G_n(P_0(x),...,P_{n-1}(x)$, for some polynomial G_n with coefficients in R. Hence combining the above two, we conclude that $R[x,...,x^{(n)}] \simeq R[P_0(x),...,P_n(x)]$ and we are done. □

Lemma 2.3.12. *If* $g \in (P_0(f), ..., P_{k-1}(f))$ *then* $\delta g \in (P_0(f), ..., P_k(f))$.

Proof. It is sufficient to show that $\delta P_{k-1}(f) \in (P_0(f), ..., P_k(f))$. We will prove this by induction. For $k = 0$, we know that $P_0(f) = f$ and $\delta f = P_1(f)$ and hence the lemma is true. Now suppose the result is true for $k-1$, that is $\delta P_{k-2}(f) \in (P_0(f), ..., P_{k-1}(f))$. Then $P_k(f) = \delta P_{k-1}(f) + h$ where $h \in (P_0(f), ..., P_{k-1}(f))$, which implies that $\delta P_{k-1}(f) \in (P_0(f), ..., P_k(f))$ and this completes the proof. □

Lemma 2.3.13. $P_k(f) = \delta^k f + g$ *where* $g \in (P_0(f), ..., P_{k-1}(f))$.

Proof. We will prove this by induction. For $k = 1$, we know that $P_1(f) = \delta f$ and hence the result is true. Suppose true for $k-1$, that is, $P_{k-1}(f) = \delta^{k-1} f + g$ where $g \in (P_0(f), ..., P_{k-2}(f))$. Then, $P_k(f) = \delta P_{k-1}(f) + h$ where $h \in (P_0(f), ..., P_{k-1}(f))$. Therefore, $P_k(f) = \delta(\delta^{k-1} f + g) + h$ where $g \in (P_0(f), ..., P_{k-2}(f))$. But by lemma 2.3.12, we conclude that $\delta(\delta^{k-1} f + g) = \delta^k f + g'$ where $g' \in (P_0(f), ..., P_k(f))$ and hence $P_k(f) = \delta^k f + g' + h$ and $g' + h \in (P_0(f), ..., P_{k-1}(f))$ and this completes the proof. □.

Lemma 2.3.14. *The following is an equality of ideals*

$$(f, \delta f, ..., \delta^k f) = (P_0(f), ..., P_k(f))$$

Proof. We make the following :

Claim. $(P_0(f), ..., P_k(f)) \subset (f, ..., \delta^k f)$.

We will prove this by induction. For $k = 0$, we know that $P_0(f) = f$. Let us assume the claim is true for $k-1$, that is, $P_{k-1}(f) \in (f, ..., \delta^{k-1} f)$. Then $P_k(f) = \delta P_{k-1}(f) + G$ where $G \in (f, ..., \delta^{k-1} f)$. Since $P_{k-1}(f) \in (f, ..., \delta^{k-1} f)$ implies that $\delta P_{k-1}(f) \in (f, ...\delta^{k-1} f, \delta^k f)$ and hence $P_k(f) \in (f, ...\delta^k f)$.

Claim. $(f, ..., \delta^k f) \subset (P_0(f), ..., P_k(f))$

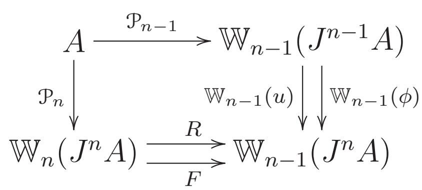
By lemma 2.3.13 we have $\delta^k f = P_k(f) - g \in (P_0(f), ..., P_k(f))$ and this proves the claim and also completes the proof. $\square$

Let $\mathbf{f}$ denote a collection of multivariate polynomials in $\mathbf{x}$.

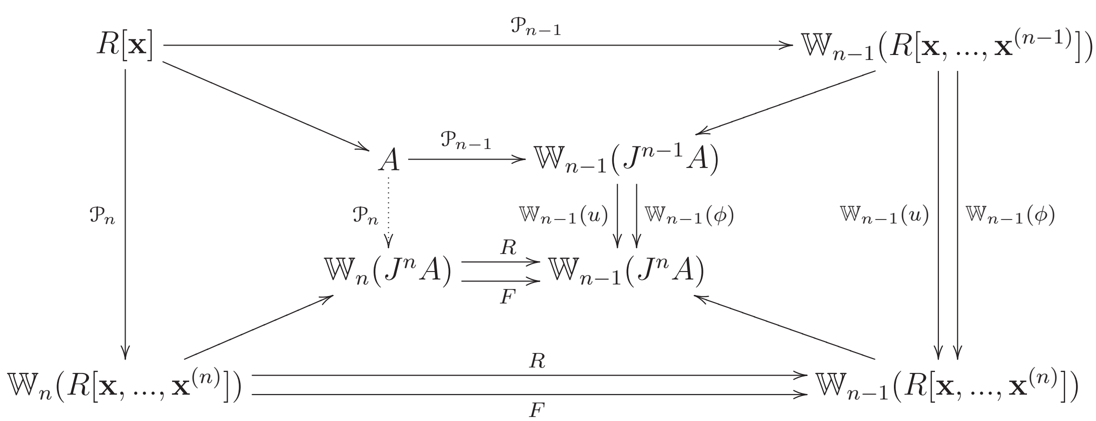
Proposition 2.3.15. $\frac{R[\mathbf{x},\mathbf{x}',...,\mathbf{x}^{(\mathbf{n})}]}{(\mathbf{f},\delta\mathbf{f},...,\delta^{\mathbf{n}}\mathbf{f})} \simeq \frac{R[P_0(\mathbf{x}),...,\mathbf{P_n}(\mathbf{x})]}{(P_0(\mathbf{f}),...,\mathbf{P_n}(\mathbf{f}))}$

Proof. This follows immediately from the above lemma. $\square$

Proposition 2.3.16. Let $A = \frac{R[\mathbf{x}]}{(\mathbf{f})}$. There exists a homomorphism $\mathcal{P}_n : A \to \mathbb{W}_n(J^n A)$ making the following diagrams commutative.



Proof. We define $\mathcal{P}_n(\overline{g(x)}) := \overline{\mathcal{P}_n(g(x))}$, where $g(x)$ is a pre-image in $R[\mathbf{x}]$ of $\overline{g(x)} \in A$. We need to show that this is well defined. It is enough to show that if f is a generator of the ideal $(\mathbf{f})$, then $\overline{\mathcal{P}_n(f(x))} = 0$, in other words, $\mathcal{P}_k(f(x)) \in (\mathbf{f}, ..., \mathbf{f}^{(n)})$ for all k. But this follows from Lemma 2.3.14. The commutativity of the above diagram follows from the commutation of the outer diagram which is true by Lemma 2.3.6



$\square$

Let C be any R-algebra and $w = (w_0, ..., w_n)$ and $w' = (w_0, ..., w_n)$ be two elements in $\mathbb{W}_n(C)$. Let the addition and the multiplication laws in the Witt ring be given as

$$\begin{aligned} (w + w')_k &= Q_k\left((w_0, ...w_k), (w'_0, ..., w'_k)\right) \\ (w.w')_k &= L_k\left((w_0, ..., w_k), (w'_0, ..., w'_k)\right) \end{aligned}$$

Given a ring homomorphism $\gamma : R[\mathbf{x}] \to \mathbb{W}_n(B)$ such that $\gamma(a) = (\gamma_0(a), ..., \gamma_n(a))$, define the ring homomorphism,

$$\begin{aligned} &\rho_\gamma : R[P_0(\mathbf{x}), ..., P_n(\mathbf{x})] \to B, \text{ by} \\ &\rho_\gamma(P_i(\mathbf{x})) := \gamma_i(\mathbf{x}) \end{aligned}$$

We will show that $\rho_\gamma(P_i(g(\mathbf{x}))) = \gamma_i(g(\mathbf{x}))$ for all i.

Lemma 2.3.17. *$\rho_\gamma(P_i(x^m))) = \gamma_i(x^m)$ for all i and m.*

Proof. We will prove this by induction on m. For $m = 1$, the result is true by definition. Suppose true for $m - 1$. Then

$$\begin{aligned} \rho_\gamma(P_i(x^m)) &= \rho_\gamma(L_i((P_0(x^{m-1}), ..., P_i(x^{m-1})), (P_0(x), ..., P_i(x)))) \\ &= L_i((\rho_\gamma(P_0(x^{m-1})), ..., \rho_\gamma(P_i(x^{m-1}))), (\rho_\gamma(P_0(x)), ..., \rho_\gamma(P_i(x)))) \\ &= L_i((\gamma_0(x^{m-1}), ..., \gamma_i(x^{m-1})), (\gamma_0(x), ..., \gamma_i(x))) \\ &= \gamma_i(x^m) \end{aligned}$$

and this completes the proof. □

Lemma 2.3.18. *If $g(\mathbf{x}) = cx_1^{i_1}...x_k^{i_k}$, where $c \in R$, then $\rho_\gamma(P_i(g(\mathbf{x}))) = \gamma_i(g(\mathbf{x}))$*

Proof. We will prove this by induction on the number of indeterminates k.

$$\begin{aligned}
\rho_\gamma(P_i(c\prod_{j=1}^{k} x_j^{i_j})) &= \rho_\gamma(L_i((P_0(c\prod_{j=1}^{k-1} x_j^{i_j}), ...P_i(c\prod_{j=1}^{k-1} x_j^{i_j})), (P_0(x_k^{i_k}), ..., P_i(x_k^{i_k}))))\\
&= L_i\left((\rho(P_0(c\prod_{j=1}^{k-1} x_j^{i_j})), ...\rho(P_i(c\prod_{j=1}^{k-1} x_j^{i_j}))), (\rho(P_0(x_k^{i_k})), ..., \rho(P_i(x_k^{i_k})))\right)\\
&= L_i((\gamma_0(c\prod_{j=1}^{k-1} x_j^{i_j}), ..., \gamma_i(c\prod_{j=1}^{k-1} x_j^{i_j})), (\gamma_0(x_k^{i_k}), ..., \gamma_i(x_k^{i_k})))\\
&= \gamma_i(c\prod_{j=1}^{k} x_j^{i_k})
\end{aligned}$$

and this completes the proof. □

Proposition 2.3.19. Let $g(\mathbf{x}) = \sum_I c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}$ where I is a multi-indexing set. Then, $\rho_\gamma(P_i(g(\mathbf{x}))) = \gamma_i(g(\mathbf{x}))$.

Proof. Suppose $g(\mathbf{x})$ has l summands. We will proceed by induction on the number of summands of g. When the number of summands is 1, we have already showed the claim in the above lemma. Assume the proposition is true for $l-1$ summands then

$$\begin{aligned}
\rho_\gamma(P_i(g(\mathbf{x}))) &= \rho_\gamma(Q_i(P_0(\sum_{I\backslash\{i_1,...,i_m\}} c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}), ..., P_i(\sum_{I\backslash\{i_1,...,i_m\}} c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}})),\\
&\qquad (P_0(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}}), ..., P_i(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}})))\\
&= Q_i((\rho_\gamma(P_0(\sum c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}))..., \rho_\gamma(P_0(\sum c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}))),\\
&\qquad (\rho_\gamma(P_0(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}})), ...\rho_\gamma(P_0(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}})))\\
&= Q_i((\gamma_0(\sum_{I\backslash\{i_1,...,i_m\}} c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}), ..., \gamma_i(\sum_{I\backslash\{i_1,...,i_m\}} c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}}))\\
&\qquad (\gamma_0(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}}), ..., \gamma_i(c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}})))\\
&= \gamma_i(\sum_{I\backslash\{i_1,...,i_m\}} c_I x_{i_1}^{j_{i_1}}...x_{i_k}^{j_{i_k}} + c_{i_1,...,i_m} x_{i_1}^{j_{i_1}}...x_{i_m}^{j_{i_m}})\\
&= \gamma_i(g(\mathbf{x}))
\end{aligned}$$

and this completes the proof. □

Corollary 2.3.20. If $f(x) \in R[x]$ such that $\gamma(f(x)) = 0$, then $\rho_\gamma(P_i(x)) = 0$.

Proof. $\gamma(f(x)) = (\gamma_0(f(x)), ..., \gamma_n(f(x))) = 0$. Hence by Proposition 2.3.19, $\rho_\gamma(P_i(x)) = \gamma_i(f(x)) = 0$. for all i. □

Corollary 2.3.21. If $\gamma : R[x] \to \mathbb{W}_n(B)$ such that $\gamma(f(x)) = 0$, then $\rho_\gamma : R[x, ..., x^{(n)}] \to B$ descends to a ring homomorphism, $\rho_\gamma : \frac{R[x,...,x^{(n)}]}{(f,...,\delta^n f)} \to B$.

Proof. This follows from $\frac{R[x,...,x^{(n)}]}{(f,...,\delta^n f)} \simeq \frac{R[P_0(x),...,P_n(x)]}{(P_0(f),...,P_n(f))}$ and, $\gamma(f(x)) = 0 \Rightarrow \rho_\gamma(P_i(f)) = 0$. □

Let $A = \frac{R[\mathbf{x}]}{(\mathbf{f})}$, then $J^n A = \frac{R[\mathbf{x},...,\mathbf{x}^{(n)}]}{(\mathbf{f},...\delta^n \mathbf{f})} \simeq \frac{R[P_0(\mathbf{x}),...,P_n(\mathbf{x})]}{(P_0(\mathbf{f}),...,P_n(\mathbf{f}))}$. It follows immediately from the above corollary that ρ_γ descends to a ring homomorphism from $J^n A$ to B.

Definition 2.3.22. Define $\Theta_n : Hom\ (A, \mathbb{W}_n(B)) \to Hom\ (J^n A, B)$ as $\Theta_n(\gamma) := \rho_\gamma$.

Definition 2.3.23. Define, $\Psi_n : Hom\ (J^n A, B) \to Hom\ (A, \mathbb{W}_n(B))$ as

$$\Psi_n(\rho)(a) := (\rho(P_0(a)), ..., \rho(P_n(a)))$$

Theorem 2.3.24. *$\Theta_n \circ \Psi_n = \mathbb{1}$ and $\Psi_n \circ \Theta_n = \mathbb{1}$, in other words $\theta_n = \Psi_n^{-1}$. We also have $\theta_{n-1}(R \circ \Psi_n(\mathbb{1}_{J^n A})) = u, \quad \theta_{n-1}(F \circ \Psi_n(\mathbb{1}_{J^n A})) = \phi$.*

Proof. Let $A = \frac{R[\mathbf{x}]}{(\mathbf{f})}$. Given a $\rho : J^n A \to B$, then $\Psi_n(\rho)(\overline{\mathbf{x}}) = (\rho(P_0(\mathbf{x})), ..., \rho(P_n(\mathbf{x})))$ where $\overline{\mathbf{x}}$ is the image of $\mathbf{x}$ in A. Hence we get $\Theta_n(\Psi_n(\rho))(P_i(\mathbf{x})) = \rho(P_i(\mathbf{x}))$ for all i, but this is ρ itself, which proves, $\Theta_n \circ \Psi_n = \mathbb{1}$.

On the other hand, given a $\gamma : A \to \mathbb{W}_n(A)$, $\gamma(a) = (\gamma_0(a), ..., \gamma_n(a))$ then, $\Theta_n(\gamma)(P_i(\mathbf{x})) = \gamma_i(\mathbf{x})$ for all i by definition. Now

$$\begin{aligned} \Psi_n(\Theta_n(\gamma))(\mathbf{x}) &= (\Theta_n(\gamma)(P_0(\mathbf{x})), ..., \Theta_n(\gamma)(P_n(\mathbf{x}))) \\ &= (\gamma_0(\mathbf{x}), ..., \gamma_n(\mathbf{x})) \\ &= \gamma(\mathbf{x}) \end{aligned}$$

and this proves the latter identity, $\Psi_n \circ \Theta_n = \mathbb{1}$.

The last part of the theorem follows from proposition 2.3.16. □

Remark 2.3.25. We would like to remark that the above proof can be shortened if one assumes that $J^n A$ is torsion free. However, this assumption would be a very strong one as it could be very well that even if A is not torsion free, $J^n A$ could have p-torsion. The following is an example- consider $A = \frac{\mathbb{Z}[x]}{(x^p)}$ then $J^1 A = \frac{\mathbb{Z}[x,x']}{(x^p, pf)}$ where f is a polynomial in x and x'. Here the image of f in $J^1 A$ is a torsion element.

Corollary 2.3.26. If $X = \mathbb{A}^k$ then $\mathfrak{J}^n X \simeq \mathbb{W}_n^k$.

Proof. $X = Spec\ R[\mathbf{x}]$ where $\mathbf{x}$ represents k indeterminate variables. Then $Hom\ (J^n(R[\mathbf{x}]), B) \simeq Hom\ (R[\mathbf{x}], \mathbb{W}_n(B))$. But $Hom\ (R[\mathbf{x}], \mathbb{W}_n(B)) \simeq \mathbb{W}_n(B)^k$ and hence the result follows. □

If B is a p-adically complete p-torsion free ring such that it has a perfect residue field k then we know that $B/p^{n+1}B \simeq \mathbb{W}_n(k)$ [26]. We obtain the following, which shows the bijection between the k points of the jet-space with the *Greenberg transform* of X.

Corollary 2.3.27. If X is a scheme over R then $\mathfrak{J}^n X(k) \simeq X(B/p^{n+1}B)$.

If B is a p-adically complete ring, then we have $Hom\ (\widehat{J^n A}, B) \simeq Hom\ (J^n A, B)$ from the universality properties of completions. Also, if we further assume that A is p-adically complete, then we obtain

$$Hom\ (\widehat{J^n A}, B) \simeq Hom_\delta(A, \mathbb{W}_n(B))$$

which is an isomorphism in the category of p-adically complete rings because by 2.2.5, $\mathbb{W}_n(B)$ is p-adically complete.

Hence, given an affine scheme X, we have shown that

$$Hom(Spf\ B, J^n X)) \simeq Hom(Spf\ \mathbb{W}_n(B), \hat{X})$$

where $\hat{X}$ denotes the p-adic formal completion of X.

Definition 2.3.28. We will call a ring A a δ-ring if there exists a p-derivation $\delta : A \to A$.

The following Proposition is an application of our map $\mathcal{P}$.

Proposition 2.3.29. Let A be a δ-ring which is also p-torsion free, then its nilradical is a δ-closed ideal, i.e, it is preserved under δ.

Proof. The derivation δ on A induces the ring homomorphism $\mathcal{P} : A \to \mathbb{W}_1(A)$. Hence if $a \in A$ is nilpotent then $\mathcal{P}(a)$ is nilpotent too. Recall that the image of a by Teichmuller lift $A \xrightarrow{[\_]} \mathbb{W}_1(A),\ a \mapsto (a, 0, 0, ...)$, is also nilpotent in $\mathbb{W}_1(A)$. There fore $\mathcal{P}(a) - [a]$ is nilpotent. But $\mathcal{P}(a) - [a] = (0, \delta a)$ and we have $(0, \delta a)^N = 0$ for some N which implies that $p^{N+1}\delta a^N = 0 \ \Rightarrow\ \delta a^N = 0$ because A is p-torsion free. □

2.4 Prolongations of formal groups

Let $X = Spf\ \widehat{A}$ be an affine p-adic formal scheme where A is noetherian and finitely generated and $\widehat{A}$ denotes its p-adic completion. Then A can be represented as $A = R[\mathbf{x}]/(\mathbf{f})$; $\mathbf{x}$ represents the collection of finite number of indeterminates and $(\mathbf{f})$ the ideal generated by a collection of polnomials $\mathbf{f}$. Then we define the p-adic jetspaces of J^nX as $J^nX = Spf\ \widehat{J^nA}$ where $J^nA = R[\mathbf{x}, ..., \mathbf{x^{(n)}}]/(\mathbf{f}, ..., \mathbf{f^{(n)}})$ as before. Consider the p-adic formal group $\hat{\mathbb{G}}_a = Spf\ \widehat{R[x]}$. Then for any p-adically complete R-algebra B, we have

$$Hom\ (\widehat{J^n(R[x]}), B) \simeq Hom\ (\widehat{R[x]}, \mathbb{W}_n(B))$$

But then $Hom\ (R[x]\hat{,} \mathbb{W}_n(B)) \simeq \mathbb{W}_n(B)$ since it is sufficient to specify the image of the generator x which implies that $Hom\ (\widehat{J^n(R[x]}), \_) \simeq \mathbb{W}_n(\_)$. Hence we obtain

$$J^n\hat{\mathbb{G}}_a \simeq \mathbb{W}_n$$

as formal schemes. Note that the structure of $J^n\hat{\mathbb{G}}_a$ as a ring object is precisely that of the structure of Witt vectors.

2.5 Morphisms between two compositions

Suppose we are in the category of R-algebras. Consider the functor $\mathbb{D}_1 : Rings \to Rings$ where $\mathbb{D}_1(B) = B[\epsilon]/(\epsilon^2)$. $\mathbb{D}_1$ attaches the ring of dual numbers to a given ring B. Then given $\alpha := (\alpha_0, ..., \alpha_n) \in B^{n+1}$, we will construct a functorial homomorphism $\Psi_\alpha : \mathbb{D}_1 \circ \mathbb{W}_n(B) \to \mathbb{W}_n \circ \mathbb{D}_1(B)$.

Proposition 2.5.1. Let B be torsion free. If $\alpha \in \mathbb{W}_n(\mathbb{D}_1(B))$ such that $\alpha^2 = 0$ then α is of the form $(\alpha_0\epsilon, ..., \alpha_n\epsilon)$ where $\alpha_i \in B$ for all i.

Proof. We will prove this by induction. Since B is torsion free implies that $\mathbb{D}_1(B)$ is torsion free too. Hence the ghost map $w : \mathbb{W}_n \circ \mathbb{D}_1(B) \to \mathbb{D}_1(B)^{n+1}$ is injective. In the case of $n = 1$ the result follows from an easy computation. Suppose the result is true for $n-1$. Let $\alpha = (\alpha_0\epsilon, ..., \alpha_{n-1}\epsilon, \beta + \alpha_n\epsilon)$ be a square zero term in $\mathbb{W}_n \circ \mathbb{D}_1(B)$. Then we have $w_n(\alpha)^2 = (p^n\beta + p^n\alpha_n)^2\epsilon = 0$. This equation gives us the solution of $\beta = 0$ and we are done. □

Proposition 2.5.2. $(\alpha_0\epsilon, ..., \alpha_n\epsilon)$ is an element in $\mathbb{W}_n \circ \mathbb{D}_1(B)$ whose square is zero.

Proof. It is sufficient to consider the case when B is torsion free. Then the result follows from proposition 2.5.1. □

Both $\mathbb{D}_1 \circ \mathbb{W}_n(B)$ and $\mathbb{W}_n \circ \mathbb{D}_1(B)$ are $\mathbb{W}_n(B)$ algebras. Hence giving a homomorphism from $\mathbb{D}_1 \circ \mathbb{W}_n(B)(= \mathbb{W}_n(B)[\epsilon]/(\epsilon^2))$ to $\mathbb{W}_n \circ \mathbb{D}(B)$ is equivalent to sending ϵ to a square-zero element in $\mathbb{W}_n \circ \mathbb{D}(B)$. By proposition 2.5.2, the ring homomorphism Ψ_α defined as $\Psi_\alpha(\epsilon) := (\alpha_0\epsilon, ..., \alpha_n\epsilon)$ for $\alpha \in R^{n+1}$ gives us the following

Proposition 2.5.3. There exists a family of canonical ring homomorphisms $\Psi_\alpha : \mathbb{D}_1 \circ \mathbb{W}_n(B) \to \mathbb{W}_n \circ \mathbb{D}_1(B)$, for $\alpha = (\alpha_1, ..., \alpha_n) \in B^{\times n+1}$ such that $\Psi_\alpha(\epsilon) = (\alpha_0\epsilon, ..., \alpha_n\epsilon)$

Example. In the case when $n = 1$ and $\alpha = (\alpha_0, \alpha_1) \in B^2$ one can easily check that

$$\Psi_\alpha((a_0, a_1) + (b_0, b_1)\epsilon) = (a_0 + \alpha_0 b_0 \epsilon, a_1 + (\alpha_1 b_0^p + p b_1 \alpha_1 + \alpha a_0^{p-1} b_0)\epsilon). \quad (2.5.1)$$

The following is a list of the first few terms where α_i's are chosen to be equal to 1,

$$b'_0 = b_0$$

$$\begin{aligned} b'_1 &= \textstyle\sum_{i=0}^{1} p^i b_i^{p^r-i} - a_i^p b'_i \\ &= b_0^p + p b_1 - a_0^p b_0 \\ &= w_1(\{b_i\}) - a_0^p b_0 \end{aligned}$$

$$\begin{aligned} b'_2 &= \textstyle\sum_{i=0}^{2} p^i b_0^{p^r-i} - \sum_{i=0}^{1} a_i^{p^{1-i}} b'_i \\ &= b_0^{p^2} + p b_1^p + p^2 b_2 - (a_0^p b_0 + a_1 b_0^p + p a_1 b_1 - a_0^p a_1 b_0) \\ &= w_2(\{b_i\}) - (a_0^p b_0 + a_1 b_0^p + p a_1 b_1 - a_0^p a_1 b_0) \end{aligned}$$

Theorem 2.5.4. *Let $X = Spec\ A$. Then there exists a morphism $\Psi : \mathbb{A}^{n+1} \times J^n TX \to TJ^n X$.*

Proof. For any ring B, note that

$$J^n TX(Spec\ B) \simeq Hom\ (A, \mathbb{D}_1 \circ \mathbb{W}_n(B)) \quad (2.5.2)$$

and

$$TJ^n X(Spec\ B) \simeq Hom\ (A, \mathbb{W}_n \circ \mathbb{D}_1(B)). \quad (2.5.3)$$

Now define $\Psi : (\mathbb{A}^{n+1} \times J^n TX)(Spec\ B) \to TJ^n X(Spec\ B)$ as $\Psi(\alpha, \chi) := (\chi \circ \Psi_\alpha)$. □

2.6 Prolongation of derivatives

Let $A = \mathcal{O}(X)$ be the co-ordinate ring of the affine scheme X, smooth over R. Assume X possesses a system $T_1, ..., T_d$ of étale coordinates. Then recall from [10] that

$$\widehat{J^nA} := \mathcal{O}(J^nX) = A[T_1', ..., T_d', ..., T_1^{(n)}, ..., T_d^{(n)}]\hat{} \tag{2.6.1}$$

For a scheme $X = Spec\ A$, let Ω be its sheaf of diffentials. Let $TA = Symm\ \Omega$. Then $TY = Spec\ TA$ is the physical tangent scheme of X and TA is the co-ordinate ring of functions of TX. Let B^* be a prolongation sequence and let $\partial : A \to B^0$ be a derivation. Then we can canonically prolong the derivation as follows

Theorem 2.6.1. *There exists a compatible system of derivatons ∂ making the following diagram commute:*

$$\begin{array}{ccc} J^nA & \xrightarrow{\partial} & B^n \\ \uparrow & & \uparrow \\ \vdots & & \vdots \\ J^1A & \xrightarrow{\partial} & B^1 \\ \uparrow & & \uparrow \\ A & \xrightarrow{\partial} & B^0 \end{array}$$

Proof. Given a derivation $A \xrightarrow{\partial} B^0$, we get a ring homomorphism $TA \to B^0$ by universal property of the tangent ring. And with the universal property of the jet spaces, we obtain

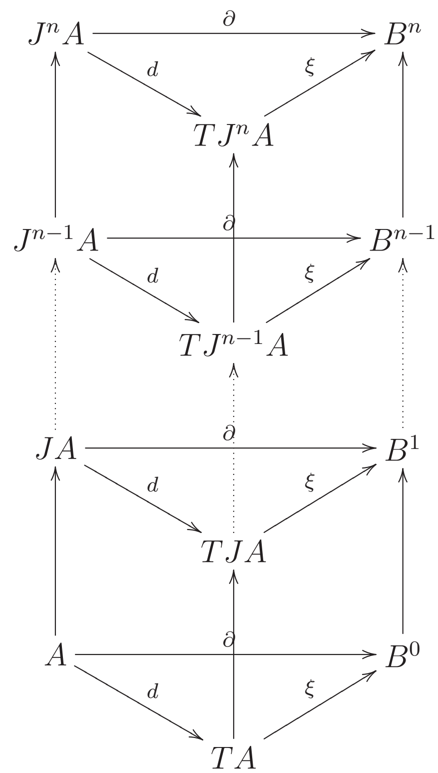
$$J^nTA \to B^n \tag{2.6.2}$$

But there is a canonical morphism $\psi : TJ^nA \to J^nTA$ hence by composing we obtain

$$\begin{array}{ccc} J^nTA & \longrightarrow & B^n \\ \uparrow & & \\ TJ^nA & & \end{array}$$

Hence we obtain a morphism $TJ^nA \to B^n$ which by the universal property of the tangent ring gives us the required derivation. $\square$

The above theorem can be diagrammatically represented as



where the vertical arrows $TJ^{n-1}A \to TJ^nA$ are obtained from the canonical prolongation sequence $J^{n-1}A \to J^nA$ after applying the Tangent functor T to it. Now let us reinterpret the above prolongation of derivatives for the sake of computation.

Let $B^0 \to ... \to B^{n-1} \overset{(u,\delta)}{\to} B^n ...$ be a prolongation sequence. Then for each n, the pair of maps (u, δ) can be interpreted as a ring homomorphism $B^{n-1} \overset{(u,\delta)}{\to} \mathbb{W}_1(B^n)$. This induces the ring homomorphism $\mathbb{D}_1(B^{n-1}) \overset{(u,\delta)}{\to} \mathbb{D}_1(\mathbb{W}_1(B^n))$, and hence composing with Ψ we get the following

$$\begin{array}{ccc} \mathbb{D}_1(B^{n-1}) & \xrightarrow{(u,\delta)} & \mathbb{D}_1(\mathbb{W}_1(B^n)) \\ & & \downarrow \Psi \\ & & \mathbb{W}_1(\mathbb{D}_1(B^n)) \end{array}$$

Hence, we have created a new prolongation sequence $\mathbb{D}_1(B^*)$ from B^*. And from the computation above we get

$$\Psi \circ (u, \delta)(a + b\epsilon) = (u(a) + u(b)\epsilon,\ \delta a + (\phi(b) + u(a)^{p-1}u(b))\epsilon) \tag{2.6.3}$$

In particular,

$$(\Psi \circ (u, \delta))\epsilon = (\epsilon, \epsilon) \tag{2.6.4}$$

Hence if we start with a derivation $(f, \partial) : A \to B^0$, in other words a ring homomorphism $(f, \partial) : A \to \mathbb{D}_1(B^0)$ and since $\mathbb{D}_1(B^*)$ is a prolongation sequence, by universal property, we have a morphism of prolongation sequences $J^*A \xrightarrow{(f,\partial)} \mathbb{D}_1(B^*)$ satisfying

$$\begin{array}{ccccc} J^nA & \xrightarrow{(f,\partial)} & \mathbb{D}_1(B^n) & \qquad & \epsilon \\ \delta\uparrow & & \uparrow\delta & & \delta\uparrow \\ J^{n-1}A & \xrightarrow[(f,\partial)]{} & \mathbb{D}_1(B^{n-1}) & & \epsilon \end{array}$$

and $(f, \partial)(x) = f(x) + (\partial x)\epsilon$.

Proposition 2.6.2.

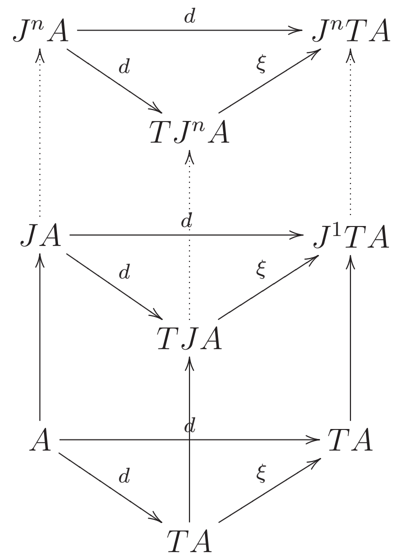
$$\xi(dx) = \partial(\delta x) = \phi(\partial x) - f(x)^{p-1}\partial x \tag{2.6.5}$$

Proof. From the commutation of the above diagram we get

$$\begin{aligned} (f, \partial)(\delta x) &= \delta \circ (f, \partial)(x) \\ f(\delta x) + \partial(\delta x)\epsilon &= \delta(f(x) + (\partial x)\epsilon \\ &= f(\delta x) + (\phi(\partial x) - f(x)^{p-1}\partial x)\epsilon \end{aligned}$$

and comparing the ϵ coordinate we get the required result. □

In particular, if we chose the prolongation sequence $B^* = \{J^*TA\}$ then we obtain



where ξ satisfies by Proposition 2.6.5

$$\xi(dx) = \phi(dx) - x^{p-1}dx \tag{2.6.6}$$

Recall from [10], 3.21 the following definition

$$(J^nTA)^+ := \sum_{j=0}^{n} \sum_{f \in A} \widehat{J^nA}(df)^{\phi^j} \subset \widehat{J^nTA} \tag{2.6.7}$$

Then $(J^nTA)^+$ is a free $\widehat{J^nA}$ module with basis

$$\{(dT_i)^{\phi^j} \mid 1 \leq i \leq d, 0 \leq j \leq n\}$$

Proposition 2.6.3. The homomorphism $\xi : T(\widehat{J^nA}) \to \widehat{J^nTA}$ induces the isomorphism

$$\xi : T(\widehat{J^nA}) \simeq (J^nTA)^+ \tag{2.6.8}$$

Proof. We will prove this by iduction on n. For $n = 0$, it is clear as $(J^0TA)^+ \simeq TA$. From 2.6.5, for all i we obtain

$$\xi(d(\delta T_i^{(n-1)})) = \phi(\xi(d(T_i^{(n-1)}))) - (T_i^{(n-1)})^{p-1}\xi(d(T_i^{(n-1)}))$$

But by induction $\xi(d(T_i^{(n-1)})) = (dT_i)^{\phi^{n-1}} + O(n-2)$ which implies that $\phi(\xi(d(T_i^{(n-1)}))) = (dT_i)^{\phi^n} + O(n-1)$ and we are done. $\square$

To rephrase what we have shown above

$$T(\widehat{J^nA}) \simeq \widehat{J^nA}[dT_1, ..., d(T_1^{(n-1)}), ..., d(T_d^{(n-1)})] \stackrel{\xi}{\simeq} \widehat{J^nA}[dT_1, ..., (dT_1)^{\phi^n}, ..., (dT_d)^{\phi^n}] \tag{2.6.9}$$

Hence, given a derivation $\partial : A \to B^0$, in order to prolong it between $\{J^*A\}$ and B^*, it is sufficient to specify where the generators of TJ^nA go. For example, the *δ-conjugate operators* ∂_n in [10] are obtained by sending

$$\begin{array}{lll} i)\ d(T_i^{(j)}) & \to & 0,\ \forall\ 0 \leq i \leq d,\ 1 \leq j \leq n-1. \\ ii)\ d(T_i^{(n)}) & \to & \phi^n \partial T_i^{(n)}. \end{array} \tag{2.6.10}$$

Condition $ii)$ is equivalent to $\partial_n(\phi^n T_i) = p^n \phi^n \partial T_i$.

We could also define a new set of operators ∂_n^+ in the same spirit as above by specifying

$$\begin{array}{lll} i)\ d(T_i^{(j)}) & \to & 0,\ \forall\ 0 \leq i \leq d,\ 1 \leq j \leq n-1. \\ ii)\ d(T_i^{(n)}) & \to & \delta^n \partial T_i^{(n)}. \end{array} \tag{2.6.11}$$

Condition $ii)$ is equivalent to $\partial_n^+(\phi^n(T_i)) = p^n \delta^n \partial T_i$.

2.7 Base Change Property.

We record the following "base-change" property of arithmetic jet spaces. One notes that the arithmetic jetspace, if not p-adically completed, does not behave well under localization. If $X = Spec\ A$ and $s \in A$, consider the open subset $X_s = Spec\ A_s \subset X$. Then by [10], $\mathcal{J}^1(X_s) = Spec\ (J^1A)_{s\phi(s)} \neq (\mathcal{J}^1X)_s$ where recall that $\mathcal{J}^1X = Spec\ J^1A$.

Hence one can see that the non-completed jet-space apparently does not behave well under the localisation, whereas it is so in the case of differential algebra [5].

However, unlike the geometric jet spaces, arithmetic jet spaces have more than one canonical map between $J^nX \to X$, namely the Frobenius ϕ and its higher powers. Once we take them into account, it is easy to see that the following is true.

Proposition 2.7.1. The following diagram is commutative

$$\begin{array}{ccc} J^nX_s & \xrightarrow{(\pi,\phi,\ldots,\phi^n)} & X_s^{n+1} \\ \big\downarrow & & \big\downarrow \\ J^nX & \xrightarrow[(\pi,\phi,\ldots,\phi^n)]{} & X^{n+1} \end{array}$$

Proof. Since $J^nX_s = Spec\ A_{(s,\phi(s),\ldots,\phi^n(s))}$ by [10], the result follows immediately. □

2.8 Canonical lift as an intersection of subschemes

Recall from [10] that for a given p-torsion free ring B which has a lift of Frobenius, one can define the lift of B points of a scheme X to B points of J^nX

$$\nabla^n : X(B) \to J^nX(B)$$

If $X = \mathbb{A}^n$ then for any $x \in \mathbb{A}^n(B)$ we have, $\nabla^n x = (x, \delta x, ..., \delta^n x)$. We will show that the canonical lift of the point can be realised as an intersection of subschemes. We will state it in an "intersection theory" setting. Consider the morphism $J^nX \overset{(\pi,\phi,\ldots,\phi^n)}{\longrightarrow} X^{n+1}$.

Lemma 2.8.1.

$$\phi^n(x) = p^n x^{(n)} + f_{n-1}(x, ..., x^{(n-1)}) \tag{2.8.1}$$

where $f_{(n-1)}$ is a polynomial of order $n-1$.

Proof. We will prove this by induction. For $n = 1$ it is clear. Assuming true for $n-1$,

$$\begin{aligned} \phi^n(x) &= \phi(p^{n-1}x^{(n-1)} + f_{n-2}) \\ &= p^{n-1}\phi(x^{(n-1)}) + \phi(f_{n-2}) \\ &= p^{n-1}(px^{(n)} + (x^{(n-1)})^p + \phi(f_{n-2}) \\ &= p^n x^{(n)} + (p^{n-1}(x^{(n-1)})^p + \phi(f_{n-2})) \end{aligned}$$

Call $f_{n-1} = p^{n-1}(x^{(n-1)})^p + \phi(f_{n-2})$ completes the proof. □

Proposition 2.8.2. $\nabla^n P = (\pi, \phi, ..., \phi^n)^*(P)$ where $P \in X(B)$.

Proof. Assume $X = Spec\ A$ is affine and suppose $A = R[x]/(f)$ where x represents a collection of indeterminates. Then
$J^n A = Spec\ R[x, ..., x^{(n)}]/(f, ..., \delta^n f)$. Suppose $P \in X(B)$ is given by the evaluation $x = a$, where $a \in B$. Then the pull-back of the cycle $P = \{x = a\}$ via $\pi, \phi, ..., \phi^n$ yields the following set of equations

$$\begin{aligned} &x = a \\ &\phi(x) = \phi(a) \\ &\vdots \qquad \vdots \\ &\phi^n(x) = \phi^n(a) \end{aligned} \tag{2.8.2}$$

We claim that the above system of equations yield the desired solution. We will use induction on n to prove. For $n = 0$, there is nothing more to prove. Assume true for $n-1$, that is, the first n equations listed above yields the solution

$$\begin{aligned} &x = a \\ &x' = \delta a \\ &\vdots \qquad \vdots \\ &x^{(n-1)} = \delta^{n-1}(a) \end{aligned} \tag{2.8.3}$$

Consider the final equation $\phi^n(x) = \phi^n(a)$. By 2.8.1 we can rewrite it as

$$p^n x^{(n)} + f(x, ..., x^{(n-1)}) = p^n \delta^n a + f(a, ...\delta^{(n-1)} a) \tag{2.8.4}$$

But by 2.8.3, we know that $x = a, ..., x^{(n-1)} = \delta^{n-1} a$ which implies $p^n x^{(n)} = p^n \delta^n a$ and since B is p torsion free, we obtain the solution $x^{(n)} = \delta^n a$ as required and this completes the proof. $\square$

Chapter 3

Differential Modular Forms

In the first few sections $(3.1 - 3.3)$ we present a review of the theory of modular forms, differential modular forms and conjugate operators acting on them. Then we present the result on ind-étale extension for an ordinary "framed" curve and its jet space in section 3.4. In section 3.5 we first show that the tangent ring of the jet space of the modular curve is isomorphic to the subring of differential modular forms of even weight. Finally, we show the main result of this chapter, Theorem 3.5.13, which shows how the ring of differential modular forms M^{∞} maps to the ring of generalized p-adic modular forms $\mathcal{W}$. In the end of this chapter, we show that there can not be a modular form ϵ such that $\epsilon^{p-1} = E_{p-1}$, in other words, the $(p-1)$-th root of E_{p-1} is not a modular form.

3.1 Prolongation Sequences

Let $\mathcal{C}$ be the category of p-adic formal schemes. By a prolongation sequence X^*, we will mean a sequence of morphisms in $\mathcal{C}$

$$X^0 \overset{\varphi_0}{\leftarrow} X^1 \leftarrow ... \leftarrow X^{n-1} \overset{\varphi_{n-1}}{\leftarrow} X^n \leftarrow ... \tag{3.1.1}$$

together with p-derivations of φ_n^* such that

$$\varphi_{n+1}^* \circ \delta_n = \delta_{n+1} \circ \varphi_n^* \tag{3.1.2}$$

where φ_n^* is the pull-back ring homomorphism on the structure sheaves $\mathcal{O}_{X^n}$ and $\mathcal{O}_{X^{n+1}}$. These prolongation sequences form a category $\mathcal{C}^*$ and we refer to [9] for a more detailed and general discussion. However, for our purpose, we restrict ourselves with prolongation sequences defined over $(Spf\ R)^*$ which is the prolongatiion sequence

$$Spf\ R \leftarrow Spf\ R \leftarrow ...$$

with the p-derivation δ defined by $\delta x = \frac{\phi(x) - x^p}{p}$ where $\phi : R \to R$ is the unique lift of frobenius.

Let X be a p-adic completion of a scheme. Then consider the prolongation sequence $J^*X = \{J^nX\}_n$ where J^nX is the n-th jetspace of X. Consider the forgetful functor $\mathcal{C}^* \to \mathcal{C}$ given by $X^* \to X^0$. Then for any prolongation sequence Z^* we have the following universal property

Proposition 3.1.1. $Hom_{\mathcal{C}}(Z^0, X) \simeq Hom_{\mathcal{C}^*}(Z^*, J^*X)$

Recall from [9], that given a prolongation sequence Y^*, one can consider a new 'shifted by m' prolongation sequence Y^{*+m} given by

$$Y^m \leftarrow Y^{m+1} \leftarrow ...$$

Definition 3.1.2. A δ-morphism from X to Y of order $\leq m$ is a morphism $f : J^{*+m}X \to J^*Y$

Diagramatically, f denotes the following compatible sequences of morphisms between p-adic formal schemes.

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ J^{m+1}X & \xrightarrow{f} & J^1Y \\ \downarrow & & \downarrow \\ J^mX & \xrightarrow{f} & Y \\ \downarrow & & \\ X & & \end{array}$$

Let us denote by $Hom^m(X,Y)$ the space of all δ-morphism from X to Y of order $\leq m$.

Definition 3.1.3. Define the ring of δ-functions of order $\leq m$ on X as $\mathcal{O}(J^mX) := Hom^m(X, \hat{\mathbb{A}}^1)$.

Proposition 3.1.4. Giving an element f in $Hom^m(X,Y)$ is equivalent to attaching to any prolongation sequence S^* a map

$$f_{S^*} : X(S^0) \to Y(S^m)$$

which is functorial in S^*.

Proof. Given $f \in Hom^m(X,Y)$ we have

$$\begin{array}{rcccl} X(S^0) & := & Hom\ (S^0, X) & \simeq & Hom\ (S^m, J^mX) \\ & & & & \downarrow \circ f \\ & & & & Hom\ (S^m, Y) \end{array}$$

To obtain the inverse of the above association, given $f_{S^*} : X(S^0) \to Y(S^m)$, chose $S^* = J^*X$. Then we have

$$f_{J^*X} : X(X) = Hom\ (X,X) \quad \to \quad Hom\ (J^mX, Y) = Y(J^mX)$$

and pick the image of the identity morphism in $Hom\ (J^mX, Y)$ and call it f and this is the required inverse and this completes the proof. $\square$

3.2 A review of modular and δ-modular forms and Katz's ring

Before proceeding with explaining our next results let us recall a few basic facts of modular curves and forms. We refer to [17] for detailed discussion. Consider the modular curve $Y_1(N)$ defined over $\mathbb{Z}[1/N, \zeta_N]$ which is the representable object for the functor from rings over $\mathbb{Z}[1/N, \zeta_N]$ to sets defined as: given a $\mathbb{Z}[1/N, \zeta_N]$-ring B, we consider the isomorphism classes of pairs (E, ι) where E is an elliptic curve defined over B and $\iota : (\mathbb{Z}/N\mathbb{Z})_B \subset E$ is a level $\Gamma_1(N)$-structure.

Let $\mathcal{E} \to Y_1(N)$ be the universal elliptic curve and let $e : Y_1(N) \to \mathcal{E}$ be the identity section. Denote by $L = e^*\Omega_{\mathcal{E}/Y_1(N)}$ where $\Omega_{\mathcal{E}/Y_1(N)}$ is the sheaf of relative 1-forms on $\mathcal{E}$. Let $X_1(N)$ denote the Deligne-Rapoport compactification of $Y_1(N)$ and take the natural extension of L to $X_1(N)$, and call it L again.

Let $X \subset X_1(N)$ be an open embedding (not necessarily a proper open subscheme). Consider the restriction of L on X and call it L again. Then over any $\mathbb{Z}[1/N, \zeta_N]$-algebra B, the modular forms of weight κ, denoted by $M_X(B, \kappa, N)$, identifies with the space of global sections $H^0(X_B, L_B^{\otimes \kappa})$, where L_B denotes the sheaf obtained by pullback. Denote

$$M_X = \bigoplus_{\kappa} M_X(B, \kappa, N).$$

The cusp $P = \infty$ is a $\mathbb{Z}[1/N, \zeta_N]$ point on $X_1(N)$ and there is a natural Fourier expansion map $E : M_X \to R((q))$ associated to P. The Fourier expansion $E : M_X \to R((q))$ is defined by evaluating at the $Tate(q)$ curve given by $E(f) := f(q) = f(Tate(q), \omega_{can}, \iota_{can})$. This Fourier expansion map E is injective. We will call a tuple (X, L, P, E) as a *Fourier framed curve.*

Recall another definition of modular forms. For any $\mathbb{Z}[1/N, \zeta_N]$-algebra B, let E/B denote an elliptic curve defined over B, $\omega \in H^0(E, \Omega_{E/B})$ a basis of the free

B-module $H^0(E, \Omega_{E/B})$ and ι as above. By a modular form of weight κ we will understand a rule f that associates to any tuple $(E/B, \omega, \iota)$ an element $f(E/B, \omega, \iota) \in B$ which depends only on the isomorphism class of the tuple, commutes with base change and satisfies

$$f(E/B, \lambda\omega, \iota) = \lambda^{-\kappa} f(E/B, \omega, \iota) \tag{3.2.1}$$

for all $\lambda \in B^{\times}$. This definition identifies with the one given previously using the global sections of higher tensor powers of L [22].

Now for $p \nmid N$, we choose a homomorphism $\mathbb{Z}[1/N, \zeta_N] \to R$ and denote by $Y_1(N)_R, L_R, P_R$ the objects over R obtained by base change. M contains the normalized Eisenstein forms E_4, E_6, E_{p-1} belonging to the spaces $M(R, 4, N), M(R, 6, N), M(R, p-1, N)$ respectively. Note that E_{p-1} is a characteristic 0 lift of the Hasse invariant, a quantity that measures super-singularity.

A *differential modular form* of order n is a rule f which attaches to every tuple $(E/S^0, \omega, \iota, S^*)$ an element of S^n where S^* is a prolongation sequence of p-adically complete rings while the other quantities in the tuple are as explained before. Then f, similar in the classical case, need to satisfy

1) $f(E/S^0, \omega, \iota, S^*)$ depends only on the isomorphism class of the tuple only.

2) f commutes with base change $u^* : S^* \to \tilde{S}^*$, then

$$f(E \otimes_{S^0} \tilde{S}^0/\tilde{S}^0, u^{0*}\omega, u^0 \times \iota, \tilde{S}^*) = u^*(f(E/S^0, \omega, \iota, S^*)) \tag{3.2.2}$$

Let us denote the space of differential modular forms of order n by M^n.

Let us call $Z = \mathit{Spec}\ M$. Then Z parametrises $(E/S^0, \omega, \iota)$ upto isomorphism. We reproduce the following Proposition from [10].

Proposition 3.2.1. $M^n \simeq \widehat{J^n M}$

Proof. We have

$$\hat{Z}(S^0) \simeq Z(S^0) \simeq \{(E/S^0, \omega, \iota)\}/\text{iso} \tag{3.2.3}$$

Then with $f \in M^n$ and for any S^* flat over R^*, we have a map

$$f : Z(S^0) \to S^n = \hat{\mathbb{A}}^1(S^n).$$

By Proposition 3.1.4, we obtain $f \in Hom\ (J^n X, \hat{\mathbb{A}}^1) = \mathcal{O}(J^n Z) \simeq \widehat{J^n M}$ and this completes the proof. □

Let

$$M^\infty = \lim_{\rightarrow} M^n.$$

We call M^∞ as the ring of δ- modular forms. Clearly, $M \subset M^\infty$. But then the question is, are there interesting new examples in M^∞ that shed a new light on M? Is there a nice theory of Fourier (Serre-Tate) expansion?

We will exhibit a few examples of new 'objects' that live in M^∞ which have no apparent counterpart in the world of classical modular forms. But firstly we would like to define δ-modular forms of a given weight.

For any polynomial $w \in \mathbb{Z}[\phi], w = \sum a_i \phi^i$ define element
$\chi_w(t) \in R[t, t^{-1}, t', ..., t^{(n)}]$ by the formula

$$\chi_w(t) = t^w := \prod (\phi^i(t))^{a_i}$$

Such a χ_w is a multiplicative δ-character [9]. By a δ-modular form of order $\leq n$ and weight w we will understand a rule f that associates of any triple $(E/B^0, \omega, \iota, B^*)$ an element $f(E/B^0, \omega, \iota, B^*) \in B^n$, which depends on the isomorphism class of the triple only, commutes with base change and satisfies

$$f(E/B^0, \lambda\omega, \iota, B^*) = \chi_w(\lambda)^{-1} f(E/B^0, \omega, \iota, B^*) \tag{3.2.4}$$

The space of such δ-modular forms will be denoted by $M^n(w)$. A Fourier framed curve is called *ordinary* if there exists an element $f \in M^1(\phi - 1)$ which is invertible in the ring M^1, such that $E^1(f) = 1$.

We shall say $f \in M^\infty$ is *isogeny covariant* if for any triple $(E/S^0, \omega, i, S^*)$ and for any étale isogeny $\pi : E' \to E$ (of elliptic curves over S^0) we have

$$f(E'/S^0, \omega', i', S^*) = [\deg \pi]^{-deg\ w/2} f(E/S^0, \omega, i, S^*) \tag{3.2.5}$$

where $\omega' = \pi^* \omega$ and $deg\ w = \sum a_i$.

Let

$$\Psi := \frac{1}{p} \log \frac{q^\phi}{q^p} = \sum_{n \geq 1} (-1)^{n-1} n^{-1} p^{n-1} \left(\frac{q'}{q^p} \right)^n \in R((q))\hat{}[q']\hat{} \tag{3.2.6}$$

Proposition 3.2.2. [9] There exists a unique form $f^1 \in M^1(-1-\phi)$ whose Fourier expansion is given by

$$E^1(f^1) = \Psi.$$

Given a ring B, we will denote its reduction mod p by $\overline{B}$.

Proposition 3.2.3. [1, 9, 10] Assume the reduction mod p of X, $\overline{X}$, is contained in the ordinary locus of the modular curve. Then there exists a unique form $f^\partial \in M(\phi - 1)$ which is invertible in the ring M^1 such that

$$E^1(f^\partial) = 1.$$

Furthermore its reduction mod p, $\overline{f^\partial} \in \overline{M^1(\phi - 1)}$ coincides with the image of the Hasse invariant $\overline{H} \in \overline{M^0(p-1)}$.

The δ-modular forms in Proposition 3.2.2 and 3.2.3 are isogeny covariant. Since M comes with a Fourier expansion map $E : M \to R((q))$, by universality property of jet spaces as in Theorem 1.0.2, extends naturally to

$$E^n : M^n \to R((q))\hat{}[q', , , q^{(n)}]\hat{} =: S^n_{for}$$

where $q^{(i)}$'s are new indeterminates. However, unlike the classical modular forms, E^n is not injective. For example, $f^\partial - 1$ and its higher p-derivatives $\delta^i(f^\partial - 1)$ for all $i \leq n-1$ are in the kernel of E^n where f^∂ is introduced in the following Theorem 3.2.3. Although, if we restrict to δ-modular forms of a fixed weight w, denoted $M^n(w)$, then E^n is injective.

3.3 Conjugate operators on modular forms

Recall the *Euler derivation* $\mathcal{D} : M \to M$ defined as follows; Let x be a local generator of L over any open set $X \subset X_1(N)$ such that L is trivial over X. Then any $f_n \in L^{\otimes n}$ can be written in the form $f_n = \varphi x^n,\ \varphi \in S^n$. Then $\mathcal{D}$ acts as

$$\mathcal{D} f_n := x \frac{d}{dx} f \tag{3.3.1}$$

Then a simple computation shows that $\mathcal{D} f_n = n f_n$. Hence the effect of $\mathcal{D}$ on f_n is independent of the trivialisation of the modular curve and therefore $\mathcal{D}$ glues over all the trivialisation to give us a globally defined $\mathcal{D} : L^{\otimes n} \to L^{\otimes n}$, preserving the weight of the modular form. Hence, for a general $f \in M$, one can uniquely write f as a sum of f_n's, $f = \sum_n f_n$. and hence one defines

$$\mathcal{D} f := \sum_n \mathcal{D} f_n = \sum_n n f_n \tag{3.3.2}$$

Recall the Ramanujan modular form $P \in M(2)$ which has the following Fourier expansion

$$P(q) := E_2(q) := 1 - 24 \sum_{m \geq 1} \left(\sum_{d|m} d \right) q^m$$

and is of weight 2. Then $P\mathcal{D}$ is a derivation satisfying $P\mathcal{D} : L^{\otimes n} \to L^{\otimes n+2}$, that is it takes a form of weight n and carries it to a form of weight $n+2$. Recall the conjugate operators in 2.6.10 introduced in [10]

Proposition 3.3.1. If $f \in M^r(w)$ where $w = \sum_i = 0^r a_i \phi^i$ is the weight of f. Then

$$(P\mathcal{D})_j(f) = a_j p^j P^{\phi^j} f.$$

Proof. Write $f = \varphi x^{\sum_{i=0}^r a_i \phi^i}$ where $\varphi \in S^r$ and since $\mathcal{D}$ is trivial on $S := \mathcal{O}(X)$. Its conjugates would also be trivial on S^r for all r.

$$\begin{aligned} (P\mathcal{D})_j f &= \varphi x^{a_0} ... \widehat{x^{a_j \phi^j}} ... x^{a\phi^r} [(P\mathcal{D})_j(\phi^j(x^{a_j}))] \\ &= \varphi x^{a_0} ... \widehat{x^{a_j \phi^j}} ... x^{a\phi^r} p^j \phi^j (P\mathcal{D}(x^{a_j})) \\ &= p^j a_j P^{\phi^j} f \qquad \square \end{aligned}$$

Let ∂ be the *Serre operator* which satisfies $\partial : L^{\otimes n} \to L^{\otimes n+2}$ and ∂^* be the *Theta operator* on modular forms whose effect on the Fourier expansion is given by

$$(\partial^* f)(q) = \theta(f(q)) \tag{3.3.3}$$

where $\theta = q\frac{d}{dq}$. Then the above three operators are tied together by

$$\partial^* = \partial + P\mathcal{D} \tag{3.3.4}$$

Proposition 3.3.2. If $f \in M^r(w)$ where w is as before then

$$E(\partial_j f) = \theta_j(f(q)) - a_j p^j f(q) P(q)^{\phi^j}$$

Proof. Combining Proposition 3.3.1 and 3.3.4, gives us the result. $\square$

3.4 Ind-étale extensions

We will first present a general result. Let $X = Spec\ S$ be an affine smooth curve over R and L an invertible sheaf on X. Now consider

$$V = Spec\ \left(\oplus_{n \in \mathbb{Z}} L^{\otimes n}\right) \to X$$

which is the physical line bundle attached to L with the zero section removed and hence has a $\mathbb{G}_m$-action on the fibers over X. Set

$$\begin{aligned} S &:= S_X := \mathcal{O}(X), \\ M &:= M_X := \mathcal{O}(V) = \bigoplus_{n\in\mathbb{Z}} L^{\otimes n} \end{aligned} \tag{3.4.1}$$

Also further assume that we are given an R-point $P \in X(R)$. Set $S_{for} = R[[t]]$. Assume we are given an isomorphism between $Spf\ R[[t]]$ and the completion of X along the image of P. Then we have an induced homomorphism $E : S \to S_{for}$ which is injective and the reduction mod p of this map $\bar{E} : \bar{S} \to \bar{S}_{for}$ is also injective. We will also assume that we are given an extension of the ring homomorphism E to

$$E : M \to S_{for} \tag{3.4.2}$$

We summarize all the above data by calling a tuple X, L, P, E a *framed curve*. Consider the following rings:

$$\begin{aligned} S^r &:= S^r_X := \mathcal{O}^r(X), \quad r \geq 0 \\ M^r &:= M^r_X := \mathcal{O}^r(V), \quad r \geq 0 \\ S^\infty &:= \lim_{\to} S^r, \\ M^\infty &:= \lim_{\to} M^r. \end{aligned} \tag{3.4.3}$$

An element $f \in M^r$ is said to be of weight $w \in W$ if, and only if, the induced δ-function $f : V(R) \to R$ satisfies

$$f(\lambda \cdot a) = \lambda^w f(a)$$

for all $\lambda \in R^\times$, $a \in V(R)$, where $(\lambda, a) \mapsto \lambda \cdot a$ is the natural $\mathbb{G}_m$-action $R^\times \times V(R) \to V(R)$. We denote by $M^r(w) = M^r_X(w)$ the R-module of all elements of $M^r = M^r_X$ of weight w. If L is trivial on X and x is a basis of L then we have the identifications

$$\begin{aligned} M &= S[x, x^{-1}], \\ M^r &= S^r[x, x^{-1}, x', ..., x^{(r)}]\hat{}, \\ M^r(w) &= S^r \cdot x^w \subset M^r \end{aligned}$$

By [10], Proposition 3.14, the reduction mod p of S^r, denoted $\overline{S^r}$, are integral domains, and the maps $\overline{S^r} \to \overline{S^{r+1}}$ are injective. In particular the rings S^r are integral domains and the maps $S^r \to S^{r+1}$ are injective with torsion free cokernels. The analogous statements hold for M^r. So, in particular, $\overline{S^\infty}$ and $\overline{M^\infty}$ are integral domains. Let $t', t'', ...$ and $q', q'', ...$ be new variables and consider the prolongation sequence $(S^r_{for})_{r\geq 0}$,

$$S^r_{for} = R[[t]][t', ..., t^{(r)}]\hat{},$$

respectively

$$S^r_{for} = R((q))\hat{}[q', ..., q^{(r)}]\hat{}.$$

We set

$$S^\infty_{for} := \lim_{\to} S^r_{for}.$$

Then the expansion maps induce, by universality, morphisms of prolongation sequences,

$$E^r : M^r \to S^r_{for}; \tag{3.4.4}$$

the maps E^r will be referred to as *δ-expansion maps* for M^r. They induce a *δ-expansion map*

$$E^\infty : M^\infty \to S^\infty_{for}. \tag{3.4.5}$$

We have the following *δ-expansion principle* for S^r:

Proposition 3.4.1. The induced map

$$\overline{E^r} : \overline{S^r} \to \overline{S^r_{for}}$$

is injective. In particular, $E^r : S^r \to S^r_{for}$, and hence the δ-expansion maps

$$E^r : M^r(w) \to S^r_{for}$$

are injective, with torsion free cokernel.

(The words "torsion free", without the specification "as an A-module", will always mean "torsion free as a $\mathbb{Z}$-module".)

Proof. It follows from [10], Proposition 4.43. □

The rings $\mathbb{S}^r$

Next, for a framed curve $X = Spec\ S$, we define

$$\begin{aligned} \mathbb{S}^r &:= Im(E^r : M^r \to S^r_{for}) \\ \mathbb{S}^\infty &:= \lim_{\to} \mathbb{S}^r = Im(E^\infty : M^\infty \to S^\infty_{for}). \end{aligned} \tag{3.4.6}$$

The ring $\mathbb{S}^\infty$ will later morally play the role of "coordinate ring of the δ-Igusa curve". The following is trivial to check (using the definitions and Proposition 3.4.1):

Proposition 3.4.2.

1) The homomorphisms $\overline{S^r} \to \overline{\mathbb{S}^r}$, $\overline{S^\infty} \to \overline{\mathbb{S}^\infty}$ are injective. In particular the homomorphisms $S^r \to \mathbb{S}^r$, $S^\infty \to \mathbb{S}^\infty$ are injective with torsion free cokernel.

2) The homomorphisms $\mathbb{S}^r \to \mathbb{S}^{r+1}$ are injective.

Remark 3.4.3. The ring $\overline{\mathbb{S}^\infty}$ is not a priori an integral domain and the map $\overline{\mathbb{S}^\infty} \to \overline{S^\infty_{for}}$ is not a priori injective. The ring $\overline{\mathbb{S}^\infty}$, however, has a natural quotient which is an integral domain, namely:

$$\widetilde{\mathbb{S}^\infty} := Im(\overline{M^\infty} \to \overline{S^\infty_{for}}). \tag{3.4.7}$$

This ring is going to play a role in what follows. We will prove later that, in the concrete setting of modular curves the map $\overline{\mathbb{S}^\infty} \to \overline{S^\infty_{for}}$ is injective (δ-expansion principle) hence $\overline{\mathbb{S}^\infty}$ is an integral domain and the surjection $\overline{\mathbb{S}^\infty} \to \widetilde{\mathbb{S}^\infty}$ is an isomorphism. Cf. Theorem 3.5.13.

Definition 3.4.4. A framed curve is called ordinary if there exists an invertible $f \in M^1(\phi - 1)$, such that $E^1(f) = 1$.

Definition 3.4.5. Let A be a k-algebra where k is a field. Let $A \subset B$ a ring extension, and Γ a profinite abelian group acting on B by A-automorphisms. We say that B is a Γ-*extension* of A if one can write A and B as filtered unions of finitely generated k-subalgebras, $A = \bigcup A_i$, $B = \bigcup B_i$, indexed by some partially ordered set, with $A_i \subset B_i$, and one can write Γ as an inverse limit of finite abelian groups, $\Gamma = \varprojlim \Gamma_i$, such that the Γ-action on B is induced by a system of compatible Γ_i-actions on B_i and

$$B_i^{\Gamma_i} = A_i$$

for all i. (Then, of course, we also have $B^\Gamma = A$.) If in addition one can choose the above data such that each A_i is smooth over k and each B_i is étale over A_i we say that B is an *ind-étale* Γ-*extension* of A.

Lemma 3.4.6.

1) Assume B is a Γ-extension of A and $C := B/I$ is a quotient of B by an ideal I. Then C is integral over A.

2) Assume B is an ind-étale Γ-extension of A and let I be a prime ideal of B such that $I \cap A = 0$. Then $C := B/I$ is an ind-étale Γ'-extension of A where Γ' is a closed subgroup of Γ.

Proof. Assertion 1 is clear. Let's prove assertion 2. Using the notation in Definition 3.4.5 set $Y_i = Spec\ B_i$, $V_i := Spec\ A_i$, $Z_i := Spec\ C_i$, $C_i := B_i/B_i \cap I$. Let $\Gamma'_i := \{\gamma \in \Gamma_i; \gamma Z_i = Z_i\}$. By Lemma 3.4.7 below C_i is étale over A_i and $C_i^{\Gamma'_i} = A_i$ so one can take $\Gamma' := \varprojlim \Gamma'_i$ acting on $C = \varprojlim C_i$. □

We have used the following "well known" lemma (whose proof will be "recalled" for convenience):

Lemma 3.4.7. *Let V be a smooth affine variety over a field k, let $Y \to V$ be a finite étale map, and let G be a finite abelian group acting on Y such that $Y/G = V$. Let $Z \subset Y$ be a subvariety that dominates V and let $G' = \{\gamma \in G; \gamma Z = Z\}$. Then Z is a connected component of Y (hence is étale over V) and $Z/G' = V$.*

Proof. Since V is smooth the connected components $Z_1, ..., Z_n$ of Y are irreducible so Z is a connected component of Y, say $Z = Z_1$. Since V is connected G acts transitively on the set $\{Z_1, ..., Z_n\}$ hence the stabilizers in G of the various Z_is are conjugate in G, hence they are equal, because G is abelian. So

$$\mathcal{O}(V) = \mathcal{O}(Y)^G = (\mathcal{O}(Z_1) \times ... \times \mathcal{O}(Z_n))^G = (\mathcal{O}(Z)^{G'} \times ... \times \mathcal{O}(Z)^{G'})^{G/G'} \quad (3.4.8)$$

where $\mathcal{O}(Z_i)^{G'} \simeq \mathcal{O}(Z)^{G'}$ via *any* $\gamma \in G$ such that $\gamma Z = Z_i$ and G/G' acts on the product via the corresponding permutation representation. Since the last ring in (3.4.8) contains $\mathcal{O}(Z)^{G'}$ embedded diagonally it follows that $\mathcal{O}(Z)^{G'} = \mathcal{O}(V)$. □

For a framed curve $X = Spec\ S$, we define

$$\begin{aligned} \mathbb{S}^r &:= Im(E^r : M^r \to S^r_{for}) \\ \mathbb{S}^\infty &:= \lim_\to \mathbb{S}^r = Im(E^\infty : M^\infty \to S^\infty_{for}) \end{aligned} \quad (3.4.9)$$

The ring $\mathbb{S}^\infty$ will later morally play the role of "coordinate ring of the δ-Igusa curve".

Definition 3.4.8. We will call a ring A a δ-ring if there exists a p-derivation $\delta : A \to A$.

We need a series of Lemmas. For the first two Lemmas we let A be a δ-ring and we consider the prolongation sequence $B^r = A[z, z^{-1}, z', ..., z^{(r)}]\hat{}$. We will denote an element f to be of $O(r)$ if $f \in B^r$. Such an element f will be called an element of order $\leq r$.

Lemma 3.4.9. *Let $\varphi \in A$. Then, for any $n \geq 1$, we have*

$$\delta^n\left(\frac{z^\phi}{z} - \varphi\right) = z^{-p^n}(z^{(n)})^p - z^{p^{n+1}-2p^n} z^{(n)} + O(n-1) + pO(n+1).$$

Proof. For $\varphi = 0$ this is [10], Lemma 5.19. Assume now φ arbitrary. One checks by induction that

$$\delta^n(z - \varphi) = \delta^n z + U + pV,$$

where $U = O(n-1)$, $V = O(n)$. Replacing z by $\frac{z^\phi}{z}$ we get

$$\delta^n\left(\frac{z^\phi}{z} - \varphi\right) = \delta^n\left(\frac{z^\phi}{z}\right) + U\left(\frac{z^\phi}{z}, ..., \delta^{n-1}\left(\frac{z^\phi}{z}\right)\right) + pV\left(\frac{z^\phi}{z}, ..., \delta^n\left(\frac{z^\phi}{z}\right)\right),$$

and we conclude by the case $\varphi = 0$ of the Lemma. □

Lemma 3.4.10. *Let $\lambda = 1 + p^n a$, $a \in \mathbb{Z}$. Then*

$$\delta^n(\lambda z) = z^{(n)} + az^{p^n} + pO(n).$$

Proof. Follows by induction. □

Lemma 3.4.11. *Let Q be a ring of characteristic p and consider the Q-algebra $Q' := Q[u]/(u^p - u - G)$ where $G \in Q$. Consider the action of $\mathbb{Z}/p\mathbb{Z} = \{\overline{a}\ ;\ a = 0, ..., p-1\}$ on $Q[u]$ defined by $\overline{a} \cdot u = u + \overline{a}$ and consider the induced $\mathbb{Z}/p\mathbb{Z}$-action on Q'. Then any $\mathbb{Z}/p\mathbb{Z}$-invariant element of Q' is in Q.*

Proof. Let $c \in Q'$ be the class of u. Then Q' is a free Q-module with basis $1, c, ..., c^{p-1}$. Assume $\sum_{i=0}^{p-1} \lambda_i c^i \in Q'$ is $\mathbb{Z}/p\mathbb{Z}$-invariant, where $\lambda_i \in Q$. We want to show that $\lambda_i = 0$ for $i \geq 1$. We may assume $\lambda_0 = 0$. Assume there is a $s \geq 1$ such that $\lambda_s \neq 0$ and let s be maximal with this property. Then

$$\lambda_s(c+1)^s + \lambda_{s-1}(c+1)^{s-1} + ... = \lambda_s c^s + \lambda_{s-1} c^{s-1} + ...$$

Picking out the coefficient of c^{s-1} we get $s\lambda_s = 0$ hence $\lambda_s = 0$, a contradiction. □

Theorem 3.4.12. *Let $X = Spec\ S$ be an ordinary framed curve. Then the ring $\overline{\mathbb{S}^\infty}$ is a quotient of an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty}$.*

Proof. For $r \geq 1$ set

$$N^r := \frac{M^r}{(f-1, \delta(f-1), ..., \delta^{r-1}(f-1))}.$$

Note that

$$E^i(\delta^{i-1}(f-1)) = \delta^{i-1}(E^1(f-1)) = \delta^{i-1}(0) = 0.$$

which implies that there are surjective homomorphisms $N^r \to \mathbb{S}^r$, hence surjective homomorphisms $\overline{N^r} \to \overline{\mathbb{S}^r}$ and therefore we obtain a surjective homomorphism at the limit

$$\lim_{\rightarrow} \overline{N^r} \to \lim_{\rightarrow} \overline{\mathbb{S}^r} = \overline{\mathbb{S}^\infty}. \tag{3.4.10}$$

Now let $X = \bigcup_\alpha X_\alpha$, $X_\alpha = Spec\ S_\alpha$, be an affine open covering such that L is trivial on each X_α. Let x_α be a basis of L on X_α and let $z_\alpha = x_\alpha^{-1}$. Set

$$\begin{array}{rcl} S_\alpha^r & := & S_{X_\alpha}^r = (S^r \otimes_S S_\alpha)\hat{} \\ & & . \\ M_\alpha^r & := & M_{X_\alpha}^r = (M^r \otimes_S S_\alpha)\hat{} \end{array}$$

Then we have an identification

$$M_\alpha^r = S_\alpha^r[z_\alpha, z_\alpha^{-1}, z'_\alpha, ..., z_\alpha^{(r)}]\hat{}.$$

Write $f = \varphi_\alpha x_\alpha^{\phi-1}$, with $\varphi_\alpha \in S_\alpha^1$. Since f and x are invertible in M_α^1, it follows that φ_α is also invertible in M_α^1. And since S_α^1 is an integral domain we have $M_\alpha^{1\times} = S_\alpha^{1\times} \cup \{cz_\alpha^n \mid c \in S_\alpha^{1\times}, n \in \mathbb{Z}\}$ implies that $\varphi_\alpha \in S_\alpha^{1\times}$. Set $N_\alpha^r = (N^r \otimes_S S_\alpha)\hat{}$; hence

$$N_\alpha^r := \frac{S_\alpha^r[z_\alpha, z_\alpha^{-1}, z_\alpha', ..., z_\alpha^{(r)}]\hat{}}{\left(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha, \delta\left(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha\right), ..., \delta^{r-1}\left(\frac{z_\alpha^\phi}{z_\alpha} - \varphi_\alpha\right)\right)}$$

For $i \geq 1$ set $u_{i,\alpha} := \frac{z_\alpha^{(i)}}{z_\alpha^{p^i}}$. Also, for $r \geq 1$, set

$$Q_\alpha^{r,0} := \frac{\overline{S_\alpha^r}[z_\alpha, z_\alpha^{-1}]}{(z_\alpha^{p-1} - \overline{\varphi_\alpha})} = \frac{\overline{S_\alpha^r}[z_\alpha]}{(z_\alpha^{p-1} - \overline{\varphi_\alpha})}. \tag{3.4.11}$$

(The latter equality is true because $z_\alpha = \overline{\varphi_\alpha} z^{-p} = \frac{\overline{\varphi_\alpha}}{z^{2(p-1)}} z^{p-2} = \overline{\varphi}^{-1} z^{p-2}$.) Then, by Lemma 3.4.9 we have $\overline{N_\alpha^1} = Q_\alpha^{1,0}[u_{1,\alpha}]$ and

$$\overline{N_\alpha^r} = \frac{Q_\alpha^{r,0}[u_{1,\alpha}, ..., u_{r,\alpha}]}{(u_{1,\alpha}^p - u_{1,\alpha} - G_0, ..., u_{r-1,\alpha}^p - u_{r-1,\alpha} - G_{r-2})}, \quad r \geq 2,$$

where $G_0 \in Q_\alpha^{r,0}$, and

$$G_i \in Q_\alpha^{r,i} := \frac{Q_\alpha^{r,0}[u_{1,\alpha}, ..., u_{i,\alpha}]}{(u_{1,\alpha}^p - u_{1,\alpha} - G_0, ..., u_{i,\alpha}^p - u_{i,\alpha} - G_{i-1})}, \quad i \geq 1.$$

Clearly the schemes *Spec* $Q_\alpha^{r,i}$, for various α's naturally glue to give a scheme *Spec* $Q^{r,i}$; so $Q^{r,i} \otimes_{\overline{S}} \overline{S_\alpha} = Q_\alpha^{r,i}$ for all α. Note that we have

$$Q_\alpha^{r,i} = \frac{Q_\alpha^{r,i-1}[u_{i,\alpha}]}{(u_{i,\alpha}^p - u_{i,\alpha} - G_{i-1})} \tag{3.4.12}$$

and natural inclusions

$$Q_\alpha^{r,0} \subset Q_\alpha^{r,1} \subset ... \subset Q_\alpha^{r,r-1} \subset \overline{N_\alpha^r} = Q_\alpha^{r,r-1}[u_{r,\alpha}]. \tag{3.4.13}$$

So we have natural homomorphisms

$$... \to Q_\alpha^{r,r-1} \to \overline{N_\alpha^r} \to Q_\alpha^{r+1,r} \to \overline{N_\alpha^{r+1}} \to ...$$

which shows that, for each α,

$$(\lim_{\vec{r}} \overline{N^r}) \otimes_{\overline{S}} \overline{S_\alpha} = \lim_{\vec{r}} \overline{N_\alpha^r} = \lim_{\vec{r}} Q_\alpha^{r,r-1} = (\lim_{\vec{r}} Q^{r,r-1}) \otimes_{\overline{S}} \overline{S_\alpha}.$$

These isomorphisms glue together to give an isomorphism

$$\lim_{\rightarrow} \overline{N^r} = \lim_{\rightarrow} Q^{r,r-1}.$$

We are left to proving that $\lim_{\rightarrow} Q^{r,r-1}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty} = \lim_{\rightarrow} \overline{S^r}$.

Start by noting that the maps $Q_\alpha^{r,r-1} \to Q_\alpha^{r+1,r}$ are injective. Also $\overline{S_\alpha^r} \to Q_\alpha^{r,r-1}$ are injective and étale; cf. (3.4.11) and (3.4.12). Now the group $\Gamma = \mathbb{Z}_p^\times$ acts on M_α^r via the rule $\gamma \cdot z_\alpha^{(i)} = \delta^i(\gamma z_\alpha)$ for $\gamma \in \Gamma$. This induces a Γ-action on N_α^r and hence a Γ-action on $\overline{N_\alpha^r}$. The latter factors through an action of $\Gamma_r := (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times$. Moreover, for $i \leq r-1$, $Q_\alpha^{r,i}$ is Γ_r-stable and the Γ_r-action on $Q_\alpha^{r,i}$ factors through a Γ_i-action. For a fixed r we will prove by induction on $0 \leq i \leq r-1$ that

$$(Q_\alpha^{r,i})^{\Gamma_i} = \overline{S_\alpha^r}. \tag{3.4.14}$$

This will end the proof of the Theorem; indeed from the above we trivially get that the maps $Q^{r,r-1} \to Q^{r+1,r}$ are injective, the maps $\overline{S^r} \to Q^{r,r-1}$ are injective and étale, and, with respect to the induced action,

$$(Q^{r,r-1})^{\Gamma_{r-1}} = \overline{S^r},$$

showing that $\lim_{\rightarrow} Q^{r,r-1}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty} = \lim_{\rightarrow} \overline{S^r}$.

Let us check (3.4.14). For $i = 0$ we proceed as follows. Let $b \in Q_\alpha^{r,0}$ be the class of z_α and let $\Gamma_0 = \mathbb{F}_p^\times = \langle \zeta \rangle$, ζ a primitive root. Then $Q_\alpha^{r,0}$ is a free $\overline{S_\alpha^r}$-module with basis $1, b, b^2, ..., b^{p-2}$. If $\sum_{l=0}^{p-2} \lambda_l b^l$ is Γ_0-invariant (where $\lambda_l \in \overline{S_\alpha^r}$) then $\sum_{l=0}^{p-2} \lambda_l \zeta^l b^l = \sum_{l=0}^{p-2} \lambda_l b^l$. Since ζ is primitive we get $\lambda_1 = ... = \lambda_{p-2} = 0$, and the case $i = 0$ is proved.

Now assume $(Q_\alpha^{r,i-1})^{\Gamma_{i-1}} = \overline{S_\alpha^r}$ and let us prove (3.4.14). Recall the equation 3.4.12 and consider the subgroup

$$\Delta_i := \{\gamma_0, ..., \gamma_p\} \subset \Gamma_i, \quad \gamma_a = 1 + p^i a + p^{i+1}\mathbb{Z};$$

so Δ_i is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ via $\gamma_a \mapsto \overline{a}$. Note that Δ_i acts trivially on $Q_\alpha^{r,i-1}$. By Lemma 3.4.10 the Δ_i-action on $Q_\alpha^{r,i}$ corresponds to the $\mathbb{Z}/p\mathbb{Z}$-action induced by $\overline{a} \cdot u_i = u_i + \overline{a}$, so we are in the situation described in Lemma 3.4.11 and we may conclude by that Lemma plus the equality $(Q_\alpha^{r,i-1})^{\Gamma_{i-1}} = \overline{S_\alpha^r}$. This ends the proof of (3.4.14) and hence of the Theorem. □

3.5 Application to differential modular forms

Let a_4, a_6 be indeterminates. Set

$$\begin{aligned} \Delta &= -2^4(4a_4^3 + 27a_6^2) \\ j &= 2^{12}3^3\tfrac{a_4^3}{\Delta} \\ i &= 2^6 - j = -2^{10}3^6\tfrac{a_6^2}{\Delta} \\ b &= \tfrac{a_6^2}{a_4^3} = -2^2 3^{-3} + 2^8 j^{-1} \end{aligned} \tag{3.5.1}$$

Then we have $R[j, j^{-1}, i^{-1}] = R[b, b^{-1}, (4+27b)^{-1}]$. Let Y_{ord} be the locus in $Y_1(N)_R$ where the Eisenstein form $E_{p-1} \in M(\mathbb{Z}_p, p-1, N)$ is invertible. Then b is an étale coordinate on any open embedding $Y \subset Y_{ord}$ [13] 4.31. Hence we obtain

$$\mathcal{O}^r(Y) \simeq \mathcal{O}(Y)[b', ..., b^{(r)}]\hat{} \tag{3.5.2}$$

Definition 3.5.1. Given a ring B and the module of Kahler R-differentials $\Omega_{B/R}$, call $TB = Symm\ \Omega_{B/R}$

We record the following result.

Proposition 3.5.2. Suppose b is an étale coordinate of B. Then $\Omega_{B/R} \simeq \Omega_{R[b]/R} \otimes B$.

Proof. Follows from application of the definition. □

Corollary 3.5.3. Suppose b is an étale coordinate of B then $TB \simeq B[db]$ where db is the image of b in $d : R[b] \to \Omega_{R[b]/R}$.

Corollary 3.5.4. $H^0(Spec\ B, \Omega_{Spec\ B/R}) = B < db >$ where db is as before.

One can similarly define the ring of higher geometric jet spaces T^nB. Then we have the following lemma

Lemma 3.5.5. *If $R[x] \to B$ is an étale morphism, then*

$$T^nB \simeq B[dx, ..., d^nx]$$

where $dx, ..., d^nx$ are the new indeterminates.

Set

$$\begin{aligned} S &:= \mathcal{O}(Y) \\ S^r &:= \mathcal{O}^r(Y) \end{aligned} \tag{3.5.3}$$

Since b is an étale co-ordinate we get

$$\begin{aligned} S^r &= S[b', ..., b^{(r)}\hat{]} \\ TS &= Symm\ \Omega_{S/R} = S[db] \end{aligned} \tag{3.5.4}$$

Let $\mathcal{E} \xrightarrow{\pi} Y_1(N)$ be the universal elliptic curve and $\omega := \pi_*\Omega_{\mathcal{E}/S}$. Then $\omega^{\otimes 2} \simeq \Omega_{Y_1(N)/R}$. Then $Symm\ \omega = S[x]$ is the space of modular forms on $Y(N)$ where x is a basis for ω

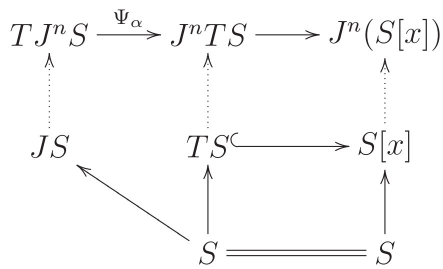
Lemma 3.5.6. *db is a modular form of weight* 2*.*

Proof. By Corollary 3.5.3 we can identify $db \in H^0(Y, \Omega_{Y/R}) \simeq H^0(Y, \omega^{\otimes 2})$ and this completes the proof. □

Hence there exists an $f \in S^\times$ such that

$$x^2 = f(db) \tag{3.5.5}$$

and we have $TS \hookrightarrow Symm\ \omega$. Note that the injection is étale.



where Ψ_α is as in Theorem 2.5.4.

We will call $w \in \mathbb{Z}[\phi]$ an even, positive weight if w can be expressed as $w = \sum_{i=0} 2a_i\phi^i \in \mathbb{Z},\ a_i \geq 0$. Let us denote the subset of all such w's by $2W^+$. Recall $S = \mathcal{O}(X)$.

Proposition 3.5.7. $T(\widehat{J^nS}) \simeq \bigoplus_{w\in 2W^+} M_X^n(w)$

Proof. By [10],

$$\bigoplus_{w\in 2W^+} M_X^n(w) = \widehat{J^nS}[x^2, \phi(x^2), ..., \phi^n(x^2)] \tag{3.5.6}$$

But $\phi(x^2) = \phi(f^{-1})\phi(db)$ and since $\phi(f^{-1})$ is invertible, we obtain the isomorphism

$$\bigoplus_{w\in 2W^+} M_X^n(w) \simeq \widehat{J^nS}[db, ..., \phi^n(db)] \simeq T(\widehat{J^nS}) \tag{3.5.7}$$

and the last isomorphism above is because b is an étale coordinate of S and 2.6.3. □

Review of the forms f^1, f^∂

The references here are [9, 1, 10]. Set

$$\Psi := \frac{1}{p}\log\frac{q^\phi}{q^p} := \sum_{n\geq 1}(-1)^{n-1}n^{-1}p^{n-1}\left(\frac{q'}{q^p}\right)^n \in S^1_{for} = R((q))\hat{\ }[q']\hat{\ }. \tag{3.5.8}$$

In the next two Propositions $X = Spec\ S$ is a modular framed curve, recall the δ-expansion maps $E^r : M^r \to S^r_{for}$, cf. (3.4.4).

Proposition 3.5.8. [9] There exists a unique form $f^1 \in M^1(-1-\phi)$ such that

$$E^1(f^1) = \Psi.$$

Proposition 3.5.9. [1, 9] Assume the reduction mod p of X, $\overline{X}$, is contained in the ordinary locus of the modular (respectively Shimura) curve. Then there exists a unique form $f^\partial \in M^1(\phi - 1)$ which is invertible in the ring M^1 such that

$$E^1(f^\partial) = 1.$$

Furthermore its reduction mod p, $\overline{f^\partial} \in \overline{M^1(\phi-1)}$, coincides with the image of the Hasse invariant $\overline{H} \in \overline{M^0(p-1)}$. In particular X is an ordinary framed curve in the sense of Definition 3.4.4.

Let $M^r_{\{h\}} = M^r[h^{-1}\widehat{]}$. Define $f^\flat \in M^1(0) = S^1$ as $f^\flat = x^{\phi+1} f^1$. Then recall from [10] that $f^\flat$ can be rewritten in the following form

$$f^\flat = a_0 b' + f_0 + p h_1$$

where $h_1 \in M^1_{\{a_4\}}$, $f_0 \in M^0(-1-p)$, $b' \in M^1_{\{a_4, a_6\}}$ and $a_0 \in M^0_{\{a_6\}}$ is invertible. Recall that we will denote an element of order $leqr$ as $O(r)$.

Lemma 3.5.10. *For any $k \in \mathbb{N}$, $\delta^k f^\flat = a_0^{p^k} \delta^k b' + O(k) + p\, O(k+1)$*

Proof. We proceed by induction. For $k = 0$, $f^\flat$ is precisely as in the statement of the lemma. Now let us assume that the statement is true for $k = i$. We will show that it is true for $k = i+1$.

$$\begin{aligned}
\delta^{i+1} f^\flat &= \delta(a_0^{p^i} \delta^i b' + O(i) + p\, O(i+1) \\
&= \delta(a_0^{p^i} \delta^i b') + \delta(O(i)) + \delta(p\, O(i+1)) \\
&\quad + \frac{1}{p}\{(a_0^{p^i} \delta^i b')^p + (O(i))^p + (p\, O(i+1))^p \\
&\qquad\qquad - (a_0^{p^i} \delta^i b' + O(i) + p\, O(i+1))^p\}
\end{aligned}$$

Both $\delta(O(i))$ and $\frac{1}{p}\{(a_0^{p^i}\delta^i b')^p + (O(i))^p + (p\, O(i+1))^p - (a_0^{p^i}\delta^i b' + O(i) + p\, O(i+1))^p\}$ are $O(i+1)$.

$$\begin{aligned}\delta^{i+1} f^\flat &= (a_0^{p^i})^p \delta^{i+1} b' + (\delta^i b')(\delta a_o^{p^i}) + p\delta(a_0^{p^i})(\delta^{i+1} b') \\ &\quad + p^p \delta(O(i+1)) + (O(i+1))^p \delta p + p\ \delta(O(i+1))\delta p + O(i+1)\end{aligned}$$

Now $(\delta^i b')(\delta a_0^{p^i})$ and $(O(i+1))^p \delta p$ are $O(i+1)$ hence

$$\begin{aligned}\delta^{i+1} f^\flat &= (a_0^{p^i})^p \delta^{i+1} b' + p\ \delta(a_0^{p^i})(\delta^{i+1} b') + p\ \delta(O(i+1)) + O(i+1) \\ &= a_0^{p^{i+1}} \delta^{i+1} b' + O(i+1) + p\ O(i+2) \text{ [Since } \delta(O(i+1)) \text{ and} \\ &\qquad\qquad \delta(a_0^{p^i})(\delta^{i+1} b') \text{ are both } O(i+2)]\end{aligned}$$

This concludes our proof. □

Lemma 3.5.11. *Any ordinary k-point of $Y = Y_1(N)_R$ has an affine open neighborhood $X \subset Y$ such that E_{p-1} is invertible on X, L is trivial on X and the natural homomorphism*

$$\overline{S} \to \frac{\overline{S^r}}{(\overline{f^\flat}, \overline{\delta f^\flat}, ..., \overline{\delta^{r-1} f^\flat})}$$

is an isomorphism.

Proof. Since $S^r = S[b', ..., b^{(n)}]\hat{}$ since b is the étale coordinate of S, we conclude the proof by Lemma 3.5.10. □

Review of Katz generalized p-adic modular functions

The references here are [23, 19].

Let B be a p-adically complete ring, $p \geq 5$, and let N be an integer coprime to p. Consider the functor

$$\{p\text{-adically complete } B\text{-algebras}\} \to \{\text{sets}\} \tag{3.5.9}$$

that attaches to any A the set of isomorphism classes of triples $(E/A, \varphi, \iota)$, where E is an elliptic curve over A, φ is a trivialization, and ι is an arithmetic level N structure. Recall that a *trivialization* is an isomorphism between the formal group of E and the formal group of the multiplicative group; an *arithmetic level N structure* is defined as an inclusion of flat group schemes over B of μ_N into $E[N]$. The functor (3.5.9) is representable by a p-adically complete ring $\mathcal{W}(B, N)$. The elements of this ring are called by Katz [23] *generalized p-adic modular forms*. Note that $\mathcal{W}(B, N) = \mathcal{W}(\mathbb{Z}_p, N)\widehat{\otimes}B$. Moreover there is a $\mathbb{Z}_p^\times$-action on $\mathcal{W}(B, N)$ coming from the action of $\mathbb{Z}_p^\times$ on the formal group of the multiplicative group. The fourier expansion E : $\mathcal{W} \to \widehat{B((q))}$ of $f \in \mathcal{W}$ is defined as the evaluation of f at the $Tate(q)$ curve given by $E(f) := f(q) = f(Tate(q), \varphi_{can}, \iota_{can})$. E is injective and has a flat cokernel over B. Also $\mathcal{W}(\mathbb{Z}_p, N)$ possesses a natural ring endomorphism $Frob$ which reduces modulo p to the p-power Frobenius endomorphism of $\mathcal{W}(\mathbb{Z}_p, N) \otimes \mathbb{Z}/p\mathbb{Z}$. So if $R = \hat{\mathbb{Z}}_p^{ur}$, as usual, and if ϕ is the automorphism of R lifting Frobenius then $Frob\widehat{\otimes}\phi$ is a lift of Frobenius on

$$\mathcal{W} := \mathcal{W}(R, N) = \mathcal{W}(\mathbb{Z}_p, N)\widehat{\otimes}R$$

which we denote by ϕ_0. Moreover the homomorphism $\mathcal{W}(R, N) \to R((q))\hat{}$ commutes with the action of ϕ_0 where ϕ_0 on $R((q))\hat{}$ is defined by $\phi_0(\sum a_n q^n) := \sum \phi(a_n) q^{np}$.

The ring of modular forms M injects into $\mathcal{W}$ via, if $f \in M$ then $f(E/A, \varphi, \iota) := f(E/A, \varphi^*(dt/t+1), \iota)$ where $dt/(t+1)$ is the invariant differential on $\mathbb{G}_m^{for}$ whose pull back via φ is a differential on the Elliptic curve E.

For any $\mathbb{Z}[1/N, \zeta_N]$-algebra B the space $M(B, \kappa, N)$ of modular forms over B of weight κ and level $\Gamma_1(N)$ has an embedding

$$M(B, \kappa, N) \subset \mathcal{W}(B, N).$$

The space $M(B, \kappa, N)$ is stable under the $\mathbb{Z}_p^\times$-action on $\mathcal{W}(B, N)$ and $\lambda \in \mathbb{Z}_p^\times$ acts on $M(B, \kappa, N)$ via multiplication by λ^k. Recall that we denoted by $Y_{ord} \subset Y =$

$Y_R = Y_1(N)_R$ the locus in Y where the Eisenstein form $E_{p-1} \in M(\mathbb{Z}_p, p-1, N)$ is invertible. Then, since E_{p-1} is invertible in $\mathcal{W}$ we get a homomorphism

$$M_{Y_{ord}} = \bigoplus_{k \in \mathbb{Z}} L_{Y_{ord}}^{\otimes k} \to \mathcal{W}.$$

More generally, if X is any affine open subset of Y_{ord} then one can find $g \in M_{Y_{ord}}$ of weight 0, $\overline{g} \neq 0$, and a homomorphism

$$M := M_X := \bigoplus_{\kappa \in \mathbb{Z}} L_X^{\otimes \kappa} \to \mathcal{W}_g = \mathcal{W}[1/g]. \tag{3.5.10}$$

(So if $X = Y_{ord}$ we may take $g = 1$.) Since g has weight 0, $\widehat{\mathcal{W}_g}$ has an induced $\mathbb{Z}_p^\times$-action and the homomorphism (3.5.10) is $\mathbb{Z}_p^\times$-equivariant if $\lambda \in \mathbb{Z}_p^\times$ acts on each $L^{\otimes k}$ via multiplication by λ^k.

Finally recall Katz's *ring of divided congruences* [19],

$$\mathcal{D} := \mathcal{D}(R, N) := \{f \in \bigoplus_{\kappa \geq 0} M(R, \kappa, N) \otimes_R K;\ E(f) \in R[[q]]\},$$

where $K := R[1/p]$. This ring naturally embeds into Katz's *ring of holomorphic generalized p-adic modular forms*,

$$\mathcal{V} := \mathcal{V}(R, N) = \{f \in \mathcal{W}(R, N);\ E(f) \in R[[q]]\},$$

and the image of $\mathcal{D}$ in $\mathcal{V}$ is p-adically dense. For simplicity we sometimes identify $\mathcal{D}, \mathcal{V}, \mathcal{W}$ with subrings of $R((q))\hat{}$; i.e. we view

$$\mathcal{D} \subset \mathcal{V} \subset \mathcal{W} \subset R((q))\hat{}.$$

We will need the following:

Lemma 3.5.12. $\mathcal{D} + R[\Delta^{-1}]$ *is p-adically dense in* $\mathcal{W}$.

Proof. It is enough to check that $\mathcal{V} + R[\Delta^{-1}]$ is p-adically dense in $\mathcal{W}$.

We first claim that for any $f \in \mathcal{W}$ there exists a sequence of polynomials $F_n \in R[t]$ in a variable t such that $F_{n+1} - F_n \in p^n R[t]$ for $n \geq 0$ and such that

$$f - F_n(\Delta^{-1}) \in p^n R((q))\hat{} + R[[q]].$$

To check the claim we construct F_n by induction. We may take $F_0 = 0$. Now, assuming F_n was constructed, write

$$f - F_n(\Delta^{-1}) = p^n G + p^{n+1} H + S, \quad G \in \sum_{i=1}^{N} R q^{-i}, \quad H \in R((q))\hat{}, \quad S \in R[[q]].$$

Since $\Delta^{-1} - q^{-1} \in R[[q]]$ we can find a polynomial $\Gamma \in R[t]$ of degree $\leq N$ such that $G - \Gamma(\Delta^{-1}) \in R[[q]]$. Then set $F_{n+1} := F_n + p^n \Gamma$ which ends the inductive step of our construction.

Now let F_n be as in our claim above and set $F := \lim F_n \in R[t]\hat{}$. Then clearly $f - F(\Delta^{-1}) \in R[[q]] \cap \mathcal{W} = \mathcal{V}$. This implies that $\mathcal{V} + R[\Delta^{-1}]$ is p-adically dense in $\mathcal{W}$ and we are done. $\square$

Theorem 3.5.13. *Assume $X = Spec\ S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. The following hold:*

1) The map $\overline{\mathbb{S}^\infty} \to \overline{S^\infty_{for}}$ is injective; in particular $\overline{\mathbb{S}^\infty}$ is an integral domain, and the map $\overline{\mathbb{S}^\infty} \to \widetilde{\overline{\mathbb{S}^\infty}}$ is an isomorphism. Moreover the ring $\overline{\mathbb{S}^\infty}$ is an ind-étale $\mathbb{Z}_p^\times$-extension of $\overline{S^\infty}$.

2) The kernel of $\overline{M^\infty} \to \overline{\mathbb{S}^\infty}$ is generated by

$$\overline{f^\partial - 1},\ \overline{\delta(f^\partial - 1)},\ \overline{\delta^2(f^\partial - 1)}, ...$$

3) The kernel of $\overline{\mathbb{S}^\infty} \to \overline{\mathcal{W}}_g$ is generated by the images of

$$\overline{f^1},\ \overline{\delta f^1},\ \overline{\delta^2 f^1}, ...$$

4) The kernel of $\overline{M^\infty} \to \overline{\mathbb{W}}_g$ *is generated by the elements*

$$\overline{f^\partial - 1},\ \overline{f^1},\ \overline{\delta(f^\partial - 1)},\ \overline{\delta f^1},\ \overline{\delta^2(f^\partial - 1)},\ \overline{\delta^2 f^1}, \ldots$$

Proof of Theorem 3.5.13. We are going to use the notation in the proof of Theorem 3.4.12. In particular recall the rings $Q^{r,r-1}$ which are finite étale extensions of $\overline{S^r}$, with

$$(Q^{r,r-1})^{\Gamma_{r-1}} = \overline{S^r}. \tag{3.5.11}$$

Note that assertion 4 follows from assertions 2 and 3.

We claim that in order to prove assertions 1 and 2 it is enough to show that all the rings $Q^{r,r-1}$ are integral domains. Indeed if this is so then

$$Q^\infty := \lim_{\rightarrow} Q^{r,r-1}$$

is an integral domain. We have surjections

$$Q^\infty \to \overline{\mathbb{S}^\infty} \to \widetilde{\mathbb{S}^\infty}, \tag{3.5.12}$$

where the last ring is an integral domain. Let I be the kernel of the composition (3.5.12). Since the composition $\overline{S^\infty} \to Q^\infty \to \widetilde{\mathbb{S}^\infty}$ is injective (cf. Proposition 3.4.1), upon viewing $\overline{S^\infty}$ as a subring of Q^∞, it follows that $I \cap \overline{S^\infty} = 0$. Since Q^∞ is an integral domain and an integral extension of $\overline{S^\infty}$ it follows that $I = 0$. This forces the surjections in (3.5.12) to be isomorphisms, and so assertions 1 and 2 of the Theorem follow.

Next note that since *Spec* $Q^{r,r-1}$ is étale and finite over *Spec* $\overline{S^r}$ and since the latter is smooth over k, it follows that *Spec* $Q^{r,r-1}$ is smooth over k so, in particular its connected components are irreducible and they are finite and étale over *Spec* $\overline{S^r}$. So in order to prove that $Q^{r,r-1}$ is an integral domain it is enough to prove that *Spec* $Q^{r,r-1}$ is connected.

Consequently in order to prove the Theorem we need to prove connectivity of $Spec\ Q^{r,r-1}$ and assertion 3. We will prove these two facts simultaneously. To prove either of these facts it is enough to prove that these facts hold for each of the open sets of a given open cover of X. So we may assume, after shrinking X, that the conclusion of Lemma 3.5.11 holds for X, in particular L is trivial on the whole of X so $f^{\flat}$ is defined and f^1 and $f^{\flat}$ differ by a unit. Consider the scheme $Spec\ T^r$ defined by the cartesian diagram

$$\begin{array}{ccc} Spec\ T^r & \rightarrow & Spec\ Q^{r,r-1} \\ \downarrow & & \downarrow \\ Spec\ \overline{S} & \rightarrow & Spec\ \overline{S^r} \end{array}$$

where the bottom horizontal arrow is defined by the surjection

$$\overline{S^r} \rightarrow \frac{\overline{S^r}}{(\overline{f^{\flat}}, \overline{\delta f^{\flat}}, ..., \overline{\delta^{r-1} f^{\flat}})} = \overline{S},$$

cf. Lemma 3.5.11. The natural $\mathbb{Z}_p^{\times}$-equivariant homomorphism $M^{\infty} \rightarrow \widehat{\mathbb{W}}_g$ maps $f^1, \delta f^1, \delta^2 f^1, ...$ into 0; cf. Proposition 3.5.8. So this homomorphism also maps $f^{\flat}, \delta f^{\flat}, \delta^2 f^{\flat}, ...$ into 0. One the other hand this homomorphism also maps $f^{\partial} - 1, \delta(f^{\partial}-1), \delta^2(f^{\partial}-1), ...$ into 0. So we get an induced $\mathbb{Z}_p^{\times}$-equivariant homomorphism $\overline{N^r} \rightarrow \overline{\mathbb{W}}_g$, hence (by restriction) we get a $\mathbb{Z}_p^{\times}$-equivariant homomorphism $Q^{r,r-1} \rightarrow \overline{\mathbb{W}}_g$, and hence we get an induced $\mathbb{Z}_p^{\times}$-equivariant homomorphism

$$T^r = \frac{Q^{r,r-1}}{(\overline{f^{\flat}}, ..., \overline{\delta^{r-1} f^{\flat}})} \rightarrow \overline{\mathbb{W}}_g.$$

Since $Spec\ \overline{\mathbb{W}}_g$ is irreducible the closure Z of the image of $Spec\ \overline{\mathbb{W}}_g \rightarrow Spec\ T^r$ is contained in one of the connected components of $Spec\ T^r$. Since Z dominates $Spec\ \overline{S}$ and since $Spec\ T^r$ is finite and étale over $Spec\ \overline{S}$, it follows that Z is a connected component of $Spec\ T^r$. Note that Z is a $\mathbb{Z}_p^{\times}$-invariant subset of $Spec\ T^r$, hence Γ_{r-1}-invariant. Recall that by (3.5.11) Γ_{r-1} acts transitively on the fibers of $Spec\ Q^{r,r-1} \rightarrow Spec\ \overline{S^r}$. Hence Γ_{r-1} acts transitively on the fibers of $Spec\ T^r \rightarrow$

$Spec\ \overline{S}$. Since each connected component of $Spec\ T^r$ surjects onto $Spec\ \overline{S}$ and since Z is Γ_{r-1}-invariant it follows that $Spec\ T^r$ must be connected. Since $Spec\ T^r$ is smooth over k and connected it follows that T^r is an integral domain. Since $Spec\ T^r$ is connected it must coincide with Z hence $Spec\ \overline{\mathbb{W}}_g \to Spec\ T^r$ is dominant. Since T^r is an integral domain, $T^r \to \overline{\mathbb{W}}_g$ is injective. So $\lim_{\to} T^r \to \overline{\mathbb{W}}_g$ is injective. But

$$\lim_{\to} T^r = \lim_{\to} Q^{r,r-1}/(\overline{f^\flat}, \overline{\delta f^\flat}, \overline{\delta^2 f^\flat}, ...) = \overline{\mathbb{S}^\infty}/(\overline{f^\flat}, \overline{\delta f^\flat}, \overline{\delta^2 f^\flat}, ...).$$

This proves assertion 3.

On the other hand since each connected component of $Spec\ Q^{r,r-1}$ surjects onto $Spec\ \overline{S^r}$ and $Spec\ T^r$ is connected it follows that $Spec\ Q^{r,r-1}$ itself is connected. This ends the proof of the Theorem. □

Corollary 3.5.14. Assume $X = Spec\ S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. The following hold:

1) The inclusion $\mathbb{S}^\infty \subset S^\infty_{for}$ has torsion free cokernel.

2) The kernel of $M^\infty \to S^\infty_{for}$ is the p-adic closure of the ideal generated by the elements

$$f^\partial - 1,\ \delta(f^\partial - 1),\ \delta^2(f^\partial - 1), ...$$

3) The kernel of $\mathbb{S}^\infty \to R((q))\hat{}$ is the p-adic closure of the ideal generated by the images of the elements

$$f^1,\ \delta f^1,\ \delta^2 f^1, ...$$

4) The kernel of $M^\infty \to R((q))\hat{}$ is the p-adic closure of the ideal generated by the elements

$$f^\partial - 1,\ f^1,\ \delta(f^\partial - 1),\ \delta f^1,\ \delta^2(f^\partial - 1),\ \delta^2 f^1, ...$$

Remark 3.5.15. Conclusion 1 in Corollary 3.5.14 should be viewed as a δ-*expansion principle*. Conclusions 2 and 4 should be viewed as δ-analogues of the theorem of

Swinnerton-Dyer and Serre according to which the kernel of the Fourier expansion map

$$\bigoplus_{\kappa\geq 0} M(\mathbb{F}_p,\kappa,N)\to\mathbb{F}_p[[q]]$$

is generated by $E_{p-1}-1$; cf. [20], p. 459.

Corollary 3.5.16. Assume $X = Spec\ S$ is a modular Fourier-framed curve with E_{p-1} invertible on X. Let $f(q)\in R((q))$ be contained in the image of the map $M\otimes_R K\to K((q))$. Then $f(q)$ is contained in the image of the map $M^\infty\to R((q))\hat{}$.

Proof. Write $f(q)=E(\frac{G}{p^\nu})=\frac{G(q)}{p^\nu}$, where $G\in M$. The image of G in $R((q))\otimes\mathbb{Z}/p^\nu\mathbb{Z}$ is 0 so the image of G in $\mathbb{S}^\infty\otimes\mathbb{Z}/p^\nu\mathbb{Z}$ is in the kernel of $\mathbb{S}^\infty\otimes\mathbb{Z}/p^\nu\mathbb{Z}\to S^\infty_{for}\otimes\mathbb{Z}/p^\nu\mathbb{Z}$. But the latter morphism is injective; indeed this is trivially checked by induction on ν, using Theorem 3.5.13. It follows that the image of G in $\mathbb{S}^\infty\otimes\mathbb{Z}/p^\nu\mathbb{Z}$ is 0, hence the image of G in $\mathbb{S}^\infty$ belongs to $p^\nu\mathbb{S}^\infty$. Hence the image of G in $R((q))\hat{}$ belongs to $p^\nu\cdot Im(M^\infty\to R((q))\hat{})$. It follows that $f(q)$ belongs to the image of $M^\infty\to R((q))\hat{}$. □

Recall that we denoted by Y_{ord} the locus in $Y=Y_1(N)_R$ where E_{p-1} is invertible.

Corollary 3.5.17. Consider the modular Fourier-framed curve $X = Spec\ S = Y_{ord}$. Then the image of $M^\infty\to R((q))\hat{}$ contains $\mathcal{D}$ and hence is p-adically dense in $\mathcal{W}$.

Proof. By Corollary 3.5.16 the image of $M^\infty\to R((q))\hat{}$ contains the ring $\mathcal{D}$. But this image also contains the ring $R[\Delta^{-1}]$. We conclude by Lemma 3.5.12. □

3.5.1 On $(p-1)$-th root of E_{p-1}.

We will show that the $p-1$-th root of E_{p-1} does not belong to the modular forms of $X_1(N)$. It will be useful to recall one of the possible constructions of the Igusa curve I. Let L be the line bundle on $X_1(N)_R$ such that the sections of the powers of L

identify with the modular forms of various weights on $\Gamma_1(N)$; cf. [20] p. 450 where L was denoted by ω. Let $E_{p-1} \in H^0(X_1(N)_R, L^{p-1})$ be the normalized Eisenstein form of weight $p-1$ and let (ss) be the supersingular locus on $X_1(N)_R$ (i.e. the zero locus of E_{p-1}). Let $X \subset X_1(N)$ be an open embedding such that L is trivial on X. As usual, let x be a basis of L on X. Recall we have $M = S[x, x^{-1}]$. Then $E_{p-1} = \varphi x^{p-1}$ where $\varphi \in \mathcal{O}(X)$. Let $S = \mathcal{O}(X)$. Define $S_! = Spec\ S[t]/(t^{p-1} - \varphi)$. Let $X_! = Spec\ S_!$. If we denote the reduction mod p by $\overline{X_!}$, then $\overline{X}_!$ is birationally equivalent to I (cf. [20], pp. 460, 461) and is the integral closure of $\overline{X}$ in the fraction field of $\overline{X}_!$. Hence $Spec\ S[t]/(t^{p-1} - \varphi)$ is irreducible since I is. Note that $t^{p-1} - \varphi$ are monic polynomials whose derivatives are invertible in $S[t]/(t^{p-1} - \varphi)$.

Lemma 3.5.18. *There exist no $\epsilon \in \widehat{M}$ such that $\epsilon^{p-1} = E_{p-1}$.*

Proof. We will prove this by contradiction. Suppose there exist an $\epsilon \in \widehat{M}$ satisfying $\epsilon^{p-1} = E_{p-1}$. Then define an algebra homomorphism $\frac{S[t]}{t^{p-1}-\varphi} = S_! \to \widehat{M} = \widehat{S[x, x^{-1}]}$, by $t \to \epsilon x^{-1}$. Hence after reduction mod p, we have the following

$$\begin{array}{ccc} Spec\ \overline{M} & \xrightarrow{f} & Spec\ \overline{S_!} \\ \downarrow & \swarrow_{\psi} & \\ Spec\ \overline{S} & & \end{array}$$

Note that $Spec\ \overline{M} \simeq Spec\ \overline{S} \times \mathbb{G}_m$. Consider the restriction map of f (call it f again) to $Spec\ \overline{M} \times \{\text{closed point}\}$. Hence we have the following commutative diagram

$$\begin{array}{ccc} Spec\ \overline{S} & \xrightarrow{f} & Spec\ \overline{S_!} \\ \downarrow & \swarrow_{\psi} & \\ Spec\ \overline{S} & & \end{array}$$

The image of f can not be a point because $\psi \circ f$ is an isomorphism. Hence $Spec\ \overline{S}$ must be isomorphic to $Spec\ S_!$ since it is irreducible. But that implies $t^{p-1} - \varphi$ is not irreducible which is a contradiction and we are done. $\square$

Chapter 4

Hecke Operators mod p

We will study the space of q-expansions of differential modular forms mod p under the action of Hecke operators. But first, we need to extend the action of Hecke operators from classical Fourier series to differential Fourier series in our context which is done in Section 4.1. The relation between the coefficients of the eigen forms under Hecke operators away from p is also established. The concept of δ-symmetry is discussed and the space of δ-symmetric power series in $k[[q]][q']$ is computed. In Section 4.3, we put together 4.1 and 4.2 and by comparing the coefficients we obtain a multiplicity one theorem. In Section 4.4, we apply it in the case when the power series in $k[[q]][q']$ is the image of a differential modular form.

4.1 Hecke operators away from p

4.1.1 Classical Hecke operators

Throughout the chapter the divisors of a given non-zero integer are always taken to be positive, the greatest common divisor of two non-zero integers m, n is denoted by

(m,n), and we use the convention $(m,n) = n$ for $m = 0$, $n \neq 0$. Fix throughout the chapter an integer $N \geq 4$ and let $\epsilon : \mathbb{Z}_{>0} \to \{0,1\}$ be the "trivial primitive character" mod N defined by $\epsilon(A) = 1$ if $(A,N) = 1$ and $\epsilon(A) = 0$ otherwise.

For each integer $n \geq 1$ and each integer $N \geq 4$ consider the set

$$\{(A,B,D); A,B,D \in \mathbb{Z}_{\geq 0}, AD = n, (A,N) = 1, B < D\}$$

Triples A, B, D will always be assumed to be in the set above. Recall (cf., say, [24]) the action of the n-th Hecke operator $T_\kappa(n)$ on classical modular forms $f = \sum_{m\geq 0} a_m q^m$ on $\Gamma_0(N)$ of weight $\kappa \geq 2$ with complex coefficients $a_m \in \mathbb{C}$ given by

$$\begin{aligned} T_\kappa(n)f &:= n^{\kappa-1} \textstyle\sum_{A,B,D} D^{-\kappa} f(\zeta_D^B q^{A/D}) \\ &= \textstyle\sum_{m\geq 0} \left(\sum_{A|(n,m)} \epsilon(A) A^{\kappa-1} a_{\frac{mn}{A^2}} \right) q^m. \end{aligned}$$

Here $q = e^{2\pi\sqrt{-1}z}$, $\zeta_D := e^{2\pi\sqrt{-1}/D}$.

4.1.2 Hecke operators $T_\kappa(n)$ on δ-series

Now assume n and N are coprime to p and assume $q, q', q'', ..., q^{(r)}, ...$ are indeterminates.

Definition 4.1.1. For each integer $\kappa \in \mathbb{Z}$ the *Hecke* operator $f \mapsto T_\kappa(n)f$ on $R((q))[q', ..., q^{(r)}]\hat{}$ is defined as follows. For $f = f(q, q', ..., q^{(r)})$,

$$T_\kappa(n)f := n^{\kappa-1} \sum_{A,B,D} D^{-\kappa} f(\zeta_D^B q^{A/D}, \delta(\zeta_D^B q^{A/D}), ..., \delta^r(\zeta_D^B q^{A/D})). \tag{4.1.1}$$

Here $\zeta_D = \zeta_n^{n/D} \in R$ where $\zeta_n \in R$ is a fixed primitive n-th root of unity and the right hand side of (4.1.1) is a priori in the ring

$$R((q_n))\hat{}[q_n', ..., q_n^{(r)}]\hat{}, \quad q_n = q^{1/n}. \tag{4.1.2}$$

However, by [10] Proposition 3.13,

$$q_n', ..., q_n^{(r)} \in R[q, q^{-1}, q', ..., q^{(r)}]\hat{}$$

hence the ring (4.1.2) equals

$$R((q_n))\hat{}[q', ..., q^{(r)}]\hat{}.$$

Since $T_\kappa(n)f$ is invariant under the substitution $q_n^{(i)} \mapsto \delta^i(\zeta_n q_n)$ it follows that $T_\kappa(n)f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}$. So the operators $T_\kappa(n)$ send $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$ into itself. As we shall see below for $n \geq 2$ the operators $T_\kappa(n)$ do *not* send $R[[q]][q', ..., q^{(r)}]\hat{}$ into itself. The operators $T_\kappa(n)$ on $R((q))[q', ..., q^{(r)}]\hat{}$ induce operators still denoted by $T_\kappa(n)$ on $k((q))[q', ..., q^{(r)}]$.

Recall the operator V on $R((q))\hat{}$ defined by $V(\sum a_n q^n) = \sum a_n q^{pn}$. It induces an operator still denoted by V on $k((q))$.

For $r = 0$, $T_\kappa(n)$ commute with the operator V on $R((q))\hat{}$.

4.1.3 Order $r = 1$

We have the following formula for the Hecke action on δ-series of order 1:

Proposition 4.1.2. Assume that

$$f = \sum_{m,m'} a_{m,m'} q^m (q')^{m'} \tag{4.1.3}$$

where $m \in \mathbb{Z}$, $m' \in \mathbb{Z}_{\geq 0}$. Then we have the following congruence mod (p):

$$T_\kappa(n)f \equiv \sum_{m,m'} \left(\sum_{A|(n,m)} n^{-m'} \epsilon(A) A^{\kappa+2m'-1} a_{\frac{mn}{A^2}-m'p,m'} \right) q^{m-m'p} (q')^{m'}. \tag{4.1.4}$$

Proof. Note that

$$\begin{aligned}\delta(\zeta_D^B q^{A/D}) &= \tfrac{1}{p}[\phi(\zeta_D^B q^{A/D}) - (\zeta_D^B q^{A/D})^p] \\ &= \tfrac{1}{p}[\zeta_D^{Bp}(q^p + pq')^{A/D} - \zeta_D^{Bp} q^{Ap/D}] \qquad (4.1.5) \\ &\equiv \tfrac{A}{D}\zeta_D^{Bp} q^{(A-D)p/D} q' \quad mod\ (p).\end{aligned}$$

Then the formula in the statement of the Proposition follows by a simple computation, using the fact that

$$\sum_{B=0}^{D-1} \zeta_D^{m+m'p}$$

is D or 0 according as D divides or does not divide $m + m'p$. □

Corollary 4.1.3. Let

$$\overline{f} = \sum_{m'} \overline{f}_{m'}(q) \left(\frac{q'}{q^p}\right)^{m'} \in k((q))[q'], \quad \overline{f}_{m'}(q) \in k((q)). \qquad (4.1.6)$$

Then for any integer κ and any integer $n \geq 1$ coprime to p we have:

$$T_\kappa(n)\overline{f} = \sum_{m'} n^{-m'}(T_{\kappa+2m'}(n)\overline{f}_{m'}(q)) \left(\frac{q'}{q^p}\right)^{m'}.$$

In particular for $\overline{\lambda}_n \in k$ we have $T_\kappa(n)\overline{f} = \overline{\lambda}_n \overline{f}$ if and only if

$$T_{\kappa+2m'}(n)\overline{f}_{m'} = n^{m'}\overline{\lambda}_n \overline{f}_{m'} \ \text{ for all } \ m' \geq 0.$$

Proof. This follows immediately from Proposition 4.1.2. □

Let us say that a series in $k((q))[q', ..., q^{(r)}]$ is *holomorphic at infinity* if it belongs to $k[[q]][q', ..., q^{(r)}]$. Also denote by v_p the p-adic valuation on $\mathbb{Z}$.

Corollary 4.1.4. Assume that, for a given $\kappa \in \mathbb{Z}$ the series $\overline{f} \in k[[q]][q']$ has the property that $T_\kappa(n)\overline{f}$ is holomorphic at infinity for all $n \geq 1$ coprime to p. Then $\overline{f}$

has the form

$$\overline{f}(q,q') = \varphi_0(q) + \sum_{m'\geq 1} (V^{v_p(m')+1}(\varphi_{m'}(q))) \left(\frac{q'}{q^p}\right)^{m'}, \tag{4.1.7}$$

with

$$\varphi_0 \in k[[q]], \quad \varphi_{m'}(q) \in q^{m'/p^{v_p(m')}} k[[q]] \ \text{ for } \ m' \geq 1. \tag{4.1.8}$$

Proof. Note that, since $T_\kappa(1)\overline{f} = \overline{f}$, $\overline{f}$ is holomorphic at infinity so equation (4.1.8) follows from (4.1.7). Let $\overline{f}$ be the reduction mod p of a series as in (4.1.3). It is enough to show if two integers $m_0 \geq 1$ and $m' \geq 1$ satisfy $v_p(m_0) \leq v_p(m')$ then $\overline{a}_{m_0,m'} = 0$. Pick such integers m_0, m' and set $i = v_p(m_0)$, $m_0 = p^i\mu$, $m' = p^i\mu'$, $n = \mu + p\mu'$. Clearly n is coprime to p. Picking out the coefficient of $q^{p^i - p^{i+1}\mu'}(q')^{p^i\mu'}$ in the equation in Proposition 4.1.2 we get

$$\overline{a}_{m_0,m'} = \overline{a}_{p^i n - p^{i+1}\mu', p^i\mu'} = 0$$

and we are done. □

Corollary 4.1.5. Let κ be an integer, let $\overline{f} \in k[[q]][q']$ be holomorphic at infinity, and assume that for any integer $n \geq 1$ coprime to p we are given a $\overline{\lambda}_n \in k$. Then $T_\kappa(n)\overline{f} = \overline{\lambda}_n\overline{f}$ for all $(n,p) = 1$ if and only if $\overline{f}$ has the form (4.1.7) and

$$T_{\kappa+2m'}(n)\varphi_{m'}(q) = n^{m'}\overline{\lambda}_n\varphi_{m'}(q) \ \text{ for all } \ m' \geq 0.$$

Proof. This follows directly from the previous corollaries plus the commutation of $T_\kappa(n)$ and V on $k[[q]]$. □

4.1.4 Order $r = 2$

Let us record the formula giving the Hecke action on δ-series of order 2. This formula will not be used in the sequel.

Proposition 4.1.6. If $f = \sum_{m,m,m''} a_{m,m',m''} q^m (q')^{m'} (q'')^{m''} \in R((q))[q', q'']\hat{}$ then we have the following congruence mod p:

$$T_\kappa(n)f \equiv \sum A^{\kappa-1} \left(\tfrac{A}{D}\right)^{m'+m''} \times a_{m,m',m''} \times q^{A(m+m'p+m''p^2)/D}$$

$$\times \left(\tfrac{q'}{q^p}\right)^{m'} \times \left[\tfrac{q''}{q^{p^2}} + \tfrac{\delta(A/D)}{A/D} \cdot \left(\tfrac{q'}{q^p}\right)^p + \tfrac{1}{2}\left(\tfrac{A}{D} - 1\right) \cdot \left(\tfrac{q'}{q^p}\right)^{2p}\right]^{m''}$$

where the sum in the right hand side runs through all m, m', m'', A, D with $A \geq 1, AD = n, (A, N) = 1, D|m + m'p + m''p^2$.

Proof. A computation similar to the one in the proof of Proposition 4.1.2. □

Note that the formula in Proposition 4.1.6 acquires a simpler form for special ns. Indeed assume $n = \ell$ is a prime. If $\ell \equiv 1$ mod p then $\frac{A}{D} - 1 = 0$ in k. If $\ell \equiv 1$ mod p^2 then $\delta(A/D) = 0$ in k. Finally if $\ell \equiv 1$ mod p but $\ell \not\equiv 1$ mod p^2 then $\delta(A/D) \neq 0$ in k.

4.1.5 Frobenii

Consider the ring endomorphisms $F, F_k, F_{/k}$ of $k((q))[q', ..., q^{(r)}]$ defined as follows: F is the p-power Frobenius (the "absolute Frobenius"); F_k is the ring automorphism that acts as the p-power Frobenius on k and is the identity on the variables $q, q', ..., q^{(r)}$; $F_{/k}$ is the ring endomorphism that is the identity on k and sends $q, q', ..., q^{(r)}$ into $q^p, (q')^p, ..., (q^{(r)})^p$ respectively (the "relative Frobenius"). So we have $F = F_k \circ F_{/k} = F_{/k} \circ F_k$. Of course $V = F_{/k}$ on $k((q))$. Also clearly $T_\kappa(n)$ commute with F. By Proposition 4.1.2 $T_\kappa(n)$ also commute with F_k on $k((q))[q']$; so $T_\kappa(n)$ commute with $F_{/k}$ on $k((q))[q']$.

4.2 Hecke operator at p

4.2.1 Taylor and Laurent δ-symmetry

Following [11] we consider the $R-$algebras

$$A \ := \ R[[s_1, ..., s_p]][s_p^{-1}]\hat{}[s'_1, ..., s'_p, ..., s_1^{(r)}, ..., s_p^{(r)}]\hat{},$$

$$B \ := \ R[[q_1, ..., q_p]][q_1^{-1}...q_p^{-1}]\hat{}[q'_1, ..., q'_p, ..., q_1^{(r)}, ..., q_p^{(r)}]\hat{},$$

where $s_1, ..., s_p, s'_1, ..., s'_p, ...$ and $q_1, ..., q_p, q'_1, ..., q'_p, ...$ are indeterminates. In [11], Lemma 9.10 we proved that the natural algebra map

$$A \to B, \quad s_j^{(i)} \mapsto \delta^i S_j,$$

where $S_1, ..., S_p$ are the fundamental symmetric polynomials in $q_1, ..., q_p$, is injective with torsion free cokernel. We will view this algebra map as an inclusion.

Definition 4.2.1. An element $G \in B$ is called *Laurent $\delta-$symmetric* [11] if it is the image of some element $G_{(p)} \in A$ (which is then unique). An element $f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}$ will be called *Laurent $\delta - p$-symmetric* if

$$\Sigma_p f := \sum_{j=1}^{p} f(q_j, ..., q_j^{(r)}) \in B$$

is Laurent $\delta-$symmetric.

In the same way one can consider the algebras

$$A \ := \ R[[s_1, ..., s_p]][s'_1, ..., s'_p, ..., s_1^{(r)}, ..., s_p^{(r)}]\hat{},$$

$$B \ := \ R[[q_1, ..., q_p]][q'_1, ..., q'_p, ..., q_1^{(r)}, ..., q_p^{(r)}]\hat{}.$$

As before the natural algebra map

$$A \to B, \quad s_j^{(i)} \mapsto \delta^i S_j,$$

is injective with torsion free cokernel.

Definition 4.2.2. An element $G \in B$ will be called *Taylor $\delta-$symmetric* if it is the image of some element $G_{(p)} \in A$ (which is then unique). An element $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ will be called *Taylor $\delta - p$-symmetric* if

$$\Sigma_p f := \sum_{j=1}^{p} f(q_j, ..., q_j^{(r)}) \in B$$

is Taylor $\delta-$symmetric.

Clearly a Taylor $\delta - p$-symmetric series is also Laurent $\delta - p$-symmetric.

Remark 4.2.3. 1) Any element of $R[[q]]$ (respectively $R((q))$) is Taylor (respectively Laurent) $\delta - p$-symmetric.

2) The Taylor (respectively Laurent) $\delta-p$-symmetric elements in $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) form a p-adically closed R-submodule.

3) If f is Taylor (respectively Laurent) $\delta - p$-symmetric then $\phi(f)$ is Taylor (respectively Laurent) $\delta - p$-symmetric.

4) If $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) and pf is Taylor (respectively Laurent) $\delta-p$-symmetric then f is Taylor (respectively Laurent) $\delta-p$-symmetric.

5) By 1)-4) any element f in $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) of the form

$$f = \frac{\sum_{i=0}^{m} \phi^i(g_i)}{p^\nu}$$

where g_i are in $R[[q]]$ (respectively in $R((q))$) is Taylor (respectively Laurent) $\delta - p$-symmetric. In particular for any g in $R[[q]]$ (respectively in $R((q))$) we have that $\delta g = \frac{\phi(g)-g^p}{p}$, and more generally $\frac{\phi^i(g)-g^{p^i}}{p}$ are Taylor (respectively Laurent) $\delta - p$-symmetric.

6) Let $\mathcal{F} \in R[[T_1, T_2]]^g$ be a formal group law, and let $\psi \in R[[T]][T, ..., T^{(r)}]\hat{}$ be

such that

$$\psi(\mathcal{F}(T_1,T_2),...,\delta^r\mathcal{F}(T_1,T_2)) = \psi(T_1,...,T_1^{(r)}) + \psi(T_2,...,T_2^{(r)})$$

in the ring

$$R[[T_1,T_2]][T_1',T_2',,...,T_1^{(r)},T_2^{(r)}]\hat{}.$$

(Such a ψ is called a δ-*character* of $\mathcal{F}$.) Let $\varphi(q) \in qR[[q]]$ and let

$$f := \psi(\varphi(q),...,\delta^r(\varphi(q))) \in R[[q]][q',...,q^{(r)}]\hat{}.$$

Then f is Taylor $\delta - p$-symmetric. Cf the argument in [12].

Note that if $\mathcal{F}$ is defined over $\mathbb{Z}_p$ then $\mathcal{F}$ posses a δ-character ψ of order r at most the height of $\mathcal{F}$ mod p such that

$$\psi(T,0,...,0) \in T + T^p\mathbb{Z}_p[[T]];$$

cf. [**?**], proof of Proposition 4.26.

Applying the above considerations to the multiplicative formal group we get that for any $\varphi(q) \in qR((q))$ the series

$$\frac{1}{p}\log\left(\frac{\phi(\varphi(q)+1)}{(\varphi(q)+1)^p}\right)$$

is Taylor $\delta - p$-symmetric. (Here, as usual, $\log(1+T) = T - T^2/2 + T^3/3 - ...$)

7) The series

$$\Psi = \frac{1}{p}\log\left(\frac{\phi(q)}{q^p}\right) \tag{4.2.1}$$

is Laurent $\delta - p$-symmetric; cf. [11], proof of Proposition 9.13.

8) In [11] we also defined the concept of δ-symmetric element in

$$R[[q_1,...,q_p,...,q_1^{(r)},...,q_p^{(p)}]]$$

(without the qualification "Taylor" or "Laurent"). We will not use this concept in the present paper. But note that if a series is Taylor δ-symmetric then it is also δ-symmetric in the sense of [11] (and Laurent δ-symmetric in the sense of the present paper).

Definition 4.2.4. For any Taylor (respectively Laurent) $\delta - p$-symmetric

$$f \in R[[q]][q', ..., q^{(r)}]\hat{} \quad \text{(respectively } f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}\text{)}$$

we define

$$Uf := p^{-1}(\Sigma_p f)_{(p)}(0, ..., 0, q, ..., 0, ..., 0, q^{(r)})$$

which is an element in $p^{-1}R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $p^{-1}R((q))\hat{}[q', ..., q^{(r)}]\hat{}$).

The operator pU takes $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) into $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$). On the other hand the restriction of U to $R((q))\hat{}$ (respectively $R[[q]]$) takes values in $R((q))\hat{}$ (respectively $R[[q]]$) and is equal to the classical U-operator

$$U(\sum a_m q^m) = \sum a_{mp} q^m.$$

Definition 4.2.5. Define for any $f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}$ the series

$$Vf := f(q^p, ..., \delta^r(q^p)) \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}.$$

So for any Taylor (respectively Laurent) $\delta - p$-symmetric f in $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) and any $\kappa \in \mathbb{Z}$ we may define

$$pT_\kappa(p)f = pUf + p^\kappa Vf$$

which is an element in $p^\kappa R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively in $p^\kappa R((q))\hat{}[q', ..., q^{(r)}]\hat{}$).

The restriction of $pT_\kappa(p)$ to $R((q))$ is, of course, p times the "classical" Hecke operator $T_\kappa(p)$ on $R((q))$ defined by

$$T_\kappa(p)(\sum a_m q^m) = \sum a_{pm} q^m + p^{\kappa-1} \sum a_m q^{pm}.$$

Recall:

Proposition 4.2.6. [11] The series Ψ in (4.2.1) satisfies

$$pU\Psi = \Psi, \quad V\Psi = p\Psi.$$

For the next definition recall that the homomorphism

$$\overline{A} := A \otimes_R k \to \overline{B} := B \otimes_R k$$

is injective (in both situations described in the beginning of the section).

Definition 4.2.7. An element $\overline{G} \in \overline{B}$ is called *Taylor δ-symmetric mod p* (respectively *Laurent $\delta-$symmetric mod p*) if it is the image of some element $\overline{G}_{(p)} \in \overline{A}$ (which is then unique). An element $\overline{f} \in k[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $\overline{f} \in k((q))[q', ..., q^{(r)}]$) will be called Taylor (respectively Laurent) $\delta - p$*-symmetric* if

$$\Sigma_p \overline{f} := \sum_{j=1}^{p} \overline{f}(q_j, ..., q_j^{(r)}) \in \overline{B}$$

is Taylor $\delta-$symmetric mod p (respectively Laurent δ-symmetric mod p).

Clearly any Taylor $\delta - p$-symmetric series is Laurent $\delta - p$-symmetric.

Remark 4.2.8. 1) The Taylor (respectively Laurent) $\delta - p$-symmetric elements in $k[[q]][q', ..., q^{(r)}]$ (respectively in $k((q))[q', ..., q^{(r)}]$) form a k-subspace closed under F_k and F (hence also under $F_{/k}$).

2) If $f \in R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) is congruent mod p to a Taylor (respectively Laurent) $\delta - p$-symmetric element then the image of $\overline{f}$ of f in $k[[q]][q', ..., q^{(r)}]$ (respectively in $k((q))[q', ..., q^{(r)}]$) Taylor (respectively Laurent) $\delta - p$-symmetric.

Definition 4.2.9. For any Taylor (respectively Laurent) $\delta - p$-symmetric

$$\overline{f} \in k[[q]][q', ..., q^{(r)}]\hat{} \text{ (respectively } k((q))[q', ..., q^{(r)}])$$

we may define

$$\text{“}pU\text{”}\overline{f} := (\Sigma_p \overline{f})_{(p)}(0, ..., 0, q, ..., 0, ..., 0, q^{(r)})$$

which is an element of $k[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $k((q))[q', ..., q^{(r)}]$).

The operator “pU” clearly commutes with the operators F and F_k and hence it also commutes with the operator $F_{/k}$ (cf. section 4.1.5). If

$$f \in R[[q]][q', ..., q^{(r)}]\hat{} \text{ (respectively } f \in R((q))\hat{}[q', ..., q^{(r)}]\hat{}\text{)}$$

is Taylor (respectively Laurent) $\delta - p$-symmetric and $\overline{f}$ is the reduction mod p of f viewed as an element in $k[[q]][q', ..., q^{(r)}]$ (respectively in $k((q))[q', ..., q^{(r)}]$) then “pU”$\overline{f}$ is the reduction mod p of pUf; this justifies the notation in “pU”$\overline{f}$.

Note that the operator $U : R((q))\hat{} \to R((q))\hat{}$ induces an operator still denoted by U, $U : k((q)) \to k((q))$ (which is, of course, the classical U-operator $U\overline{f} = \sum \overline{a}_{mp} q^m$, for $\overline{f} = \sum \overline{a}_m q^m \in k((q))$). On the other hand note that “pU”$\overline{f} = 0$ for all $\overline{f} \in k((q))$. Finally note that if $\kappa \geq 1$ then the operator $T_\kappa(p)$ on $R((q))$ induces an operator $T_\kappa(p)$ on $k((q))$; if $\kappa \geq 2$ then $T_\kappa(p)$ on $k((q))$ coincides with U on $k((q))$.

Definition 4.2.10. Define the ring endomorphism V of

$$k[[q]][q', ..., q^{(r)}] \text{ (respectively } k((q))[q', ..., q^{(r)}]\text{)}$$

as the reduction mod p of the operator V over R. (Note that $V(q') = 0$ and $F_{/k}(q') = (q')^p$ so in particular $V \neq F_{/k}$ on $k((q))[q']$.) As in the case of characteristic zero, for any $\kappa \in \mathbb{Z}_{\geq 0}$ and any Taylor (respectively Laurent) $\delta - p$-symmetric series $\overline{f}$ in $k[[q]][q', ..., q^{(r)}]$ (respectively $k((q))[q', ..., q^{(r)}]$) we define

$$\text{“}pT_\kappa(p)\text{”}\overline{f} = \text{“}pU\text{”}\overline{f} + \overline{p^\kappa} \cdot V\overline{f}$$

which is again an element of $k[[q]][q', ..., q^{(r)}]$ (respectively $k((q))[q', ..., q^{(r)}]$). (Note that $\overline{p^\kappa}$ is 0 or 1 according as κ is >0 or 0.)

The operator V clearly commutes with F and F_k (and hence also with $F_{/k}$). So the operators "$pT_\kappa(p)$" commute with $F, F_k, F_{/k}$.

Also for f any Taylor (respectively Laurent) $\delta - p$-symmetric series in $R[[q]][q', ..., q^{(r)}]\hat{}$ (respectively $R((q))\hat{}[q', ..., q^{(r)}]\hat{}$) with reduction mod p $\overline{f}$ we have that "$pT_\kappa(p)$"$\overline{f}$ is the reduction mod p of $pT_\kappa(p)f$ which, again, justifies our notation.

4.3 Structure of Laurent and Taylor δ-symmetric series

In what follows we address the problem of determining what series are Laurent (respectively Taylor) $\delta - p$-symmetric and determining the action of our operators "pU" on them. We will use the following notation: for all $\varphi = \sum \overline{a}_n q^n \in k((q))$ we define

$$\varphi^{(-1)} := \theta^{p-2}\varphi = \sum_{(n,p)=1} \frac{\overline{a}_n}{n} q^n \in k((q)) \tag{4.3.1}$$

where $\theta = q\frac{d}{dq}$ is the Serre theta operator.

Theorem 4.3.1. *If an element $\overline{f} \in k[[q]][q']$ is Taylor $\delta - p$-symmetric then it has the form*

$$\overline{f} = \varphi_0(q) + \sum_{s\geq 0} (V^{s+1}(\varphi_{p^s}(q))) \left(\frac{q'}{q^p}\right)^{p^s} \in k((q))[q'] \tag{4.3.2}$$

with $\varphi_0(q) \in k[[q]]$, $\varphi_1(q), \varphi_p(q), \varphi_{p^2}(q), ... \in qk[[q]]$

Conversely we will prove:

Theorem 4.3.2. *Any element of the form*

$$\overline{f} = \varphi_0(q) + \sum_{s\geq 0} (V^{s+1}(\varphi_{p^s}(q))) \left(\frac{q'}{q^p}\right)^{p^s} \in k((q))[q']$$

with $\varphi_0(q), \varphi_1(q), \varphi_p(q), \varphi_{p^2}(q), ... \in k((q))$ *is Laurent* $\delta - p$*-symmetric and*

$$\text{“}pU\text{”}\overline{f} = -\sum_{s\geq 0} V^s(\varphi_{p^s}^{(-1)}(q)) + \sum_{s\geq 0}(V^{s+1}(U(\varphi_{p^s}(q)))) \left(\frac{q'}{q^p}\right)^{p^s}.$$

If in addition $\overline{f} \in k[[q]][q']$ *(i.e. if* $\varphi_0(q) \in k[[q]]$ *and* $\varphi_1(q), \varphi_p(q), \varphi_{p^2}(q), ... \in qk[[q]]$*) then* $\overline{f}$ *is Taylor* $\delta - p$*-symmetric.*

Corollary 4.3.3. Let $\overline{f} \in k((q))[q']$ be Laurent $\delta - p$-symmetric and let $\overline{\lambda}_p \in k$. Then “$pT_\kappa(p)$”$\overline{f} = \overline{\lambda}_p \cdot \overline{f}$ if and only if:

1) $U(\varphi_{p^s}(q)) = \overline{\lambda}_p \cdot \varphi_{p^s}(q)$ for all $s \geq 0$ and

2) $\overline{p^\kappa} \cdot V(\varphi_0(q)) - \sum_{s\geq 0} V^s(\varphi_{p^s}^{(-1)}(q)) = \overline{\lambda}_p \cdot \varphi_0(q)$.

Corollary 4.3.4. If $\overline{f} \in k[[q]][q']$ is Taylor $\delta - p$-symmetric then the series “pU”$\overline{f}$ and “$pT_\kappa(p)$”$\overline{f}$ are again Taylor $\delta - p$-symmetric.

We will first prove Theorem 4.3.2.

Lemma 4.3.5. *For any* $n \in \mathbb{Z}$ *and* $s \in \mathbb{Z}_{\geq 0}$ *the element*

$$\overline{f} = q^{np^{s+1}}(q')^{p^s} = q^{(n+1)p^{s+1}} \left(\frac{q'}{q^p}\right)^{p^s} \in k((q))[q']$$

is Laurent $\delta - p$*-symmetric (and actually Taylor* $\delta - p$*-symmetric if* $n \geq 0$*.) Moreover*

$$\text{“}pU\text{”}\overline{f} = \begin{cases} q^{(n+1)p^s}\left(\frac{q'}{q^p}\right)^{p^s} & \textit{if } \ p | n+1 \\ \\ -\frac{q^{(n+1)p^s}}{n+1} & \textit{if } \ p \not| n+1 \end{cases}$$

Proof. It is enough to consider the case $s = 0$; the general case follows by applying the p-power Frobenius.

For $n = -1$ note that

$$q^{-p}q' \equiv \Psi \ \ mod \ \ (p)$$

and so $q^{-p}q'$ is Laurent $\delta - p$-symmetric because Ψ is Laurent $\delta - p$-symmetric. Also "pU"$\overline{f} = \overline{f}$ because $pU\Psi = \Psi$.

Assume now $n \neq -1$. We have

$$\begin{aligned} \delta(q^{n+1}) &= \tfrac{1}{p}[(q^p + pq')^{n+1} - q^{p(n+1)}] \\ &= \tfrac{1}{p}\left[p(n+1)q^{pn}q' + \textstyle\sum_{j\geq 2} \frac{p^j}{j!}(n+1)...(n-j+2)q^{p(n+1-j)}(q')^j\right] \end{aligned}$$

For $j \geq 2$ (and since $p \geq 5$) we have

$$v_p\left(\frac{p^j}{j!}\right) \geq j - v_p(j!) \geq j - \frac{j}{p-1} > 1.$$

It follows that

$$\delta(q^{n+1}) = (n+1)[q^{pn}q' + pF_{n+1}(q, q')], \quad F_{n+1}(q, q') \in R[q, q^{-1}, q']. \tag{4.3.3}$$

In particular $\delta(q^{n+1})$ is divisible by $n+1$ in $R((q))\hat{}[q']\hat{}$ and we have the following congruence in $R((q))\hat{}[q']\hat{}$:

$$\frac{1}{n+1}\delta(q^{n+1}) \equiv q^{np}q' \quad mod \quad (p). \tag{4.3.4}$$

By Remark 4.2.8, assertions 4) and 5), the left hand side of the latter congruence is Laurent $\delta - p$-symmetric (and also Taylor $\delta - p$-symmetric if $n \geq 0$) and hence $q^{pn}q'$ is Laurent $\delta - p$-symmetric (and also Taylor $\delta - p$-symmetric if $n \geq 0$).

To compute "pU"$\overline{f}$ start with the following computation in $R((q))\hat{}[q']\hat{}$:

$$\begin{aligned}
p^2(n+1)U\left(\tfrac{\delta(q^{n+1})}{n+1}\right) &= pU(p\delta(q^{n+1})) \\
&= pU(\phi(q^{n+1})) - pU(q^{p(n+1)}) \\
&= \phi(pU(q^{n+1})) - pU(q^{p(n+1)}) \\
&= \begin{cases} -pq^{n+1} & \text{if } p \not| n+1 \\ p\phi(q^{\frac{n+1}{p}}) - pq^{n+1} & \text{if } p | n+1 \end{cases} \\
&= \begin{cases} -pq^{n+1} & \text{if } p \not| n+1 \\ p^2\delta(q^{\frac{n+1}{p}}) & \text{if } p | n+1 \end{cases} \\
&= \begin{cases} -pq^{n+1} & \text{if } p \not| n+1 \\ p^2\frac{n+1}{p}\left[q^{p(\frac{n+1}{p}-1)}q' + pF_{\frac{n+1}{p}}(q,q')\right] & \text{if } p | n+1 \end{cases}
\end{aligned}$$

from which we get the following congruences mod p in $R((q))\hat{}[q']\hat{}$:

$$pU(q^{pn}q') \equiv pU\left(\frac{\delta(q^{n+1})}{n+1}\right) \equiv \begin{cases} -\frac{q^{n+1}}{n+1} & \text{if } p \not| n+1 \\ q^{n+1-p}q' & \text{if } p | n+1. \end{cases}$$

and we are done. □

Lemma 4.3.6. *Consider the polynomials*

$$s_1, ..., s_p, s'_1, ..., s'_p, D \in k[q_1, ..., q_p, q'_1, ..., q_p], \quad D := \prod_{i<j}(q_i - q_j).$$

Then the polynomials

$$D^p q'_1, ..., D^p q'_p$$

are linear combinations of

$$1, s'_1, ..., s'_p$$

with coefficients in $k[q_1, ..., q_p]$.

Proof. For $j = 1, ..., p$ let s_{ij} be obtained from s_i by setting $q_j = 0$; so s_{ij} is the ith fundamental symmetric polynomial in $\{q_1, ..., q_p\} \backslash \{q_j\}$. Taking δ in the equalities

$$q_1 + ... + q_p = s_1, ..., q_1 ... q_p = s_p$$

in $R[q_1, ..., q_p, q'_1, ..., q'_p]$ and reducing mod p we get the following equalities in $k[q_1, ..., q_p, q'_1, ..., q'_p]$:

$$\begin{aligned} q'_1 + ... + q'_p &= s'_1 - \gamma_1 \\ s^p_{11} q'_1 + ... + s^p_{1p} q'_p &= s'_2 - \gamma_2 \\ &\cdots\cdots \\ s^p_{p-1,1} q'_1 + ... + s^p_{p-1,p} q'_p &= s'_p - \gamma_p \end{aligned}$$

for some $\gamma_1, ..., \gamma_p \in k[q_1, ..., q_p]$. View this as a linear system of equations with unknowns $q'_1, ..., q'_p$. We shall be done if we prove that the determinant of the matrix of this system is $\pm D^p$. This follows by taking determinants in the obvious identity of matrices

$$\begin{pmatrix} q_1^{p-1} & -q_1^{p-2} & ... & 1 \\ q_2^{p-1} & -q_2^{p-2} & ... & 1 \\ ... & & & \\ q_p^{p-1} & -q_p^{p-2} & ... & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & ... & 1 \\ s_{11} & s_{12} & ... & s_{1p} \\ ... & & & \\ s_{p-1,1} & s_{p-1,2} & ... & s_{p-1,p} \end{pmatrix} = (D_{ij})$$

where

$$D_{ij} = \prod_{s \neq j} (q_i - q_s)$$

and noting that (D_{ij}) is a diagonal matrix with determinant D^2. □

Lemma 4.3.7. *Assume the notation of Lemma* 4.3.6 *and* $n \geq 0$. *Then the element*

$$\sum_{i=1}^{p} q_i^{np} q_i' \in k[[q_1, ..., q_p]][q_1', ..., q_p']$$

is a linear combination of

$$1, s_1', ..., s_p'$$

with coefficients in the ideal

$$(s_1, ..., s_p)^{[(n+1)/p]-1} k[s_1, ..., s_p].$$

Proof. By Lemma 4.3.6 we can write

$$\sum_{i=1}^{p} q_i^{np} q_i' = A_0 + \sum_{j=1}^{p} A_j s_j'$$

where $A_j \in k[q_1, ..., q_p, D^{-1}]$ for $j = 0, ..., p$. On the other hand, by (4.3.4) $\sum_{i=1}^{p} q_i^{np} q_i'$ is the reduction mod p of

$$\frac{1}{n+1} \sum_{i=1}^{p} \delta(q_i^{n+1}) \in R[q_1, ..., q_p, q_1', ..., q_p].$$

We claim that the following holds:

$$\sum_{i=1}^{p} \delta(q_i^{n+1}) \in (s_1, ..., s_p, s_1', ..., s_p')^{[(n+1)/p]} R[s_1, ..., s_p, s_1', ..., s_p']. \tag{4.3.5}$$

Assuming (4.3.5) is true let us show how to conclude the proof of the Lemma. By (4.3.5) we get that

$$\sum_{i=1}^{p} q_i^{np} q_i' \in (s_1, ..., s_p, s_1', ..., s_p)^{[(n+1)/p]} k[s_1, ..., s_p, s_1', ..., s_p'].$$

So we have

$$\sum_{i=1}^{p} q_i^{np} q_i' = \sum B_{i_1 \dots i_p} (s_1')^{i_1} \dots (s_p')^{i_p}$$

where

$$B_{i_1 \dots i_p} \in (s_1, ..., s_p)^{[(n+1)/p] - i_1 - \dots - i_p} k[s_1, ..., s_p].$$

Since $s_1', ..., s_p'$ are algebraically independent over $k[q_1, ..., q_p]$ we get

$$\begin{aligned} A_0 &= B_{0 \dots 0} \\ A_1 &= B_{10 \dots 0} \\ A_2 &= B_{010 \dots 0}, \text{ etc} \end{aligned}$$

hence

$$A_j \in (s_1, ..., s_p)^{[(n+1)/p] - 1} k[s_1, ..., s_p], \quad j = 0, ..., p$$

which ends the proof of the Lemma.

To check (4.3.5) above note that

$$\sum_{i=1}^{p} \delta(q_i^{n+1}) = \delta\left(\sum_{i=1}^{p} q_i^{n+1}\right) + \frac{\left(\sum_{i=1}^{p} q_i^{n+1}\right)^p - \sum_{i=1}^{p} q_i^{(n+1)p}}{p}.$$

The second term in the right hand side of the above equation is a homogeneous polynomial in $q_1, ..., q_p$ of degree $(n+1)p$ hence it is a weighted homogeneous polynomial in $s_1, ..., s_p$ of weight $(n+1)p$ where $s_1, ..., s_p$ are given weights $1, ..., p$ respectively. Hence this polynomial is a sum of monomials in $s_1, ..., s_p$ of degree $\geq n+1$. Similarly $\sum_{i=1}^{p} q_i^{n+1}$ is a sum of monomials in $s_1, ..., s_p$ of degree $\geq [(n+1)/p]$. This implies that $\delta(\sum_{i=1}^{p} q_i^{n+1})$ is a sum of monomials in $s_1, ..., s_p, s_1', ..., s_p'$ of degree $\geq [(n+1)/p]$ which proves (4.3.5). $\square$

Proof of Theorem 4.3.2. In view of Lemma 4.3.5 (which treats the case of monomials) we see that in order to prove that $\overline{f}$ in the statement of the Theorem is Laurent (respectively Taylor) $\delta-p$-symmetric it is enough to show that any series of the form

$$\overline{f}=\sum_{n=0}^{\infty}\overline{c}_n q^{pn} q' \in k[[q]][q']$$

is Taylor $\delta-p$-symmetric. By Lemma 4.3.7 we may write

$$\sum_{i=1}^{p} q_i^{np} q_i' = G_{0n} + \sum_{j=1}^{p} G_{jn} s_j'$$

where

$$G_{jn} \in (s_1,...,s_p)^{[(n+1)/p]-1} k[s_1,...,s_p], \quad j=0,...,p.$$

Since $G_j := \sum_{n=0}^{\infty} \overline{c}_n G_{jn}$ are convergent in $k[[s_1,...,s_p]]$ we have

$$\sum_{i=1}^{p} \overline{f}(q_i) = G_0 + \sum_{j=1}^{p} G_j s_j' \in k[[s_1,...,s_p]][s_1',...,s_p']$$

which proves that $\overline{f}$ is Taylor $\delta-p$-symmetric. The assertion about "pU"$\overline{f}$ follows from Lemma 4.1.4 by taking limits. $\square$

Next we proceed to proving Theorem 4.3.1. We need some preliminaries. Let $C_p(q_1,q_2) := \frac{q_1^p+q_2^p-(q_1+q_2)^p}{p} \in \mathbb{Z}[q_1,q_2]$. We start with a version of Lemma 4.3.6:

Lemma 4.3.8. *Consider the elements $\sigma = q_1+q_2 \in k[q_1,q_2]$ and $\pi = q_1q_2 \in k[q_1,q_2]$ and let $\gamma \in k[q_1,q_2]$ be the image of $C_p(q_1,q_2) \in \mathbb{Z}[q_1,q_2]$. Then*

$$q_1' = \frac{\pi' - q_1^p\sigma' + q_1^p\gamma}{(q_2-q_1)^p}, \quad q_2' = -\frac{\pi' - q_2^p\sigma' + q_2^p\gamma}{(q_2-q_1)^p}$$

in the ring

$$k[q_1,q_2,q_1',q_2',\frac{1}{q_2-q_1}].$$

Proof. Applying δ to the defining equations of σ and π we get

$$\begin{aligned} q_1'+q_2' &= \sigma'-\gamma \\ q_2^p q_1' + q_1^p q_2' &= \pi' \end{aligned}$$

and solve for q_1', q_2'. □

For the next Lemma let us denote by $v_{q_2-q_1} : k((q_1,q_2))^\times \to \mathbb{Z}$ the normalized valuation on the fraction field $k((q_1,q_2))$ of $k[[q_1,q_2]]$ attached to the irreducible series $q_2-q_1 \in k[[q_1,q_2]]$; in other words, if $0 \neq F(q_1,q_2) \in k[[q_1,q_2]]$ then $v_{q_2-q_1}(F)$ is the maximum integer i such that $(q_2-q_1)^i$ divides F in $k[[q_1,q_2]]$.

Lemma 4.3.9. *Let* $\Phi(q) = \sum_{m=0}^{\infty} \beta_m q^m \in k[[q]]$, $\Phi \notin k$, *Supp* $\Phi := \{m \in \mathbb{Z}_{\geq 0}; \beta_m \neq 0\}$. *Then*

$$v_{q_2-q_1}(\Phi(q_2)-\Phi(q_1)) = p^{\min\{v_p(m);0\neq m\in Supp\ \Phi\}}.$$

Proof. We have

$$\begin{aligned}\Phi(q_2)-\Phi(q_1) &= \textstyle\sum_{(n,p)=1}\sum_{i=0}^{\infty}\beta_{np^i}(q_2^{np^i}-q_1^{np^i}) \\ &= \textstyle\sum_{i=0}^{\infty}(q_2-q_1)^{p^i}G(q_1,q_2)\end{aligned}$$

where

$$G_i(q_1,q_2) = \sum_{(n,p)=1} \beta_{np^i}(q_2^{(n-1)p^i} + q_2^{(n-2)p^i}q_1^{p^i} + \ldots + q_1^{(n-1)p^i}).$$

Let $i_0 = \min\{v_p(m); 0 \neq m \in Supp\ \Phi\}$. Then $\beta_{np^i} = 0$ for all $(n,p)=1$ and $i < i_0$ and there exists n_0, $(n_0,p)=1$ such that $\beta_{n_0p^{i_0}} \neq 0$. It is enough to show that $G_{i_0}(q_1,q_2)$ is not divisible by q_2-q_1 in $k[[q_1,q_2]]$ equivalently that $G(q,q) \neq 0$. But

$$G_{i_0}(q,q) = \sum_{(n,p)=1} n\beta_{np^{i_0}}q^{(n-1)p^{i_0}} \neq 0.$$

□

Proof of Theorem 4.3.1. We proceed by induction on the degree $\deg(\overline{f})$ of $\overline{f}$ viewed as a polynomial in q' with coefficients in $k[[q]]$. If this degree is 0 we are done. Assume now the degree is ≥ 1. We may assume $\overline{f}(0,0)=0$.

By hypothesis,

$$\overline{f}(q_1, q_1') + ... + \overline{f}(q_p, q_p') = G$$

in $k[[q_1, ..., q_p]][q_1', ..., q_p']$, where $G \in k[[s_1, ..., s_p]][s_1', ..., s_p']$. Setting $q_3 = ... = q_p = 0$ and $q_3' = ... = q_p' = 0$ we get

$$\overline{f}(q_1, q_1') + \overline{f}(q_2, q_2') = G(\sigma, \pi, 0, ..., 0, \sigma', \pi', 0, ..., 0). \tag{4.3.6}$$

Note that $k[[q_1, q_2]]$ is a finite $k[[\sigma, \pi]]$-algebra so σ', π' are algebraically independent over $k((q_1, q_2))$. By Lemma 4.3.8 the left hand side of (4.3.6) is a polynomial H in σ', π' with coefficients in $k((q_1, q_2))$. On the other hand since H is in the right hand side of (4.3.6) H has coefficients in $k[[q_1, q_2]]$. Hence each non-zero coefficient of the polynomial H has $v_{q_2-q_1}$-adic valuation ≥ 0. Now write

$$\overline{f}(q, q') = \sum_{m'} \Phi_{m'}(q)(q')^{m'}, \quad \Phi_{m'} \in k[[q]].$$

Also write each m' as $m' = n'p^{i'}$ with n' not divisible by p. Using Lemma 4.3.8 we have $H = \sum_{m'} H_{m'}$ where

$$H_{m'} = \frac{F_{m'}}{(q_2 - q_1)^{n'p^{i'}+1}} \tag{4.3.7}$$

where $F_{m'} \in k((q_1, q_2))[\sigma', \pi']$ is given by

$$\begin{aligned} F_{m'} &= \Phi_{m'}(q_1)\left((\pi')^{p^{i'}} - q_1^{p^{i'}+1}(\sigma')^{p^{i'}} + q_1^{p^{i'}+1}\gamma^{p^{i'}}\right)^{n'} \\ &\quad +(-1)^{n'}\Phi_{m'}(q_2)\left((\pi')^{p^{i'}} - q_2^{p^{i'}+1}(\sigma')^{p^{i'}} + q_2^{p^{i'}+1}\gamma^{p^{i'}}\right)^{n'}. \end{aligned}$$

Note that the coefficient of $(\pi')^{m'}$ in $F_{m'}$ is

$$\Phi_{m'}(q_1) + (-1)^{n'}\Phi_{m'}(q_2) \tag{4.3.8}$$

while the coefficient of $(\pi')^{m'-p^{i'}}(\sigma')^{p^{i'}}$ in $F_{m'}$ is

$$-n'\left(q_1^{p^{i'}+1}\Phi_{m'}(q_1) + (-1)^{n'}q_2^{p^{i'}+1}\Phi_{m'}(q_2)\right). \tag{4.3.9}$$

Let now $m' = \deg(\overline{f})$. If n' is even the polynomial (4.3.8) has $v_{q_2-q_1}$-adic valuation 0 which contradicts the fact that the non-zero coefficients of H have $v_{q_2-q_1}$-adic valuation ≥ 0. So n' is odd. By Lemma 4.3.9 the $v_{q_2-q_1}$-adic valuation of (4.3.8) equals

$$p^{\min\{v_p(m);0\neq m\in Supp\ \Phi_{m'}\}}, \quad \text{if } \Phi_{m'} \notin k.$$

Also the $v_{q_2-q_1}$-adic valuation of (4.3.9) equals

$$p^{\min\{v_p(m);m\in Supp(q^{p^{i'+1}}\Phi_{m'})\}} = p^{\min\{v_p(m+p^{i'+1});m\in Supp\ \Phi_{m'}\}}.$$

By the fact that the non-zero coefficients of H have $v_{q_2-q_1}$-adic valuation ≥ 0 we get that

$$p^{\min\{v_p(m);0\neq m\in Supp\ \Phi_{m'}\}} \geq n'p^{i'+1} \quad \text{if } \ \Phi_{m'} \notin k \tag{4.3.10}$$

and

$$p^{\min\{v_p(m+p^{i'+1});m\in Supp\ \Phi_{m'}\}} \geq n'p^{i'+1}. \tag{4.3.11}$$

From (4.3.10) we get

$$v_p(m) \geq i'+1 \ \text{ for all } \ 0\neq m \in Supp\ \Phi_{m'}, \ \text{ if } \ \Phi_{m'} \notin k. \tag{4.3.12}$$

We claim now that $n' = 1$. Assume $n' \geq 2$. By (4.3.10)

$$v_p(m) > i'+1 \ \text{ for all } \ 0\neq m \in Supp\ \Phi_{m'}, \ \text{ if } \ \Phi_{m'} \notin k.$$

Hence

$$v_p(m+p^{i'+1}) = i'+1 \ \text{ for all } \ m \in Supp\ \Phi_{m'}.$$

By (4.3.11) $p^{i'+1} \geq 2p^{i'+1}$, a contradiction. This ends the proof that $n' = 1$.

By (4.3.12)

$$\Phi_{m'}(q)(q')^{m'} = (V^{i'+1}\varphi)(q')^{p^{i'}}$$

for some $\varphi \in k[[q]]$. By Lemma 4.3.5 $\Phi_{m'}(q)(q')^{m'}$ is Taylor $\delta - p$-symmetric hence so is $\overline{f} - \Phi_{m'}(q)(q')^{m'}$ which has smaller degree than $\overline{f}$. We conclude by the induction hypothesis. □

4.4 Multiplicity one

We begin by recalling the well known situation for series in $k[[q]]$. Then we proceed with our main results about δ-series in $k[[q]][q']$.

Throughout this section we fix $\kappa \in \mathbb{Z}_{\geq 0}$.

Definition 4.4.1. A series $\varphi \in qk[[q]]$ is said to be an *eigenvector of all Hecke operators* $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$ if $\varphi \neq 0$ and the following hold:

$$\begin{cases} T_{\kappa+2}(n)\varphi = \overline{\lambda}_n \cdot \varphi, \quad (n,p) = 1 \\ \\ T_{\kappa+2}(p)\varphi = \overline{\lambda}_p \cdot \varphi. \end{cases} \tag{4.4.1}$$

Of course the last equation in (4.4.1) is equivalent to

$$U\varphi = \overline{\lambda}_p \cdot \varphi.$$

Proposition 4.4.2. Assume $\varphi \in qk[[q]]$ is an eigenvector of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$ Then there exists $\gamma \in k^{\times}$ such that

$$\varphi(q) := \gamma \cdot \sum_{(n,p)=1} \sum_{i \geq 0} \overline{\lambda}_n \overline{\lambda}_p^i \cdot q^{np^i}. \tag{4.4.2}$$

Proof. Pick out coefficient of q in the first equation (4.4.1) and the coefficient of q^m, $m \geq 1$ in the second equation (4.4.1). (Here we use the convention that $0^0 = 1$.) □

Definition 4.4.3. A δ-series $\overline{f} = \overline{f}(q,q') \in k[[q]][q']$ is said to be an *eigenvector of all Hecke operators* $nT_\kappa(n)$, *"*$pT_\kappa(p)$*"*, $(n,p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$ if $\overline{f}$ is

Taylor $\delta - p$-symmetric and satisfies

$$\begin{cases} nT_\kappa(n)\overline{f} = \overline{\lambda}_n \cdot \overline{f}, \quad (n,p) = 1; \\ \\ \text{“}pT_\kappa(p)\text{”}\overline{f} = \overline{\lambda}_p \cdot \overline{f}. \end{cases} \tag{4.4.3}$$

Theorem 4.4.4. *Assume $\overline{f} = \overline{f}(q, q') \in k[[q]][q']$, $\overline{f} \notin k$, is an eigenvector of all Hecke operators $nT_\kappa(n)$, “$pT_\kappa(p)$”, $(n,p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$. Then there exists $\varphi = \varphi(q) \in qk[[q]]$ and $c, c_i \in k$, $i \geq 0$, with $\overline{p^\kappa} \cdot c_{i-1} = \overline{\lambda}_p \cdot c_i$ for $i \gg 0$, such that φ is an eigenvector of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p) = 1$, with the same eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p$ and such that*

$$\begin{aligned} \overline{f} &= c + \left(\textstyle\sum_{i\geq 0} c_i F^i_{/k}\right)\varphi^{\sharp,2}, \\ \varphi^{\sharp,2} &:= \varphi^{(-1)} - \overline{\lambda}_p \cdot V(\varphi)\tfrac{q'}{q^p} + \overline{p^\kappa} \cdot V^2(\varphi)\left(\tfrac{q'}{q^p}\right)^p. \end{aligned} \tag{4.4.4}$$

Remark 4.4.5. One can also write $\overline{f}$ in (4.4.4) as

$$\begin{aligned} \overline{f} &= c + \textstyle\sum_{i\geq 0} c_i \left[V^i(\varphi^{(-1)}) - \overline{\lambda}_p \cdot V^{i+1}(\varphi)\left(\tfrac{q'}{q^p}\right)^{p^i} + \overline{p^\kappa} \cdot V^{i+2}(\varphi)\left(\tfrac{q'}{q^p}\right)^{p^{i+1}}\right] \\ \\ &= c + \left(\textstyle\sum_{i\geq 0} c_i V^i\right)\varphi^{(-1)} + \textstyle\sum_{i\geq 0}(\overline{p^\kappa}c_{i-1} - \overline{\lambda}_p c_i)V^{i+1}(\varphi)\left(\tfrac{q'}{q^p}\right)^{p^i}, \end{aligned}$$

where $c_{-1} := 0$. Note that the condition that $\overline{p^\kappa} \cdot c_{i-1} = \overline{\lambda}_p \cdot c_i$ for $i \gg 0$ insures that the right hand side of the first equation in (4.4.4) is a polynomial in the variable q'.

Remark 4.4.6. Looking at the constant terms in (4.4.3) one sees that if $c \neq 0$ then

$$\begin{cases} \overline{\lambda}_n = n \cdot \sum_{A|n} \epsilon(A)A^{\kappa-1}, \quad (n,p) = 1; \\ \\ \overline{\lambda}_p = \overline{p^\kappa}. \end{cases} \tag{4.4.5}$$

Conversely we will prove:

Theorem 4.4.7. *Let $\kappa \in \mathbb{Z}_{\geq 0}$. Assume $\varphi = \varphi(q) \in qk[[q]]$ is an eigenvector of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p)=1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$. Let $c_i \in k$ for $i \geq 0$ with $\overline{p^\kappa} \cdot c_{i-1} = \overline{\lambda}_p \cdot c_i$ for $i \gg 0$. Also let c be an arbitrary element in k or 0 according as equations (4.4.5) hold or fail respectively. Let $\overline{f} \in k[[q]][q']$ be defined by Equation (4.4.4). Then $\overline{f}$ an eigenvector of all Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p)=1$, with the same eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p$.*

Let $k[F_{/k}]$ be the k-algebra generated by $F_{/k}$ which is a commutative polynomial ring in one variable. Note that $k[[q]][q']$ is a $k[F_{/k}]$-module and the k-linear space of series $\overline{f}(q,q') \in k[[q]][q']$ with $f(0,0)=0$ is a torsion free $k[F_{/k}]$-submodule. Note also that the ideal $qk[[q]]$ is a torsion free module over the ring $k[[F_{/k}]]$ of power series in $F_{/k}$. Finally recall that a δ-series $\overline{f}(q,q') \in k[[q]][q']$ is called *primitive* if $U(\overline{f}(q,0))=0$. Theorems 4.4.4 and 4.4.7 immediately imply:

Corollary 4.4.8. Fix $\overline{\lambda}_n \in k$ for $(n,p)=1$ and $\overline{\lambda}_p \in k$. Let $\mathcal{F}$ be the k-linear space of all the δ-series $\overline{f} = \overline{f}(q,q') \in k[[q]][q']$ with $f(0,0)=0$ which are either 0 or are eigenvectors of all Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p)=1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$. We have $\mathcal{F} \neq 0$ if and only if there exists an eigenvector $\varphi \in qk[[q]]$ of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n,p)=1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p$. Assume furthermore that this is the case and let $\varphi^{\sharp,2}$ be defined as in (4.4.4). Then $\varphi^{\sharp,2}$ belongs to $\mathcal{F}$ and is a primitive δ-series; also any primitive δ-series in $\mathcal{F}$ is a k-multiple of $\varphi^{\sharp,2}$. Furthemore the following hold:

1) If $\kappa > 0$, $\overline{\lambda}_p = 0$ then $\mathcal{F}$ is a free $k[[F_{/k}]]$-submodule of $k[[q]]$ of rank 1 with basis $\varphi^{\sharp,2} = \varphi^{(-1)}$.

2) If either $\kappa > 0$, $\overline{\lambda}_p \neq 0$ or $\kappa = 0$, $\overline{\lambda}_p = 0$ then $\mathcal{F}$ is a free $k[F_{/k}]$-submodule of $k[[q]][q']$ of rank one with basis $\varphi^{\sharp,2}$.

3) If $\kappa = 0$, $\overline{\lambda}_p \neq 0$ then $\mathcal{F}$ is a free $k[F_{/k}]$-submodule of $k[[q]][q']$ of rank 1 with

basis

$$\varphi^{\sharp,1} := \left(\sum_{i\geq 0} (\overline{\lambda}_p)^{-i} F^i_{/k}\right) \varphi^{\sharp,2}. \tag{4.4.6}$$

Remark 4.4.9. Note that

$$\varphi^{\sharp,1} = \left(\sum_{i\geq 0} (\overline{\lambda}_p)^{-i} V^i\right) \varphi^{(-1)} - \overline{\lambda}_p \cdot V(\varphi) \cdot \frac{q'}{q^p}$$

and also that $\varphi^{\sharp,1}$ is the unique element of $qk[[q]]$ satisfying the equation

$$V(\varphi^{\sharp,1}) - \overline{\lambda}_p \varphi^{\sharp,1} + \overline{\lambda}_p \varphi^{\sharp,2} = 0.$$

Proof of Theorem 4.4.4. For any series $\beta(q) \in k[[q]]$ write

$$\beta(q) = \sum_{m\geq 0} a_m(\beta) q^m.$$

By Theorem 4.3.1 and Corollaries 4.1.5 and 4.3.3 $\overline{f}$ has the form (4.3.2) and

$$\begin{aligned}
T_\kappa(n)\varphi_0 &= \tfrac{\overline{\lambda}_n}{n} \cdot \varphi_0, \quad (n,p) = 1 \\
T_{\kappa+2p^s}(n)\varphi_{p^s} &= \overline{\lambda}_n \cdot \varphi_{p^s}, \quad (n,p) = 1,\ s \geq 0 \\
U(\varphi_{p^s}) &= \overline{\lambda}_p \cdot \varphi_{p^s}, \quad s \geq 0 \\
\overline{p^\kappa} \cdot V(\varphi_0) - \textstyle\sum_{s\geq 0} V^s(\varphi_{p^s}^{(-1)}) &= \overline{\lambda}_p \cdot \varphi_0.
\end{aligned} \tag{4.4.7}$$

In particular the following equalities hold:

$$
\begin{aligned}
a_{np^s}(\varphi_0) &= \tfrac{\overline{\lambda}_n}{n} \cdot a_{p^s}(\varphi_0), \quad (n,p)=1, s \geq 0, \\
a_n(\varphi_{p^s}) &= \overline{\lambda}_n \cdot a_1(\varphi_{p^s}), \quad (n,p)=1, s \geq 0, \\
a_{mp}(\varphi_{p^s}) &= \overline{\lambda}_p \cdot a_m(\varphi_{p^s}), \quad m \geq 1, s \geq 0, \\
\overline{p^\kappa} \cdot a_{p^{s-1}}(\varphi_0) - a_1(\varphi_{p^s}) &= \overline{\lambda}_p \cdot a_{p^s}(\varphi_0), \quad s \geq 0,
\end{aligned}
\tag{4.4.8}
$$

where by convention we set $a_{p^{s-1}}(\varphi_0) = 0$ if $s = 0$. Let $c = a_0(\varphi_0)$ and $c_i = a_{p^i}(\varphi_0)$ for $i \geq 0$. By (4.4.8) we get

$$
\begin{aligned}
a_{np^i}(\varphi_0) &= \tfrac{\overline{\lambda}_n}{n} \cdot c_i, && (n,p)=1, \quad i \geq 0 \\
a_{np^i}(\varphi_{p^s}) &= \overline{\lambda}_n \overline{\lambda}_p^i \cdot (\overline{p^\kappa} \cdot c_{s-1} - \overline{\lambda}_p c_s), && (n,p)=1, \quad i \geq 0, \ s \geq 0,
\end{aligned}
\tag{4.4.9}
$$

where $c_{-1} := 0$. Define φ by the equality (4.4.2) with $\gamma = 1$.

Assume first that there is an $s \geq 0$ such that $a_1(\varphi_{p^s}) \neq 0$. Then φ_{p^s} is a non-zero multiple of φ so (4.4.1) follows from (4.4.7) and (4.4.4) follows from (4.4.9). Since $\overline{f}$ is a polynomial in q' we get that $\overline{p^\kappa} \cdot c_{s-1} - \overline{\lambda}_p c_s = 0$ for $s \gg 0$.

Assume now that $a_1(\varphi_{p^s}) = 0$ for all $s \geq 0$. Then $\varphi_{p^s} = 0$ for all $s \geq 0$ hence $\overline{f} = \varphi_0$. By the last equation in (4.4.7) and since $\varphi_0 \notin k$ we get $\overline{p^\kappa} = \overline{\lambda}_p = 0$. Then the right hand side of (4.4.4) becomes

$$
c + \sum_{i \geq 0} \sum_{(n,p)=1} c_i \frac{\overline{\lambda}_n}{n} q^{np^i}. \tag{4.4.10}
$$

By the first equation in (4.4.9) we get that (4.4.10) equals $\varphi_0 = \overline{f}$; so equation (4.4.4) holds. Clearly $U\varphi = 0$ so the second equality in (4.4.1) holds. Finally, since $\varphi_0 \notin k$ we may write $\varphi_0 = F^d_{/k} \tilde{\varphi}_0$ with $\tilde{\varphi}_0 \in k[[q]]$ and d maximal with this property; in particular $c_d \neq 0$. Note that $\theta\tilde{\varphi}_0 = c_d\varphi$. Also by (4.4.7) we have $T_\kappa(n)\tilde{\varphi}_0 = \frac{\overline{\lambda}_n}{n}\tilde{\varphi}_0$ for $(n,p) = 1$. Hence

$$T_{\kappa+2}(n)\varphi = c_d^{-1}T_{\kappa+2}(n)\theta\tilde{\varphi}_0 = c_d^{-1}n\theta(T_\kappa(n)\tilde{\varphi}_0) = c_d^{-1}\overline{\lambda}_n\theta\tilde{\varphi}_0 = \overline{\lambda}_n\varphi$$

and so the first equality in (4.4.1) holds. This ends the proof. □

Proof of Theorem 4.4.7. This follows directly from Corollary 4.1.3 and Theorem 4.3.2 using the following facts (which are direct consequences of the formulae for the Hecke operators acting on Fourier coefficients (4.1.4)):

$$\begin{aligned} T_{\kappa+2p^i}(n)\varphi &= \overline{\lambda}_n \cdot \varphi, \quad (n,p) = 1, \quad i \geq 0 \\ T_\kappa(n)(\varphi^{(-1)}) &= \tfrac{\overline{\lambda}_n}{n} \cdot \varphi, \quad (n,p) = 1. \end{aligned}$$

□

4.5 δ-modular forms

4.5.1 Review of classical modular forms

Start by recalling some basic facts about modular forms; cf. [17]. Let $N > 4$ be an integer and let B be a $\mathbb{Z}[1/N, \zeta_N]$-algebra. Let $Y = Y_1(N)$ be the affine modular curve over B classifying pairs (E, α) consisting of elliptic curves E over B-algebras plus a level $\Gamma_1(N)$ structure $\alpha : \mathbb{Z}/N\mathbb{Z} \to E$. Let Y_{ord} be the ordinary locus in Y (i.e. the locus where the Eisenstein form E_{p-1} is invertible). Let X be Y or Y_{ord}. Let L be the line bundle on X, direct image of the sheaf of relative differentials on the universal elliptic curve over X, and let

$$V = Spec\left(\bigoplus_{\kappa\in\mathbb{Z}} L^{\otimes\kappa}\right) \to X \tag{4.5.1}$$

be the $\mathbb{G}_m$-torsor associated to L.

Set $M = \mathcal{O}(V) = \bigoplus_{\kappa\in\mathbb{Z}} L^{\otimes\kappa}$. Recall that there is a Fourier expansion map

$$E : M \to B((q))$$

defined by the cusp $\Gamma_1(N)\cdot\infty$ [17], p. 112. Recall also that Y has a natural compactification, $X_1(N)$, equipped with a natural line bundle, still denoted by L, extending the line bundle L on Y, such that the space of classical modular forms, $M(\Gamma_1(N), B, \kappa) \subset L^{\otimes\kappa}$, on $\Gamma_1(N)$ of weight κ, defined over B identifies with $H^0(X_1(N), L^{\otimes\kappa})$. Recall that the diamond operators act on $M(\Gamma_1(N), B, \kappa)$; the invariant elements form the space $M(\Gamma_0(N), B, \kappa)$ of classical modular forms on $\Gamma_0(N)$ of weight κ defined over B. Recall the q-expansion principle: for any B as above there is an induced injective Fourier expansion map $E : M(\Gamma_1(N), B, \kappa) \to B[[q]]$ and if $B' \subset B$ then $M(\Gamma_1(N), B', \kappa)$ identifies with the group of all $f \in M(\Gamma_1(N), B, \kappa)$ such that $E(f) \in B'[[q]]$. Recall also the following base change property: if B' is any B-algebra and either B' is flat over B or $\kappa \geq 2$ and N is invertible in B' then the map $M(\Gamma_1(N), B, \kappa) \otimes_B B' \to M(\Gamma_1(N), B', \kappa)$ is an isomorphism; cf. [17], p.111.

4.5.2 δ-series from classical modular forms

Theorem 4.5.1. *Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $f(q) = \sum_{m\geq 1} a_m q^m \in q\mathbb{Z}_p[[q]]$ be a series satisfying $a_1 = 1$ and*

$$\begin{cases} a_{p^i n} = a_{p^i} a_n \quad \text{for } (n,p) = 1,\ i \geq 0 \\ a_{p^{i-1}} a_p = a_{p^i} + p^{\kappa+1} a_{p^{i-1}} \quad \text{for } i \geq 2. \end{cases} \tag{4.5.2}$$

Let $\varphi := \overline{f} = \sum_{m\geq 1} \overline{a}_m q^m \in q\mathbb{F}_p[[q]]$ be the reduction mod p of $f(q)$. Then the series

$$f^{\sharp,2} = f^{\sharp,2}(q, q', q'') := \frac{1}{p} \cdot \sum_{n\geq 1} \frac{a_n}{n} (p^\kappa \phi^2(q)^n - a_p \phi(q)^n + pq^n) \in \mathbb{Q}_p[[q, q', q'']] \tag{4.5.3}$$

belongs to $\mathbb{Z}_p[[q]][q', q'']^\wedge$ *and its reduction mod* p *equals*

$$\overline{f^{\sharp,2}} = \overline{f^{\sharp,2}(q,q',q'')} = \varphi^{(-1)} - \overline{a}_p V(\varphi)\frac{q'}{q^p} + \overline{p^\kappa} \cdot V^2(\varphi)\left(\frac{q'}{q^p}\right)^p \in \mathbb{F}_p[[q]][q']. \quad (4.5.4)$$

Proof. For $\kappa = 0$ the argument is in [13]; the case $\kappa > 0$ is entirely similar. (Note that the form $f^{(0)}_{[a_p]}$ in [13] is congruent mod p to f itself.) □

Remark 4.5.2. Note that conditions (4.5.2) imply that $U\varphi = \overline{a}_p \cdot \varphi$.

Example 4.5.3. Let $\kappa \in \mathbb{Z}_{\geq 0}$ and let $F \subset \mathbb{C}$ be a number field with ring of integers $\mathcal{O}_F$. Let

$$f(q) = \sum_{m\geq 1} a_m q^m \in q\mathcal{O}_F[[q]] \quad (4.5.5)$$

be the Fourier expansion of a cusp form

$$f \in M(\Gamma_0(N), \mathcal{O}_F, \kappa + 2).$$

Assume $a_1 = 1$ and assume $f(q)$ is an eigenvector for all the Hecke operators $T_{\kappa+2}(n)$ with $n \geq 1$. Assume p is a rational prime that splits completely in F, consider an embedding $\mathcal{O}_F \subset \mathbb{Z}_p$, view $f(q)$ as an element of $q\mathbb{Z}_p[[q]]$, and let $\varphi := \overline{f} = \sum_{m\geq 1} \overline{a}_m q^m \in q\mathbb{F}_p[[q]]$ is the reduction mod p of $f(q)$. Then the equalities (4.5.2) hold. So by Theorem 4.5.1 the series

$$f^{\sharp,2} = f^{\sharp,2}(q,q',q'') := \frac{1}{p} \cdot \sum_{n\geq 1} \frac{a_n}{n}(p^\kappa \phi^2(q)^n - a_p \phi(q)^n + pq^n) \in \mathbb{Q}_p[[q,q',q'']] \quad (4.5.6)$$

belongs to $\mathbb{Z}_p[[q]][q', q'']^\wedge$ and its reduction mod p equals

$$\overline{f^{\sharp,2}} := \overline{f^{\sharp,2}(q,q',q'')} = \varphi^{(-1)} - \overline{a}_p V(\varphi)\frac{q'}{q^p} + \overline{p^\kappa} \cdot V^2(\varphi)\left(\frac{q'}{q^p}\right)^p \in \mathbb{F}_p[[q]][q']. \quad (4.5.7)$$

Note also that $T_{\kappa+2}(n)\varphi = \overline{a}_n \cdot \varphi$ for $(n,p) = 1$ and $U\varphi = \overline{a}_p \cdot \varphi$. So by Theorem 4.4.7 $\overline{f^{\sharp,2}} = \varphi^{\sharp,2}$ is an eigenvector of the Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p) = 1$, with eigenvalues $\overline{a}_n, \overline{a}_p$. Also, by the same Theorem, if in addition $\overline{a}_p \neq 0$ and $\kappa = 0$, then the series $\varphi^{\sharp,1}$ in (4.4.6) is also an eigenvector of the Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p) = 1$, with eigenvalues $\overline{a}_n, \overline{a}_p$.

Example 4.5.4. Consider the Ramanujan series

$$P(q) := E_2(q) := 1 - 24\sum_{m\geq 1}\left(\sum_{d|m} d\right) q^m$$

and assume N is prime. Consider the series

$$g(q) := -\frac{1}{24}(P(q) - NP(q^N)) = \frac{N-1}{24} + f(q) \in \mathbb{Z}_{(p)}[[q]],$$

where

$$f(q) = \sum_{m\geq 1}\left(\sum_{A|m}\epsilon(A)A\right) q^m. \tag{4.5.8}$$

Then $g(q)$ is the Fourier expansion of a classical modular form in $M(\Gamma_0(N), \mathbb{Z}_{(p)}, 2)$ which is an eigenvector of the Hecke operators $T_2(n)$ for all $n \geq 1$ with eigenvalues $a_n := \sum_{A|n}\epsilon(A)A$; cf. [17], Example 2.2.6, Proposition 3.5.1, and Remark 3.5.2. Let $\varphi := \overline{f} = \sum_{m\geq 1}\overline{a}_m q^m \in q\mathbb{F}_p[[q]]$ be the reduction mod p of $f(q)$. By [24], Theorem 9.17, the equalities (4.5.2) hold with $\kappa = 0$. So by Theorem 4.5.1 the series

$$f^{\sharp,2} = f^{\sharp,2}(q,q',q'') := \frac{1}{p}\cdot\sum_{n\geq 1}\frac{a_n}{n}(\phi^2(q)^n - a_p\phi(q)^n + pq^n) \in \mathbb{Q}_p[[q,q',q'']] \tag{4.5.9}$$

belongs to $\mathbb{Z}_p[[q]][q',q'']\hat{}$ and its reduction mod p equals

$$\overline{f^{\sharp,2}} := \overline{f^{\sharp,2}(q,q',q'')} = \varphi^{(-1)} - \overline{a}_p V(\varphi)\frac{q'}{q^p} + V^2(\varphi)\left(\frac{q'}{q^p}\right)^p \in \mathbb{F}_p[[q]][q']. \tag{4.5.10}$$

Note also that $T_2(n)\varphi = \overline{a}_n\cdot\varphi$ for $(n,p) = 1$ and $U\varphi = \overline{a}_p\cdot\varphi$. So by Theorem 4.4.7 $\overline{f^{\sharp,2}} = \varphi^{\sharp,2}$ is an eigenvector of the Hecke operators $nT_0(n)$, "$pT_0(p)$", $(n,p) = 1$, with eigenvalues $\overline{a}_n, \overline{a}_p$. Also, by the same Theorem, if in addition $\overline{a}_p \neq 0$ and $\kappa = 0$, then the series $\varphi^{\sharp,1}$ in (4.4.6) is also an eigenvector of the Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n,p) = 1$, with eigenvalues $\overline{a}_n, \overline{a}_p$. Note that if $N \equiv 1$ mod p then

Equations 4.4.5 hold because

$$\begin{cases} a_n = \sum_{A|n} \epsilon(A)A \equiv n \sum_{A|n} \epsilon(A)A^{-1}, \quad \text{mod } p \text{ for } (n,p)=1, \\ \\ a_p = \sum_{A|p} \epsilon(A)A \equiv 1 \mod p. \end{cases}$$

Note also that if $N \equiv 1$ mod p it follows that $f(q) \equiv g(q)$ mod p so $\varphi(q)$ is the Fourier expansion of a modular form in $M(\Gamma_0(N), \mathbb{F}_p, 2)$

4.5.3 Application to δ-eigenforms

As noted in [15] the image of the Fourier expansion map $M^\infty \to R((q))\hat{}$ is contained in $\mathcal{W}$; this is by the universality property of $\mathcal{O}^r(V)$ and by the fact that $\mathcal{W}$ possesses a lift of Frobenius ϕ_0 and hence it is naturally a δ-subring of $R((q))\hat{}$.

Proposition 4.5.5. The image of $M^r(w)$ in $\mathcal{W}$ consists of elements of weight $deg(w)$.

Proof. It is easy to see that one may replace X in the statement above by an open set of it. So one may assume L is free on X. Let x be a basis of L. Then any element $f \in M^r(w)$ can be written as $f = f_0 \cdot x^w$ where $f_0 \in \mathcal{O}^r(X)$. Now the image of x in $\mathcal{W}$ has weight 1. Since ϕ_0 on $\mathcal{W}$ preserves the elements of a given weight it follows that the image of x^w in $\mathcal{W}$ has weight $deg(w)$. On the other hand f_0 is a p-adic limit of polynomials with R-coefficients in elements of the form $\delta^i g_0$, where $g_0 \in \mathcal{O}(X)$. Again, since ϕ_0 sends elements of weight 0 in $\mathcal{W}$ into elements of weight 0 the same is true for $\delta : \mathcal{W} \to \mathcal{W}$. Since the image of g_0 in $\mathcal{W}$ has weight 0 so does the image of $\delta^i g_0$ in $\mathcal{W}$ and hence so does the image of f. □

Next we state our main applications to "δ-eigenforms" (i.e. δ-modular forms whose δ-Fourier expansions are "δ-eigenseries"). First we will prove:

Theorem 4.5.6. *Assume $\overline{f} = \overline{f}(q, q') \in k[[q]][q']$ is not a p-th power in $k[[q]][q']$ and assume $\overline{f}$ is the reduction mod p of the δ-Fourier expansion of a δ-modular form in $M^r(w)$ with $r \geq 0$, $\kappa := deg(w) \geq 0$. Assume furthermore that $\overline{f}$ is an eigenvector of all Hecke operators $nT_\kappa(n)$, "$pT_\kappa(p)$", $(n, p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$. Then there exists $\varphi = \varphi(q) \in qk[[q]]$ which is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa')$, $\kappa' \geq 0$, $\kappa' \equiv \kappa + 2$ mod $p - 1$, and there exist $c, c_i \in k$, $i \geq 0$, with $\overline{p^\kappa} \cdot c_{i-1} = \overline{\lambda}_p c_i$ for $i \gg 0$, such that φ is an eigenvector of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n, p) = 1$, with the same eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p$ and such that $\overline{f}$ satisfies (*4.4.4*).*

Conversely we will prove:

Theorem 4.5.7. *Assume $\varphi \in qk[[q]]$ is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa')$, $\kappa' \geq 0$, $\kappa' \equiv \kappa + 2$ mod $p - 1$, and that φ is an eigenvector of all Hecke operators $T_{\kappa+2}(n)$, $T_{\kappa+2}(p)$, $(n, p) = 1$, with eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p \in k$. Assume $X = Y_{ord}$. Consider the series $\overline{f} = \overline{f}(q, q') \in k[[q]][q']$ defined by the formula (*4.4.4*) with $c = 0, c_i \in k$ for $i \geq 0$, and $c_i = 0$ for $i \gg 0$. Then $\overline{f}$ is the δ-Fourier expansion of a δ-modular form $f \in M^1(\kappa)$ and (by Theorem *4.4.7*) is an eigenvector of all Hecke operators $nT_\kappa(n)$, "$T_\kappa(p)$", $(n, p) = 1$, with the same eigenvalues $\overline{\lambda}_n, \overline{\lambda}_p$.*

Note that Theorems 4.5.6 and 4.5.7 imply Theorem 1.0.10 in the Introduction. The one-to-one correspondence in Theorem 1.0.10 is given by $\varphi \mapsto \varphi^{\sharp,2}$ with $\varphi^{\sharp,2}$ defined by (4.4.4).

4.5.4 Review of δ-modular forms [11, 15]

Let V be an affine smooth scheme over R and fix a closed embedding $V \subset \mathbb{A}^m$ into an affine space over R.

Definition 4.5.8. A map $f : V(R) \to R$ is called a *δ-function of order r on X* [7] if there exists a restricted power series Φ in $m(r+1)$ variables, with R-coefficients such that

$$f(a) = \Phi(a, \delta a, ..., \delta^r a),$$

for all $a \in V(R) \subset R^m$. We denote by $\mathcal{O}^r(V)$ the ring of δ-functions of order r on V.

(Recall that *restricted* means *with coefficients converging p-adically to* 0; also the definition above does not depend on the embedding $V \subset \mathbb{A}^m$.) Composition with δ defines p-derivations $\delta : \mathcal{O}^r(V) \to \mathcal{O}^{r+1}(V)$. The rings $\mathcal{O}^r(V)$ have the following universality property: for any R-algebra homomorphism $u : \mathcal{O}(V) \to A$ where A is a p-adically complete δ-ring there are unique R-algebra maps $u^r : \mathcal{O}^r(V) \to A$ that commute in the obvious sense with δ and prolong u.

Let now V be as in (4.5.1) with $B = R$ and $\mathbb{Z}[1/N, \zeta_N] \subset R$ a fixed embedding.

Definition 4.5.9. [15] A *$\delta-$modular function* of order r (on $\Gamma_1(N)$, holomorphic on X) is a δ-function $f : V(R) \to R$ of order r.

Let $W := \mathbb{Z}[\phi]$ be the ring generated by ϕ. For $w = \sum a_i \phi^i \in W$ ($a_i \in \mathbb{Z}$) set *deg*$(w) = \sum a_i \in \mathbb{Z}$; for $\lambda \in R^\times$ we set $\lambda^w := \prod \phi^i(\lambda)^{a_i}$.

Definition 4.5.10. A *δ-modular form of weight w* (of order r, on $\Gamma_1(N)$, holomorphic on X) is a δ-modular function $f : V(R) \to R$ of order r such that

$$f(\lambda \cdot a) = \lambda^w f(a),$$

for all $\lambda \in R^\times$ and $a \in V(R)$, where $(\lambda, a) \mapsto \lambda \cdot a$ is the natural action $R^\times \times V(R) \to V(R)$.

We denote by $M^r := \mathcal{O}^r(V)$ the ring of all δ-modular functions of order r and we set $M^\infty := \bigcup_{r \geq 0} M^r$. We denote by $M^r(w)$ the R-module of δ-modular forms of order

r and weight w; cf. [15]. (In [11] the space $M^r(w)$ was denoted by $M^r(\Gamma_1(N), R, w)$ or $M^r_{ord}(\Gamma_1(N), R, w)$ according as X is Y or Y_{ord}.) Note that $M^r(0)$ identifies with $\mathcal{O}^r(X)$ which, in its turn, embeds into $\mathcal{O}^r(X_1(N))$.

By the universality property of the rings $M^r = \mathcal{O}^r(V)$ there exists a unique δ-ring homomorphism (the *δ-Fourier expansion map*)

$$E : M^\infty \to S^\infty_{for} := \bigcup_{r\geq 0} R((q))[q', ..., q^{(n)}]\hat{}, \quad E(f) = f(q, q', q'', ...),$$

extending the Fourier expansion map $E : M \to R((q))\hat{}$. We may also consider the composition

$$M^\infty \to S^\infty_{for} \xrightarrow{\pi} R((q))\hat{}, \quad f \mapsto f(q),$$

where the map π sends $q', q'', ...$ into 0; we refer to this composition as the *Fourier expansion map.*

Recall the "δ-expansion principle":

Proposition 4.5.11. [11] The maps $E : M^r(w) \to R((q))[q', ..., q^{(r)}]\hat{}$ are injective with torsion free cokernel; hence the induced maps $\overline{E} : M^r(w) \otimes k \to k((q))[q', ..., q^{(r)}]$ are injective.

Proof. This is [11], Lemma 6.1. □

Recall also the following result:

Theorem 4.5.12. *[11] If in Example* 4.5.3 $\kappa = 0$*,* $F = \mathbb{Q}$*, and* $p \gg 0$ *then the series* $f^{\sharp,2}(q, q', q'') \in R[[q]][q', q'']\hat{}$ *in (*4.5.6*) is the image of a (unique) δ-modular form (still denoted by)* $f^{\sharp,2} \in \mathcal{O}^2(X_1(N)) \subset M^2(0)$*. If in addition f in Example* 4.5.3 *is of "CL type" then the series* $\varphi^{\sharp,1} \in k[[q]][q']$ *in that Example is the image of a δ-modular form* $f^{\sharp,1} \in \mathcal{O}^1(X_1(N)) \subset M^1(0)$*.*

Here by f being of *CL type* we mean that the Neron model of the elliptic curve over $\mathbb{Q}$ associated to f via the Eichler-Shimura construction has good ordinary reduction and its base change to R is the canonical lift of this reduction; cf. [11, 13] for more details.

Proof. Let $f^\sharp \in \mathcal{O}^r(X_1(N))$ be as in [11], Theorems 6.3 and 6.5; cf. also [13], Lemma 4.18. So r is 1 or 2 according as f is or is not of CL type. Then Theorem 4.5.12 follows from [11], Theorems 6.3 and 6.5, by letting the δ-modular form $f^{\sharp,2}$ be defined by

$$f^{\sharp,2} := \begin{cases} f^\sharp, & \text{if } f \text{ is not of CL type,} \\ \phi(f^\sharp) - a_p f^\sharp, & \text{if } f \text{ is of CL type,} \end{cases}$$

and by letting

$$f^{\sharp,1} := f^\sharp \quad \text{if } f \text{ is of CL type.}$$

□

Remark 4.5.13. It is tempting to conjecture that if in Example 4.5.3 $\kappa \geq 0$ is arbitrary, $F = \mathbb{Q}$, and $p \gg 0$ then the series $f^{\sharp,2}(q, q', q'')$ is the δ-Fourier expansion of a δ-modular form $f^{\sharp,2} \in M^r(\kappa)$ for some $r \geq 2$. An appropriate variant of this should also hold for arbitrary F. As we shall see, however, the situation is drastically different with Example 4.5.4; cf. Theorem 4.5.17.

Recall the *Serre derivation operator* $\partial : M \to M$ introduced by Serre and Katz [22]. (Cf. also [10], p.254 for a review). Recall that $\partial(L^{\otimes n}) \subset L^{\otimes(n+2)}$. Recall also that if X is contained in Y_{ord} then one has the Ramanujan form $P \in M^0(2)$. By [10], Propositions 3.43, 3.45, 3.56, there exists a unique sequence of R-derivations $\partial_j : M^\infty \to M^\infty$, $j \geq 0$, such that

$$\begin{cases} \partial_j \circ \phi^s = 0 \quad \text{on } M \text{ for } j \neq s \\ \partial_j \circ \phi^j = p^j \cdot \phi^j \circ \partial \quad \text{on } M \text{ for } j \geq 0 \end{cases} \tag{4.5.11}$$

These derivations then also have the property that

$$\begin{cases} \partial_j = 0 \quad \text{on } M^{j-1} \text{ for } j \geq 1 \\ \partial_j \circ \delta^j = \phi^j \circ \partial \quad \text{on } M \text{ for } j \geq 0 \end{cases} \tag{4.5.12}$$

and that

$$\partial_j(M^r(w)) \subset M^r(w+2\phi^j). \tag{4.5.13}$$

Recall the Ramanujan theta operator $\theta = q\frac{d}{dq} : R((q)) \to R((q))$. Then by [10], Lemma 4.18, there is a unique sequence of R-derivations $\theta_j : S^{\infty}_{for} \to S^{\infty}_{for}$ such that

$$\begin{cases} \theta_j \circ \phi^s = 0 \ \text{ on } R((q)) \text{ for } j \neq s \\ \theta_j \circ \phi^j = p^j \cdot \phi^j \circ \theta \ \text{ on } R((q)) \text{ for } j \geq 0; \end{cases} \tag{4.5.14}$$

and such that

$$\begin{cases} \theta_j = 0 \ \text{ on } R((q))[q', ..., q^{(j-1)}]\hat{} \text{ for } j \geq 1 \\ \theta_j \circ \delta^j = \phi^j \circ \theta \ \text{ on } R((q)) \text{ for } j \geq 0. \end{cases} \tag{4.5.15}$$

Proposition 4.5.14. For any $w = \sum_{i=0}^{r} a_i \phi^i \in W$, any $j \geq 0$, and any $f \in M^r(w)$ the following formula holds in S^{∞}_{for}:

$$E(\partial_j f) = \theta_j(E(f)) - a_j p^j E(f) E(P)^{\phi^j}.$$

Proof. This was proved in [10], Proposition 8.42 in the case of "δ-Serre-Tate expansions"; the case of δ-Fourier expansions is entirely similar. (The level 1 case of this Proposition was proved in [1] using the structure of the ring of modular forms of level 1.) □

Proof of Theorem 4.5.6. By Theorem 4.4.4 all we have to show is that φ in that Theorem is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa')$, $\kappa' \equiv \kappa + 2$ mod $p-1$. Since $\overline{f}$ is not a p-th power we may assume $c_0 = 1$. Now if $\overline{f}(q, q')$ is the reduction mod p of the δ-Fourier expansion

$$E(f) = f(q, q', ..., q^{(r)}) \in S^{\infty}_{for}$$

of a δ-modular form $f \in M^r(w)$ then, by Proposition 4.5.14, and Equations 4.4.4 and 4.5.15 we have the following congruences mod p in S^{∞}_{for}:

$$\begin{aligned} E(\partial_1 f) &\equiv \theta_1(E(f)) \\ &\equiv -\overline{\lambda}_p V(\varphi) q^{-p} \theta_1(\delta q) \\ &\equiv -\overline{\lambda}_p V(\varphi) q^{-p} \phi(\theta q) \\ &\equiv -\overline{\lambda}_p V(\varphi). \end{aligned}$$

By Equation (4.5.13) we have that $\partial_1 f \in M^r(w + 2\phi)$. So by Proposition 4.5.5 the image $E(\partial_1 f)(q, 0, ..., 0)$ of $E(\partial_1 f)$ in $R((q))\hat{}$ is an element of weight $\kappa + 2$ in $\mathcal{W}$. So $E(\partial_1 f)(q, 0, ..., 0)$ is congruent mod p to the Fourier expansion of a classical modular form of weight $\kappa' \equiv \kappa + 2$ mod $p - 1$. So $\overline{\lambda}_p V(\varphi)$ is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa')$. If $\overline{\lambda}_p \neq 0$ then $V(\varphi)$ is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa')$ hence so is $\varphi = UV\varphi$ (because U preserves the weight [20], p.458). If $\overline{\lambda}_p = 0$ then, by (4.4.2) we have $\varphi = \sum_{(n,p)=1} \overline{\lambda}_n q^n$ so $\varphi = \theta(\varphi^{(-1)}) = \theta(\varphi_0)$. Now φ_0 is the image of $E(f)$ in $k[[q]]$ so, as above, by Proposition 4.5.5, φ_0 is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa'')$ where $\kappa'' \equiv \kappa$ mod $p-1$. But θ sends Fourier expansions of modular forms of weight κ'' into Fourier expansions of modular forms of weight $\kappa'' + p + 1$; cf. [20], p. 458. So φ is the Fourier expansion of a modular form in $M(\Gamma_1(N), k, \kappa'' + p + 1)$, and we are done because $\kappa'' + p + 1 \equiv \kappa + 2$ mod $p - 1$. □

Proof of Theorem 4.5.7. Set $\kappa' = \kappa + 2 + (p-1)\nu$, $\nu \geq 0$. Since $\varphi^{(-1)}(q) = \theta^{p-2}\varphi(q)$ by get that $\varphi^{(-1)}(q)$ is the Fourier expansion of a modular form over k of weight $\kappa' + (p-2)(p+1) = \kappa + (p-1)(p+\nu)$ hence $V^i(\varphi^{(-1)}(q))$ is the Fourier expansion of a modular form over k of weight $\kappa_{0,i} := p^i(\kappa + (p-1)(p+\nu))$; the latter lifts to a modular form $\Phi_{0,i} \in M(\Gamma_1(N), R, \kappa_{0,i})$ which can be viewed as an element in $M^0(\kappa_{0,i})$. Also $V^{i+1}(\varphi)$ and $V^{i+2}(\varphi)$ are Fourier expansions of modular forms over k of weights $\kappa_{1,i} := p^{i+1}\kappa'$ and $\kappa_{2,i} := p^{i+2}\kappa'$ so they lift to modular forms $\Phi_{i,1} \in M(\Gamma_1(N), R, \kappa_{1,i})$ and $\Phi_{2,i} \in M(\Gamma_1(N), R, \kappa_{2,i})$ respectively. The latter can be viewed as elements of $M^0(\kappa_{1,i})$ and $M^0(\kappa_{2,i})$ respectively. Finally note that $f^1 \cdot f^\partial \in M^1(-2)$ and the Eisenstein form E_{p-1} can be viewed as an element in $M^0(p-1)$; its inverse is an element in $M^0(1-p)$. Let $\lambda_p \in R$ be a lift of $\overline{\lambda}_p$. Note that $\kappa_{0,i} \equiv \kappa \bmod p-1$; set $e_{0,i} := \frac{\kappa-\kappa_{0,i}}{p-1}$. Similarly $\kappa_{1,i} \equiv \kappa+2 \bmod p-1$ and $\kappa_{2,i} \equiv \kappa+2p \bmod p-1$; set $e_{1,i} := \frac{\kappa+2-\kappa_{1,i}}{p-1}$ and $e_{2,i} := \frac{\kappa+2p-\kappa_{2,i}}{p-1}$. Then, by Propositions 3.5.8 and 3.5.9 $\overline{f}$ is the δ-Fourier expansion of the δ-modular form

$$\sum_{i\geq 0} c_i \left[E_{p-1}^{e_{0,i}} \cdot \Phi_{0,i} - \lambda_p \cdot E_{p-1}^{e_{1,i}} \cdot \Phi_{1,i} \cdot (f^1 \cdot f^\partial) + \overline{p^\kappa} \cdot E_{p-1}^{e_{2,i}} \cdot \Phi_{2,i} \cdot (f^1 \cdot f^\partial)^p\right] \quad (4.5.16)$$

which is an element of $M^1(\kappa)$. This ends the proof. □

Example 4.5.15. We consider the special case of Example 4.5.3. Let

$$f(q) = \sum_{m\geq 1} a_m q^m \in q\mathbb{Z}[[q]] \quad (4.5.17)$$

be the Fourier expansion of a cusp form $f \in M(\Gamma_0(N), \mathbb{Z}, 2)$. Assume $a_1 = 1$ and assume $f(q)$ is an eigenvector for all the Hecke operators $T_2(n)$ with $n \geq 1$. Assume p is a prime and let $\varphi := \overline{f} = \sum_{m\geq 1} \overline{a}_m q^m \in q\mathbb{F}_p[[q]]$ be the reduction mod p of $f(q)$. Then the equalities (4.5.2) hold with $\kappa = 0$. So by Theorem 4.5.1 the series

$$f^{\sharp,2} = f^{\sharp,2}(q,q',q'') := \frac{1}{p} \cdot \sum_{n\geq 1} \frac{a_n}{n}(p^\kappa \phi^2(q)^n - a_p\phi(q)^n + pq^n) \in \mathbb{Q}_p[[q,q',q'']] \quad (4.5.18)$$

belongs to $\mathbb{Z}_p[[q]][q',q'']\hat{}$ and its reduction mod p equals

$$\overline{f^{\sharp,2}} := \overline{f^{\sharp,2}(q,q',q'')} = \varphi^{(-1)} - \overline{a}_p V(\varphi)\frac{q'}{q^p} + V^2(\varphi)\left(\frac{q'}{q^p}\right)^p \in \mathbb{F}_p[[q]][q']. \quad (4.5.19)$$

Note also that $T_2(n)\varphi = \overline{a}_n \cdot \varphi$ for $(n,p) = 1$ and $U\varphi = \overline{a}_p \cdot \varphi$. So by Theorem 4.4.7 $\overline{f^{\sharp,2}}$ is an eigenvector of the Hecke operators $nT_0(n)$, "$pT_0(p)$", $(n,p) = 1$, with eigenvalues $\overline{a}_n, \overline{a}_p$. In addition, if $p \gg 0$, by Theorem 4.5.12, the series $f^{\sharp,2}(q, q', q'')$ in (4.5.18) is the δ-Fourier expansion of a δ-modular form $f^{\sharp,2} \in \mathcal{O}^2(X_1(N)) \subset M^2(0)$.

On the other hand, as in the proof of Theorem 4.5.7, $\varphi^{(-1)}(q)$ is the Fourier expansion of a modular form over k of weight $p^2 - p$; the latter lifts to a modular form $\Phi_0 \in M(\Gamma_1(N), R, p^2 - p)$ which can be viewed as an element in $M^0(p^2 - p)$. Also $V(\varphi)$ and $V^2(\varphi)$ are Fourier expansions of modular forms over k of weights $2p$ and $2p^2$ so they lift to modular forms $\Phi_1 \in M(\Gamma_1(N), R, 2p)$ and $\Phi_2 \in M(\Gamma_1(N), R, 2p^2)$ respectively. The latter can be viewed as elements of $M^0(2p)$ and $M^0(2p^2)$ respectively. Then $\overline{f^{\sharp,2}(q, q', q'')}$ is the δ-Fourier expansion of the δ-modular form

$$f^! := E_{p-1}^{-p} \cdot \Phi_0 - a_p \cdot E_{p-1}^{-2} \cdot \Phi_1 \cdot (f^1 \cdot f^\partial) + \cdot E_{p-1}^{-2p} \cdot \Phi_2 \cdot (f^1 \cdot f^\partial)^p \in M^1(0). \quad (4.5.20)$$

Note now that $f^{\sharp,2} \in M^2(0)$ and $f^! \in M^1(0)$ have the same δ-Fourier expansion and the same weight. By Proposition 4.5.11 (the "δ-expansion principle") we get the following:

Corollary 4.5.16. In the notation of Example 4.5.15 we have the congruence $f^{\sharp,2} \equiv f^! \bmod p$ in $M^2(0)$.

Note that the right hand side of this congruence has order 1 and has a priori "singularities" both at the cusps of $X_1(N)$ and at the supersingular points. In stark contrast with that, the left hand side of the above congruence has *no* "singularity" at either the cusps or the supersingular points.

Also in stark contrast with Theorem 4.5.12 we have the following consequence of Theorem 4.5.6.

Theorem 4.5.17. *Let $f(q)$ be as in Example 4.5.4 and assume $N \not\equiv 1 \bmod p$ (for instance $p \gg 0$). Then the series $\overline{f^\sharp(q, q', q'')}$ in (4.5.10) is not the image of any element in any space $M^r(w)$ with $r \geq 0$, $deg(w) = 0$.*

Proof. Assume the notation of Example 4.5.4. By Theorem 4.5.6 it follows that the image of $f(q)$ in $\mathbb{F}_p[[q]]$ is the Fourier expansion of some modular form $\widehat{f} \in M(\Gamma_1(N), \mathbb{F}_p, 2 + (p-1)\nu)$, $\nu \geq 0$. On the other hand, by Example 4.5.4 we know that the image of $g(q)$ in $\mathbb{F}_p[[q]]$ is the Fourier expansion of a modular form $\widehat{g} \in M(\Gamma_0(N), \mathbb{F}_p, 2)$. It follows that the modular form

$$\widehat{h} := E_{p-1}^{\nu} \cdot \widehat{g} - \widehat{f} \in M(\Gamma_1(N), \mathbb{F}_p, 2 + (p-1)\nu)$$

has Fourier expansion a constant $\gamma := \frac{N-1}{24} \in \mathbb{F}_p^{\times}$. On the other hand γ, viewed as an element in $M(\Gamma_0(N), \mathbb{F}_p, 0)$ has Fourier expansion γ. By the Serre and Swinnerton-Dyer Theorem [18], p.140, the difference $\widehat{h} - \gamma$ is divisible by $E_{p-1} - 1$ in the ring $\bigoplus_{\kappa \in \mathbb{Z}} M(\Gamma_1(N), \mathbb{F}_p, \kappa)$. It follows that the weights $2 + (p-1)\nu$ and 0 are congruent mod $p - 1$, a contradiction. □

References

[1] Barcau, M: Isogeny covariant differential modular forms and the space of elliptic curves up to isogeny, Compositio Math. 137, 237-273 (2003).

[2] Borger, J. : The basic geometry of Witt vectors, I: The affine case, to appear in Algebra and Number Theory.

[3] Borger, J. : The basic geometry of Witt vectors, II: Spaces, to appear in Mathematische Annalen.

[4] Breuil C., Conrad B., Diamond F., Taylor R.: On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises, J. Amer. Math. Soc. 14 (2001), 4, 843-939.

[5] Buium, A. : Differential Algebra and Diophantine geometry, Actualités Mathématiques. [Current Mathematical Topics] Hermann, Paris, 1994.

[6] Buium, A. : Intersections in jet spaces and a conjecture of S. Lang, Annals of Math. 136 (1992) 557-567.

[7] Buium, A. : Differential characters of Abelian varieties over $p-$adic fields, Invent. Math. 122, 309-340 (1995).

[8] Buium, A.: Geometry of p-jets, Duke Math. J., 82, 2, (1996), 349-367.

[9] Buium, A. : Differential Modular forms, Crelle J., 520 (2000) , 95-167.

[10] Buium, A. : Arithmetic Differential Equations, Mathematical Surveys and Monographs 118, AMS, 2005.

[11] Buium A., Differential eigenforms, J. Number Theory 128 (2008), 979-1010.

[12] Buium, A.: Differential characters on curves, Serge Lang Memorial Volume, to appear.

[13] Buium, A., Poonen, B.: Independence of points on elliptic curves arising from special points on modular and Shimura curves, II: local results, Compositio Math., 145 (2009), 566-602.

[14] Buium, A., Saha, A., Hecke operators on differential modular forms mod p, submitted to J. of Number Theory, arXiv:1104.0129.

[15] Buium A., Saha A., The ring of differential Fourier expansions, submitted to J. of Number Theory, arXiv:1104.0124.

[16] Deligne, P., Rappoport, M., Schemas de modules de courbes elliptiques, LNM 349, Springer 1973, pp. 143-316.

[17] Diamond, F., and Im, J.: Modular forms and modular curves. In: Seminar on Fermat's Last Theorem, Conference Proceedings, Volume 17, Canadian Mathematical Society, pp. 39-134 (1995).

[18] Goren, E. Z., Lectures on Hilbert Modular Varieties and Modular Forms, CRM Monograph Series CRMM 14, 2002.

[19] Gouvêa, F., Arithmetic of p-adic modular forms, Lecture Notes in Math. 1304, Springer, 1985.

[20] Gross, B. H., A tameness criterion for Galois representations associated to modular forms mod p, Duke Math. J., **61**, 2, 445-517 (1990).

[21] Hesselholt, L., Lecture notes on the big de Rham-Witt complex, Lecture notes 2009.

[22] Katz, N.: $p-$adic properties of modular schemes and modular forms, LNM 350, Springer, Heidelberg (1973).

[23] Katz, N.: p-adic interpolation of real Eisenstein series, Ann. of Math. **104** (1976), 459-571.

[24] Knapp, A.: Elliptic Curves, Math. Notes, Princeton Univ. Press (1992).

[25] Manin, Yu. I. : Rational points on algebraic curves over function fields, Izvestja. Acad. Nauk USSR, 27 (1963), 1395-1440.

[26] Mumford, D. : Lectures on curves on an algebraic surface, Annals of Mathematics Studies, 59.

[27] Serre, J. P.: Formes modulaires et fonctions zéta $p-$adiques. In: LNM 350 (1973).

[28] Silverman, J. H.: Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1985).

[29] Silverman, J. H.: Advanced Topics in the Arithmetic of Elliptic Curves. Springer, Heidelberg, New York (1994).