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Contact homology of toric contact manifolds of Reeb type

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by

DISSERTATION

Submitted in Partial Fulfillment of the
Requirements for the Degree of

The University of New Mexico
Albuquerque, New Mexico

Dedication

I would like to dedicate this work to my family, especially Patience and Sparrow Hawk, who have patiently weathered a long painful Purgatory to allow me to complete it.

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First and foremost I would like to thank my advisor Charles Boyer for all of his teaching, care, encouragement and attention, without which I would have accomplished nothing. I would also like to thank Alex Buium for much help and many interesting discussions. Finally I would like to thank Yasha Eliashberg for his invitation to visit Stanford, and for reading this dissertation.

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ABSTRACT OF DISSERTATION

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Abstract

We use contact homology to distinguish contact structures on various manifolds. We are primarily interested in contact manifolds which admit an action of Reeb type of a compact Lie group. In such situations it is well known that the contact manifold is then a circle orbi-bundle over a symplectic orbifold. With some extra conditions we are able to compute an invariant, cylindrical contact homology, of the contact structure in terms of some orbifold data, and the first Chern class of the tangent bundle of the base space. When these manifolds are obtained by contact reduction, then the grading of contact homology is given in terms of the weights of the moment map. In many cases, we are able to show that certain distinct toric contact structures are also non-contactomorphic. We also use some more general invariants by imposing extra constraints on moduli spaces of holomorphic curves to distinguish other manifolds in dimension 5.

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Chapter 1

Introduction

1.1 Overview

Like symplectic manifolds, contact manifolds have no local invariants. Darboux's theorem tells us that locally all contact structures are the same and Gray stability tells us that there is no deformation theory. Nonetheless there are many contact manifolds which are not contactomorphic. Sometimes one can distinguish different contact structures via the first Chern class of the underlying symplectic vector bundle defined by the contact distribution. However this is insufficient. In [Gir] Giroux shows that the contact structures

$$\xi_n = ker(\alpha_n = cos(n\theta)dx + sin(n\theta)dy)$$

are pairwise noncontactomorphic, however $c_1(\xi_n) = 0$ for all n.

Calculations of this type have been entirely dependent on the specific geometric conditions of the example. However, due to the introduction of contact homology and the more general symplectic field theory of Eliashberg, Givental and Hofer [EGH00], we may now sometimes distinguish contact structures when the classical invariants fail. Using this powerful tool Ustilovsky [Ust99] was able to find infinitely many exotic contact structures on odd dimensional spheres all in the same homotopy class of almost complex structures. Similarly Otto Van Koert, in his thesis [Koe05], made a similar calculation for a larger class of Brieskorn manifolds using the Morse-Bott form of the theory. It should be mentioned that it came to the author's attention upon completion of this work that Miguel Abreu and Leonardo Macarini, using different methods, have, independently, computed a general formula for contact homology for toric contact manifolds with $c_1(\xi) = 0$.

Some of the ideas in this thesis were originally motivated by examples related to a question of Lerman about contact structures on various S^1-bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1$ [Ler03]. We are now able to distinguish these structures essentially using an extension of a theorem of Bourgeois and Eliashberg, Givental, Hofer. for S^1-bundles over symplectic manifolds which admit perfect Morse functions [Bou02] [EGH00].

Theorem 1.1.1 (Bourgeois)**.** *Let (M, ω) be a symplectic manifold with*

$$[\omega] \in H_2(M, \mathbb{Z})$$

satisfying

$$c_1(TM) = \tau[\omega]$$

for some $\tau \in \mathbb{R}$. Assume that M admits a perfect Morse function. Let V be a Boothby-Wang fibration over M with its natural contact structure. Then contact homology $HC(V, \xi)$ is the homology of the chain complex generated by infinitely many copies of $H_*(M, \mathbb{R})$, with degree shifts $2ck - 2$, $k \in \mathbb{N}$, where c is the first Chern class of $T(M)$ evaluated on a particular homology class. The differential is then given in terms of the Gromov-Witten potential of M.

This theorem exploits the fact that in the case of S^1-bundles the differential in contact homology is especially simple since there is essentially one type of orbit for each multiplicity (i.e.., simple orbits can be paramatrized so that their periods are all 1). Perfection of the Morse function kills the Morse-Smale-Witten part of the differential in Morse-Bott contact homology, so the chain complex reduces to the homology of M.

The grading in contact homology comes from the fact that the index calculations may be made via integration of $c_1(TM)$ over certain spherical two dimensional homology classes. In the case of simply connected reduced spaces the cohomology ring of the base has a particularly nice form in terms of 2 dimensional cohomology classes, obtained from the moment map as a Morse function. Moreover all the two dimensional homology of the base in these cases is generated by spheres.

The above theorem always works with no modification when $c_1(\xi) = 0$ and the base is a generalized flag manifold, since then the contact structure must be regular and is then an honest circle bundle. When $c_1 \neq 0$ one can almost use this theorem, but must make and keep track of some specific choices of spanning disks for Reeb orbits. However, everything still works in many cases and we can actually make calculations.

This inspired the author to explore, rather than a group acting transitively, any torus action of Reeb type. The Reeb type assumption ensures that the contact manifold is the total space of an S^1-orbibundle, where the base space admits a *Hamiltonian* action of a compact Lie group. Now there are more complications, since there are toric contact manifolds which only fiber in the orbifold sense over symplectic orbifolds. However, given the Hamiltonian nature of these manifolds and base orbifolds, we can still make our index calulations with some adjustments. This fact makes the Robbins-Salamon and then the Conley-Zehnder indices easier to compute without the need to find a stable trivialization of the contact distribution. In this way we are able to extend the theorem of Bourgeois. For homogeneous spaces the above theorem works automatically, but we must allow for non-monotone bases, i.e., $c_1(\xi) \neq 0$. To generalize the idea to general toric contact manifolds it is necessary to consider symplectic bases (of the circle bundle) which are orbifolds. The main new idea of this thesis is to formalize this process for index computation from [Bou02] and extend the result to an orbifold base, i.e., the case where the Reeb action is only locally free. Specifically, we extend this to orbifolds bases which also admit a Hamiltonian action, since over $\mathbb{C}$ their cohomology ring is still a polynomial ring in H^2 with spherical representatives of the "diagonal" homology classes.

Theorem 1.1.2. *Let M be a contact manifold, which is an S^1 orbi-bundle over the symplectic orbifold $\mathcal{Z}$, where $\mathcal{Z}$ admits a strongly Hamiltonian action of a compact Lie group. Suppose that the curvature form, $d\alpha$ of M as a circle orbibundle over $\mathcal{Z}$ is given by*

$$\sum w_j \pi^* c_j,$$

where c_j are the Chern classes associated to the Hamiltonian action. Assume that the c_j generate $H^*(\mathcal{Z};\mathbb{C})$ as variables in a truncated polynomial ring. Let $\tilde{w}_j$ be the coeffecient of c_j in $c_1(T(\mathcal{Z}))$. Assume further transversality of the linearized $\bar{\partial}_J$-operator, and that $\sum_j \tilde{w}_j > 1$. Then cylindrical contact homology is generated by copies of the homology of the critical submanifolds for any of the Morse-Bott functions given by the components of the moment map, with degree shifts given by twice-integer multiples of the sums of the $\tilde{w}_j$ plus the dimension of the stratum, S, of $\mathcal{Z}$ in which the given Reeb orbit is projected under π.

As corollaries we obtain contact homology for both toric and homogeneous contact manifolds. The reader should also beware that parts of such calculations are only formal without some sort of transversality of the $\bar{\partial}_J$ operator. Even if one can get the right geometric structure on the moduli spaces of J-holomorphic curves in a symplectization there are still problems with the proof of independence of the homology on the choices of contact 1-form and almost complex structure. Also one might want to use index calculations to explore more complicated versions of contact homology and SFT where one seeks non-trivial behavior of J-holomorphic curves as in Gromov-Witten theory. In these cases, especially when higher genus curves are considered, abstract transversality results appear to be crucial. Though several such statements, arguments, and special cases have been published there are still certain persistent gaps. There are many current developments around this issue and the underlying analysis [HWZ07], [CM07]. Without such a result there is no way to know if some of the counts that we make are actually correct. In section 3 we give an alternate argument for transversality for homogeneous contact manifolds and for toric contact manifolds using only elementary tools from algebraic geometry, this is possible only because the almost complex structures involved are integrable by virtue of the Hamiltonian nature of the problem. We should note that, our transversality results only work for the dimension formulae in the symplectization of the relevant contact manifolds, thus certain results about invariance of the homology algebras, which take place in more general symplectic cobordisms must take transversality of the linearized Cauchy-Riemann operator as an assumption. In dimension 5, the base manifold has dimension 4 and then we can take advantage of some of the nice characteristics of J-holomorphic curves in symplectic 4 manifolds. In particular we can take advantage of positivity of intersection of J-holomorphic curves.

We are able to phrase all of this in terms of contact reduction, in the case of circle actions we get a nice formula for the relevant Maslov indices. Even better, in favorable situations we can read off the cylindrical contact homology from the Lerman-Tolman polytope of the base orbifold. We should also note that sometimes this cylindrical set-up does not give enough information. To remedy this, i.e., to get non-trivial behavior of J-holomorphic cylinders, we add marked points intersecting the Poincaré duals of cohomology classes lifted from the base, and study the new invariants that we can get in this way. This is anther place where the transversality results are important (other than in the proof of invariance), since we really need to identify which moduli spaces of holomorphic curves come in k-dimensional families.

We use these kinds of ideas in the last chapter. There we circle orbibundles over toric symplectic 4-manifolds and use the interplay between full contact homology algebra and the Gromov-Witten potential of the base and the SFT potential in the total space of the orbibundle, we are able to get a slight generalization of a theorem in [EGH00].

Chapter 2

Symplectic and Contact Geometry

2.1 Basic definitions and results

In this section we give some basic definitions, theorems, and ideas in order to fix notation and perspective. In this entire thesis, unless otherwise stated, we assume M is oriented. First of all we define a contact structure:

Definition 2.1.1. *A* ***contact structure*** *on a manifold M of dimension $2n-1$ is a* ***maximally non-integrable*** *$2n-2$-plane distribution, $\xi \subset T(M)$. In other words ξ is a field of $2n-2$-planes which is the kernel of some 1-form α which satisfies*

$$\alpha \wedge d\alpha^{n-1} \neq 0.$$

Such an α is called a ***contact 1-form****.*

Notice that given such a 1-form α, $f\alpha$ is also a contact 1-form for ξ whenever f is smooth and non-vanishing. In the following, a contact manifold has dimension $2n-1$, hence the symplectization has dimension $2n$ and our symplectic bases all have dimension $2n-2$. Unless otherwise stated we assume that M is compact and without boundary. Given a choice of contact 1-form, α we define the *Reeb vector field* of α as the unique vector field R_α satisfying

$$i_{R_\alpha} d\alpha = 0$$

and

$$i_{R_\alpha} \alpha = 1.$$

A contact form α is called *quasi-regular* if, in foliated coordinate charts with respect to the flow of the Reeb vector field, orbits intersect each chart a finite number of times. If that number can be taken to be 1 we call the α *regular*. Note that if the manifold is compact quasiregularity makes all Reeb orbits periodic[1] if α is regular of then all of these orbits have the same period. In contrast, the standard contact form on $\mathbb{R}^{2n-1}$ is regular, but its Reeb vector field has no periodic orbits. We call a contact manifold (quasi)regular if there is a contact form α for ξ such that α is (quasi)regular. Given (M, α), we denote by V the *symplectization* of M:

$$V := (M \times \mathbb{R}, \omega = d(e^t\alpha)).$$

A contact structure defines a symplectic vector bundle with transverse symplectic form $d\alpha$. We can then choose an almost complex structure J_0 on ξ. We extend this to a complex structure J on V by

$$J|_\xi = J_0$$

and

$$J\frac{\partial}{\partial t} = R_\alpha$$

where t is the variable in the $\mathbb{R}$-direction. Note that we also get a metric, as usual, on V compatible with J given by

$$g(v, w) = \omega(v, Jw).$$

[1]By Poincaré recurrence, for example.

If (M, ξ) has a quasi-regular contact form, and J can be chosen to be integrable, then we call (M, ξ) a **Sasakian** manifold.

We have the important result from [BG00b], originally proved in the regular case in [BW58]:

Theorem 2.1.1 (Orbifold Boothby-Wang)**.** *Let (M, ξ) be a quasi-regular contact manifold. Then M is a principal S^1-orbibundle over a symplectic orbifold $(\mathcal{Z}, \omega)$ with connection 1-form α whose curvature satisfies*

$$d\alpha = \pi^* \omega.$$

*If α is **regular** then $\mathcal{Z}$ is a manifold, and M is the total space of a principal S^1-bundle.*

Definition 2.1.2. *If M is an S^1-orbibundle over a symplectic orbifold, $\mathcal{Z}$ as above, then we call $(M, \mathcal{Z})$ a **Boothby-Wang pair**. If M admits the action of a Lie group, G of Reeb type, where R_α is properly contained in $\mathfrak{g}$, then we call such a pair a **Hamiltonian Boothby-Wang pair**.*

This enables us to study the nature of M via the cohomology of $\mathcal{Z}$. As we will see, in nice enough cases, the cohomology of $\mathcal{Z}$ along with the bundle data of the Boothby-Wang fibration will determine the contact homology as well. Notice that if we really want to study the quasi-regular case via the base we are forced to consider symplectic orbifolds, a complete study of contact geometry with symmetries necessarily must include symplectic orbifolds.

2.2 Orbifolds

In this section we collect the necessary information about orbifolds. The first subsection contains basic definitions about orbifolds. The second section discusses orbifolds as stratified spaces. In the third section we describe how to define and integrate forms on orbifolds. Most of this can follows the excellent book [BG08], there is also very good information in [CR02] and [RT]

2.2.1 Basic Concepts

When a Lie group acts smoothly, freely, and properly on a manifold, we may endow the quotient space with the structure of a smooth manifold. However, in many cases of interest, we have an action of a Lie group which is only *locally free*, i.e., that the isotropy groups are finite, not necessarily trivial. In this case what we get is almost a manifold, but not quite, in the sense that certain points have been identified "too much." This leads to the definition of an orbifold. It is not so clear that this standard definition of orbifold has anything to do with actions of Lie groups, but we will see that under quite reasonable conditions every such space may be written as a quotient of a locally free action of a compact Lie group.

We begin with some basic definitions. Let X be a paracompact Hausdorff space.

Definition 2.2.1. *Let U be a connected open subset of X. An* ***orbifold chart*** *or* ***local uniformizing system*** *is a triple $(\tilde{U}, \Gamma, \varphi)$ such that*

- *$\tilde{U}$ is an open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ containing the origin,*
- *Γ is finite group, called the* ***local uniformizing group*** *or* ***local uniformizer*** *acting effectively on $\tilde{U}$,*
- *$\varphi : \tilde{U} \to U$ is a Γ-invariant continuous map,*
- *the natural quotient map $\tilde{U}/\Gamma \to U$ is a homeomorphism.*

We now need a way to relate different charts on order to glue them together into something like a manifold.

Definition 2.2.2. *Let* $(\tilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$, $(\tilde{U_\beta}, \Gamma_\beta, \varphi_\beta)$ *be two local uniformizing systems. An* ***injection*** *between* $(\tilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$ *and* $(\tilde{U_j}, \Gamma_j, \varphi_j)$ *is a smooth map*

$$\lambda : \tilde{U_\alpha} \to \tilde{U_\beta}$$

such that

$$\varphi_\beta \circ \lambda = \varphi_\alpha.$$

Remark 2.2.1. *We need to be careful when understanding this definition. First of all the injections look like standard transition maps, however, they are not. This is to be expected since locally, these objects look like branched coverings over discs. Given two different branched coverings with different degrees, we do not expect to have invertible maps locally. We should also keep in mind that an orbifold is a topological space, with additional structure. Whenever we have manifolds contained in the underlying topological space, then these objects do have their natural change of coordinate maps. We should keep this in mind as we continue through these definitions and constructions.*

Now we are ready to define an orbifold atlas on X. This is really just like the definition of a manifold, except we require some kind of compatibility with the local uniformizing groups.

Definition 2.2.3. *An orbifold atlas on* X *is a family*

$$\mathcal{U} = \{(U_\alpha, \Gamma_\alpha, \varphi_\alpha)\}$$

of local uniformizing systems such that

- $X = \bigcup_\alpha \varphi_\alpha(\tilde{U_\alpha})$,
- *given two charts*

$$(\tilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$$

and

$$(\widetilde{U_\beta}, \Gamma_\beta, \varphi_\beta)$$

with U_α, U_β *the images of* φ_α *and* φ_β *and* $x \in U_\alpha \cap U_\beta$ *there exists a neighborhood* U_ν *of* x*, a local uniformizing system*

$$(\tilde{U}_\nu, \Gamma_\nu, \varphi_\nu),$$

and injections

$$\lambda_{\alpha\nu} : (\tilde{U}_\nu, \Gamma_\nu, \varphi_\nu) \to (\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$$

and

$$\lambda_{\beta\nu} : (\tilde{U}_\nu, \Gamma_\nu, \varphi_\nu) \to (\tilde{U}_\beta, \Gamma_\beta, \varphi_\beta).$$

One orbifold atlas is a refinement of another if there is an injection from each local uniformizing system in the first atlas into the second. Two atlases are equivalent if they admit a common refinement.

Now we can finally say what an orbifold is:

Definition 2.2.4. *An* ***orbifold*** *or* ***V-manifold*** *is a paracompact Hausdorff space with an equivalence class of orbifold atlases. An orbifold is called* ***developable*** *if it is the quotient of a finite group acting properly discontinuously on a manifold.*

We will write such a structure as $\mathcal{X} = (X, \mathcal{U})$.

Remark 2.2.2. *Note that in the above definitions we did not require that the action of each local uniformizer to be effective. For many applications and much of the theory of orbifolds this is too restrictive (for example the total space of an orbibundle over an orbifold with all local uniformizers acting effectively may not have effective actions of local uniformizers.) For the purposes in this thesis, we will always assume that these actions are effective unless we mention otherwise. Such orbifolds are called* ***effective*** *or* ***reduced*** *orbifolds. This name reduced is not to be confused with other uses such as in the phrase "symplectically reduced."*

Given an orbifold $\mathcal{X} = (X, \mathcal{U})$ there is a natural stratification of X. Let $x \in X$, then take a local uniformizing system $(\tilde{U}, \Gamma, \varphi)$ with $x \in \varphi(\tilde{U})$. Then we consider $p \in \varphi^{-1}(x)$. The isotropy subgroup Γ_p then depends only on x, so we define the isotropy group Γ_x of x to be the isotropy subgroup for any element of the inverse image of x in a local uniformizing chart.

Definition 2.2.5. *A point $x \in X$ is called **regular** if Γ_x is trivial. Otherwise x is called a **singular** point.*

The set of regular points is a dense open set. The stratification of X is given as follows. x, x' are in the same stratum if their isotropy groups are conjugate. Note that the these orbifold singular points may not be singular in the sense of an ordinary manifold. In other words we could have a subset Y of the underlying topological space which has the structure of a smooth manifold, yet nonetheless, has a nontrivial local uniformizing group for each $y \in Y$. For example if we consider a product of S^2's with uniformizers given by distinct cyclic subgroups on the respective north and south poles. Then there are embedded S^2's which are smooth submanifolds of the underlying space, which have nontrivial local uniformizers for each of their points.

Maps of orbifolds can be a strange thing, since there are choices to be made. Usually there is no confusion. Whatever type of map we want to define is defined, as usual on the open sets of an orbifold atlas, where restricted to each open set, the map has the desired property, i.e., continuous, smooth, C^r. We must, of course in addition require some sort of compatibility. Here this manifests itself as a choice of lift to $\tilde{U_\alpha}$. If this family of lifts is also compatible with all injections then the map of orbifolds is called *good*.

Orbisheaves are defined similarly as a family of sheaves $\mathcal{F}(\tilde{U_\alpha})$ defined on each local uniformizing system, such that each injection $\lambda_{\beta\alpha}$ induces an isomorphism of sheaves:

$$\mathcal{F}(\lambda) : \mathcal{F}_{\tilde{U_\alpha}} \to \lambda^* \mathcal{F}_{\tilde{U_\beta}}.$$

2.2.2 Orbibundles, etc.

To generalize the basic constructions from differential geometry we need a notion of fiber bundle for orbifolds. Again this is a local construction with certain compatibility conditions with respect to all local uniformizing systems.

Definition 2.2.6. *Let $(\mathcal{X},\mathcal{U})$ be an orbifold. An **orbibundle** is a collection of fiber bundles $B(\widetilde{U_\beta})$ over $\tilde{U_\beta}$ for each local uniformizing system $(U_\beta, \Gamma_\beta, \varphi_\beta)$. These fiber bundles each have fiber F and an action of a Lie group G on F. We require the existence of a map $h_{\tilde{U_\alpha}} : \Gamma_\alpha \to G$ which satisfies the following.*

- *if $b \in \pi^{-1}(\tilde{x}_\alpha)$, for $\tilde{x_\alpha} \in \tilde{U_\alpha}$, then for each $\gamma \in \Gamma_\alpha$ we have $bh_{\tilde{U_\alpha}}(\gamma) \in \pi^{-1}(\gamma^{-1}\tilde{x}_i)$.*

- *An injection*

  $$\lambda_{\beta\alpha} : \tilde{U_\alpha} \to \tilde{U_\beta}$$

  induces a bundle map

  $$\lambda_{\beta\alpha}{}^* : B\tilde{U_\alpha} \to B\tilde{U_\beta}$$

  such that

  $$h_{\tilde{U_\alpha}}(\gamma) \circ \lambda^*_{\beta\alpha} = \lambda^*_{\beta\alpha} \circ h_{\tilde{U_\beta}}(\gamma'),$$

  whenever γ' satisfies $\lambda_{\beta\alpha} \circ \gamma = \gamma' \circ \lambda_{\beta\alpha}$.

- $(\lambda_{\nu\beta} \circ \lambda_{\beta\alpha})^* = \lambda^*_{\beta\alpha} \circ \lambda^*_{\nu\beta}$, *for injections $\lambda_{\nu\beta}, \lambda_{\beta\alpha}$, and sets $\tilde{U_\beta}, \tilde{U_\nu}, \tilde{U_\alpha}$ corresponding to appropriate local uniformizing systems.*

In particular, if the fibers are vector spaces with G acting via linear transformations, then the orbibundle is called a vector orbibundle. If the fiber is the Lie group G acting on the right, then the orbibundle is called a principal orbibundle.

Given an orbifold and an orbibundle, we may consider the total space, which has an orbifold structure. In the case of a Principal G-orbibundle we can say for sure that this total space is a smooth *manifold* if and only if all the functions $h_{\tilde{U_j}}$ are injective. If all local uniformizers act effectively on the total space as subgroups of the local uniformizers of the base orbifold, then the orbibundle is called *proper*. Clearly this always holds whenever the total space is a manifold, having trivial uniformizers.

Given this definition we can define an orbibundle map to be a family of bundle maps defined on the level of the open sets, which are compatible with all injections.

By considering $GL(n, \mathbb{R})$ orbibundles i.e., tangent orbibundles we can see that any orbifold as defined above is the quotient of a locally free action of a compact Lie group on a manifold M.

Moreover, of particular importance is the notion of a section of an orbibundle. Again this is just something that is defined locally on the family of vector bundles which is compatible with the injections and local uniformizers.

Definition 2.2.7. *An orbisection is a collection of sections $\sigma = \{\sigma_\alpha\}$ of each bundle $B(\tilde{U_\alpha})$ which satisfies*

- *for each $\gamma \in \Gamma_\alpha$, $\sigma_{\tilde{U_\beta}}(\gamma^{-1}(\tilde{x})) = h_{\tilde{U_\beta}}(\gamma)\gamma_{\tilde{U_\alpha}}(\tilde{x})$.*
- *If $\lambda_{\beta\alpha}$ is an injection then $\lambda^*_{\beta\alpha}\sigma_{\tilde{U_\beta}}(\lambda(\tilde{x})) = \sigma_{\tilde{U_\alpha}}(\tilde{x})$.*

Now with the definition of an tangent orbibundle we can generalize all functorial constructions of differential geometry to orbifolds, and define sections of such objects, vector fields and differential forms, for example. Here we have defined sections as collections of sections defined for each local uniformizing system. We can go further and define local invariant sections defined by averaging over the group. More explicitly, if $\sigma_{\tilde{U}}$ is a section defined on a local uniformizing system, $(\widetilde{U}, \Gamma, \varphi)$ we can define an invariant local section, $\sigma'_{\tilde{U}}$ by

$$\sigma'_{\tilde{U}} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \sigma_{\tilde{U}} \circ g.$$

We can then extend these local invariant sections using the definition of orbisection to global invariant orbisections. In order to integrate a differential form on an orbifold $\mathcal{X} = (X, \mathcal{U})$ first we assume that the form, σ is supported in a single uniformizing system, $(\tilde{U}, \Gamma, \varphi)$. Then

$$\int_{\varphi(\tilde{U})} \sigma = \frac{1}{|\Gamma|} \int_{\tilde{U}} \sigma_{\tilde{U}}.$$

Otherwise, we just sum over all strata using a partition of unity on the local uniformizing systems. This is the approach taken in [Sat57].

When the open sets in the local uniformizing systems are open subsets of $\mathbb{C}^n$ we can, as in the case of complex manifolds, use algebro-geometric methods. In this case we can get an invariant almost complex structure, J, in the orbifold sense from our theory of sections of functorial linear constructions mentioned above. Of biggest importance to us is generalizing the notions of divisor and line bundle. We will need to play with both Weil and Cartier divisors, but as we shall see, much of the time we will be able to work with the more intuitive Weil divisors.

Reall that the locus of regular points on an orbifold is a complex manifold, so by Hartog's theorem we can extend holomorphic functions on X_{reg} to all of X. In this way we can define divisors and their structure sheaves.

Let us define our first "new" type of divisor.

Definition 2.2.8. *Given a complex orbifold* $(\mathcal{X}, \mathcal{U})$ *the* ***branch divisor*** *is a* $\mathbb{Q}$-*divisor of the form*

$$\triangle = \Sigma_\alpha (1 - \frac{1}{m_\alpha}) D_\alpha,$$

where the sum is taken over all Weil divisors D_α which intersect the orbifold singular locus of X, and m_α is the gcd of the orders of all local uniformizers Γ_α which intersect D_α.

Note that these divisors are really just complex codimension 1 subsets of X which inherit the orbifold structure of $\mathcal{X}$. In ordinary algebraic geometry we cannot always define Weil divisors on an arbitrary scheme, we have the notion of Cartier divisor. In this orbifold theory we have *orbidivisors* or *Baily divisors.* Though we will always be working with normal complex varieties, so that Weil divisors are always defined, we need this concept in order to relate divisors to orbi-line bundles. In the following let D_x denote the stalk of the divisor sheaf.

Definition 2.2.9. *An* ***orbidivisor*** *or* ***Baily divisor*** *is a Cartier divisor defined on each local uniformizing system*

$$(\widetilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$$

which satisfies the following compatibility requirements.

- *For each $x \in X$ and each $\gamma \in \Gamma$, $f \in D_{\gamma x}$ implies $f \circ \gamma \in D_x$.*
- *The injection $\lambda_{\beta\alpha}$ respects the stalk structure, and respects the action of Γ_α.*

Since we can lift $\mathbb{Q}$-divisors of the form $\sum_\alpha \frac{b_\alpha}{m_\alpha} D_\alpha$, branch divisors, for example, to Baily divisors the following proposition tells us how to go from divisors to line bundles.

Proposition 2.2.1. *Given a Baily divisor, D, on $\mathcal{X}$ one can construct a complex line orbibundle $[D]$ which corresponds to an invertible orbisheaf $\mathcal{O}(D)$.*

This gives us a handle on important invariants, and sets up the standard relationship between the orbifold first Chern class and the canonical bundle.

2.2.3 Symplectic orbibundles

In this subsection we extend the symplectic definition of first Chern number over a Riemann surface to the world of orbifolds. We are modifying [MS95] to the situation of orbifolds.

First a symplectic orbibundle is, from the definition above, a family of symplectic vector bundles defined over U_α for each local uniformizing system $(\widetilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$ which satisfy the given compatibility condition. Given such an orbibundle and a local uniformizing system, $(\widetilde{U_\alpha}, \Gamma_\alpha, \varphi_\alpha)$, for each α we may choose an invariant compatible almost complex structure J_α, and form the invariant metric $g_\alpha(\cdot,\cdot) = \omega_\alpha(\cdot, J_\alpha \cdot)$. Our most important goal in this subsection is to prove the following lemma for orbifolds.

Lemma 2.2.1. *Let $\mathcal{E} \to \mathcal{M}$ be a symplectic vector orbibundle. Then given a smooth real (orbi)curve*

$$\gamma : [0,1] \to \mathcal{M}$$

and isomorphisms Φ_0, Φ_1 from the fibers over $\gamma(0), \gamma(1)$ to symplectic $\mathbb{R}^n$, there exists a symplectic trivialization of $\gamma^*\mathcal{E}$.

Proof. This proof is the same as in [MS95], except that we must make things agree with the orbifold structure. To do this we lift the curves to each local uniformizing neighborhood. Then in each $\widetilde{U_\alpha}$ we proceed as in the reference making sure that the gluing is compatible with injections. □

Now we want to prove that given any orbi-Riemann surface, $\mathcal{S}$ with nonempty boundary, and any symplectic orbibundle, $\mathcal{E}$ over $\mathcal{S}$ of rank k there is an isomorphism from $\mathcal{E}$ to the trivial symplectic orbibundle of rank k.

Recall that to put an orbifold structure on a Riemann surface, we are highly restricted. First of all since in a complex orbifold one may only have strata of even dimension, the strata are either connected components of the underlying space Σ, or collections of points $z_i \in \Sigma$ along with open 2-real dimensional neighborhoods about them. In the developable case, we just use invariant forms and proceed as in [MS95]. Otherwise we just work in each local uniformizing system. In this way it is clear that we can do this for discs.

Now we would like to use pair of pants induction to prove this for general orbifold Riemann surfaces with boundary. First note that if the genus is greater than 1, the orbifold structure is developable [BG08], so we just average, and use the result for manifolds. So it now suffices to check the induction step when one piece has genus 1 and has either 1 or 2 punctures. In other words we need to extend the trivialization over a genus 1 orbi-surface.

To do this we first note that the any orbifold singularites are isolated points. We may then extend away from these points using the smooth case. Now we choose a lift of the symplectic structure to the local uniformizing neighborhood. Then we get an orbibundle with invariant symplectic form (by averaging over the group), in each local uniformizing system this bundle is still trivial. Finally we glue these new discs back in.

We shall revisit this later when we consider maps of Riemann surfaces into symplectic orbifolds later. Now let us make sure that the standard definition of first Chern number works well for orbicurves. Just as in the smooth case, let us decompose our orbicurve as a union of orbicurves with boundary. Since the orbifold locus is zero dimensional, we may choose our separating curve(s) to be contained in the dense open set of regular points. We then parametrize these curves and consider a change of coordinate charts. For each t, and each local uniformizing system we get a different change of coordinates, which is a symplectic bundle map, of trivial orbibundles. We take the Maslov index over each local uniformizing system, dividing out by the order of the group each time.

Chapter 3

Symplectic and Contact Manifolds with Symmetries

3.1 Hamiltonian group actions and moment maps

In this expository chapter we discuss special group actions on symplectic and contact manifolds. Most of this can be found in [BG08]. For the original work on symplectic toric orbifolds [LT97]. The original convexity theorem is due independently to Atiyah, and Guillemin-Sternberg [GS82], for example. Let G be a compact Lie group. Suppose G acts on the symplectic orbifold $\mathcal{Z}$ via symplectomorphims. We must be careful to interpret all of this in the orbifold sense. We call such an action Hamiltonian if there is a function H, such that for each $\zeta \in \mathfrak{g}$, $dH = i_{X^\zeta}\omega$, where X^ζ is a fundamental vector field for ζ under the action. We call such an action strongly Hamiltonian if there is a G-equivariant **moment map**

$$\mu : \mathcal{Z} \to \mathfrak{g}^*$$

which satisfies

$$d\langle \mu, \zeta \rangle = i_{X^\zeta}\omega,$$

in other words the inner product of the moment map with an element ζ of $\mathfrak{g}^*$ is a Hamiltonian function for the associated vector field X^ζ. Symplectic orbifolds admitting such actions have many very nice properties. In this section we will list many of these, not the least of which is an orbifold version of the Delzant theorem [LT97].

A very important example is that of a circle acting on $\mathbb{CP}^n$.

One way to see this is look at

$$(\mathbb{CP}^n, \omega_{FS})$$

with the Fubini-Study form. Then given weights

$$w_1, \dots, w_n,$$

S^1 acts on $[z_0 : \dots : z_n]$ by

$$e^{i\theta}[z_0 : \dots : z_n] = [z_0 : e^{w_1\theta} z_1 \dots : e^{w_n\theta} z_n].$$

A moment map is given by

$$\mu([z_0, \dots, z_n]) = \frac{1}{\sum_{k=0}^{n} |z_j|^2} \sum_{j=1}^{n} w_j |z_j|^2.$$

When $n = 1$ this is just a circle acting on S^2 by rotations and the moment map is the height function. We can see this in cylindrical-polar polar coordinates on

$$(S^2, d\theta \wedge dh),$$

with the action given by

$$e^{it}(\theta, h) = (\theta + t, h),$$

with moment map given by

$$\mu(\theta, h) = h.$$

We need to collect some basic facts relating the orbifold stratification to the fixed point sets of various subgroups. We will relate all of this to critical submanifolds of the moment map.

Proposition 3.1.1. *Suppose that the compact Lie group acts on the symplectic orbifold $\mathcal{Z}$ in a Hamiltonian fashion. The, each component of the moment map, the square of each component, or the norm squared of the full moment map are all Morse-Bott functions of even index, where each critical submanifold is a symplectic suborbifold of $\mathcal{Z}$.*

The following theorem is, in a sense, the main structure theorem for toric orbifolds. In the manifold case this is the famous convexity theoem of Atiyah, Guillemin and Sternberg. In the orbifold setting this is due to Lerman and Tolman.

Theorem 3.1.1. *Let (M, ω) be a compact symplectic orbifold, of dimension $2n$ with the strongly Hamiltonian action of a k dimensional torus T. Then the image of the moment map is a convex polytope.*

In the proof of this theorem, we see that the vertices of this polytope are given by the images of the components of the fixed point sets of the action. More generally the dimension k faces are the images of components of fixed point sets of codimension k subtori of T.

Moreover we find that these orbifolds that admit such actions also admit integrable complex strutures, hence these are all Kähler orbifolds, and the orbifold strata are all even dimensional Kähler orbifolds. Even better as we shall see these orbifolds have very interesting and useful cohomology rings.

3.2 Toric symplectic and contact geometry

We now wish to look more closely at a special case, i.e., when the torus has the maximal possible dimension.

Definition 3.2.1. *A toric symplectic orbifold is a tuple $(\mathcal{X}, \omega, \rho, \mu)$, where $\mathcal{X}$ is an orbifold of dimension $2n$, ω is an invariant symplectic form, ρ is the strongly Hamiltonian action of a torus T of dimension n, and μ is the moment map associated to ρ.*

There are some useful facts about the Morse theory and orbifold stratification of symplectic toric orbifolds. First we know, by the Atiyah-Guillemin-Sternberg convexity theorem, that the image of the moment map is a convex polytope. Taking this idea further, Lerman and Tolman proved that there is $1-1$ correspondence between *labelled* polytopes and symplectic toric orbifolds. Here is their main convexity result.

Theorem 3.2.1. *Let (M, ω, T, μ) be a symplectic toric orbifold. Then the image of the moment map is rational polytope. Moreover to each facet there is an integer label giving the orbifold structure group of the points in the preimage under the moment map of the facet.*

The next few results really flesh out what this means and how to use it.

There is a very useful relationship between the stratification and the structure of the polytope. Let $\mathcal{Z} = \bigcup_k \Sigma_k$ denote the stratification of $\mathcal{Z}$ in terms of conjugacy classes of the local uniformizing groups. This gives the labelling of the *facets* or codimension 1 faces.

Theorem 3.2.2 (Lerman-Tolman)**.** *Let F^o be the interior of a facet of the moment polytope of a toric symplectic orbifold. For any x_1, $x_2 \in F^o$, $\mu^{-1}(x_1)$, $\mu^{-1}(x_2)$ have the same structure group.*

Notice that the interior of the whole polytope is the open dense set of points with the same local uniformizing group, i.e., the set of *reguar* points, and each face has an open dense set of points with the the same local uniformizing group.

Now one may wonder, what about the boundaries of these faces? Again from [LT97] we have

Lemma 3.2.1. *Let (M, ω, T, μ) be a toric symplectic orbifold. The isotropy groups and local uniformizing group of each $x \in M$ can be read off from the associated LT polytope as follows. Let $\mathcal{F}(x) = \{F^o | F^o$ is a facet of $\triangle$ containing $\mu(x)$ in its closure.$\}$ Let $\eta_{F^o} \in \mathfrak{t}$ denote the primitive outward pointing normal to the facet F^o, and m_{F^o} the associated label. Then the isotropy group of x is the linear span of the torus H whose tangent bundle is spanned by the normals η_{F^o}.*

The orbifold structure group is given by $\ell/\hat{\ell}$ where ℓ is the integer lattice given by circle subgroups of H, and $\hat{\ell}$ is the lattice generated by m_{F^o} multiples of the normal vectors, η_{F^o}.

3.2.1 Toric contact manifolds

Now we need to talk about the contact case. Here we start with the symplectization, and see that there is an isomorphism between symplectomorphisms of the cone which commute with homotheties and contactomorphisms of M.

Let (M^{2n-1}, ξ) be a contact manifold. Choose a contact 1-form α for ξ. Let $V := (\mathbb{R} \times M, \omega = d(e^t\alpha))$ be its symplectization. Denote by $Symp^0(V, \omega)$ be the subgroup of the symplectomorphism group of V, $Symp((V, \omega))$ consisting of all contactomorphisms of V commuting with homotheties. Denote by $\mathfrak{symp}((V, \omega))$ and $\mathfrak{symp}^0((V, \omega))$ the corresponding Lie algebras. Denote by $Con(M, \xi)$ the contactomorphism group of (M, ξ). The proof of the following can be found in [LM87]

Proposition 3.2.1.

$$Symp^0(V,\omega) \cong Con(M,\xi).$$

Now suppose that G acts on effectively V via symplectomorphisms and invariant with respect to homotheties. This means that there is a homomorphism

$$\rho : G \to Symp(V,\omega)^0.$$

Since the action is effective the image of ρ is a subgroup of $Symp(V,\omega)^0$. Now, ω by definition is exact, so any action of G is Hamiltonian, i.e., there exists a G-equivariant moment map

$$\tilde{\mu} : V \to \mathfrak{g}*$$

defined by

$$d\langle \widetilde{\mu(x)}, \zeta\rangle = -i_X^\zeta \omega,$$

where X^τ is the fundamental vector field of $\zeta \in \mathfrak{g}$.

Now we take a look at the defining equation of the moment map. By Cartan's magic formula we have

$$-i_{X^\zeta} d(e^t\alpha) = di_{X^\zeta} e^t\alpha - \mathcal{L}_{X^\zeta} e^t\alpha$$

which implies, since X^ζ preserves α

$$\langle \tilde{\mu}, \zeta\rangle = i_{X^\zeta} e^t\alpha$$

up to a constant. This shows that the moment map is essentially given by evaluation of the contact form on the fundamental vector field.

Now let us assume that G is a torus, T. We can then consider the kernel of the exponential map $\mathfrak{t} \to T$. We call this kernel the *integral lattice* of T and denote it by $\mathbb{Z}_T$.

Just as with compact symplectic manifolds there is a convexity theorem for symplectizations. First we need to talk about cones.

Definition 3.2.2. *A subset $\mathcal{C} \subset \mathfrak{t}^*$ is called a polyhedral cone if it can be represented by*

$$\mathcal{C} = \bigcap \{y \in \mathfrak{t}^* | \langle y, v_i \rangle \geq 0\}$$

for some finite set of vectors v_i. Such a cone is called ***rational*** *if $v_i \in \mathbb{Z}_T$ for all i.*

The v_i here are the inward pointing normal vectors of the polyhedral cone. We will also assume that the v_i are primitive in that they are the "smallest" possible elements of the integer lattice, in that multiplication by a number strictly between 0 and 1 removes the vector from the integer lattice.

Theorem 3.2.3. *Suppose M is compact. Suppose T acts effectively on V, with $\rho(T) \subset Symp(V, \omega)^0$. Assume that there exists $\tau \in \mathfrak{t}$ such that $\langle \tilde{\mu}, \tau \rangle > 0$. Then the image of the moment map is a convex polyhderal cone.*

Now we can define a moment map on μ on M via restriction of $\tilde{\mu}$ in the $\mathbb{R}$ direction to $t = 0$. For a fixed contact form α we have

$$\langle \mu, \tau \rangle := \langle \mu_\alpha, \tau \rangle := \alpha(X^\tau).$$

Definition 3.2.3. *Let G be a Lie group acting effectively via coorientation preserving contactomorphisms on M. We define the contact moment map $\Upsilon(\alpha, x)$ by*

$$\langle \Upsilon(\alpha, x), \tau \rangle = \langle \alpha, X^\tau(x) \rangle.$$

The point is that we can use Υ with any contact 1-form that we want, so that we do not have to make a definitive choice right away. Most of the time however, in applications we won't speak of Υ at all, and work with a preferred contact form.

The following definition was introduced in [BG00a].

Definition 3.2.4. *Let G be a Lie group which acts on the contact manifold (M, ξ). The action is said to be of* ***Reeb type*** *if there is a contact 1-form α for ξ and an element $\zeta \in \mathfrak{g}$, such that $X^\zeta = R_\alpha$.*

This is a very important definition for us. We will see that manifolds admitting actions of Reeb type are all S^1-orbibundles over symplectic orbifolds which admit a *Hamiltonian* torus action. In this case we can actually relate the polyhedral cone described above to the Lerman-Tolman polytope of the base orbifold.

The first step is the following. If the action of the torus T is of Reeb type, suppose that $\zeta \in \mathfrak{t}$ satisfies $X^\zeta = R_\alpha$ for some quasiregular 1-form α, then

$$\langle \Upsilon(\alpha, x), \zeta \rangle = \langle \alpha, X^\zeta \rangle = \alpha(R_\alpha) = 1.$$

Taking coordinated r_i on $\mathfrak{t}^*$ we get the equation

$$\sum_i r_i w_i = 1$$

for a hyperplane in $\mathfrak{t}^*$ where $w_i \in \mathbb{Z}$, called the *characteristic* or *Reeb hyperplane.*

Moreover, a contact manifold of Reeb type always admits a K-contact structure. This gives the following result.

Proposition 3.2.2. *If an action of a torus T^k is of Reeb type then there is a quasi-regular contact structure whose* 1*-form satisfies, ker*$(\alpha) = \xi$*. Moreover, then M is the total space of a S^1 bundle over a symplectic orbifold which admits a Hamiltonian action of a torus T^{k-1}.*

We now need the definition:

Definition 3.2.5. *A **toric contact manifold** is a co-oriented contact manifold (M^{2n-1}, ξ) with an effective action of a torus, T^n of maximal dimension n and a moment map[1] into the dual of the Lie algebra of the torus.*

[1] The contact moment map can be defined in terms of the symplectization, V, or intrinsically in terms of the annhilator of ξ in TM^*. For more information about this see [Ler02] or [BG08]

Definition 3.2.6. *Let ξ be a G-invariant contact structure, pick a contact 1-form α. The **moment cone** is defined to be*

$$C(\alpha) = \{t\gamma \in \mathfrak{g}^* | \gamma \in \mu_\alpha(M), t \in [0, \infty)\}.$$

Theorem 3.2.4. *Let $\rho : T \to Con(M, \xi)^+$ be an effective action of Reeb type. Then $C(\Upsilon)$ is a rational polyhedral cone. Moreover a choice of T-invariant contact form gives the intersection of the Reeb hyperplace with the moment cone the structure of a convex polytope in the Reeb hyperplane.*

At first this proposition may seem a bit obtuse, however it is this proposition which tells us what the moment polytope is on the base polytope. It makes fully concrete the relationship between the moment polytope of $\mathcal{Z}$ and the Reeb vector field.

3.2.2 Symplectic and contact reduction

For a complete understanding of toric geometry we need to understand all of this in terms of symplectic and contact reduction. Via the construction of Delzant for symplectic manifolds and Lerman-Tolman in the orbifold case, we see that we can get many example via reduction on $\mathbb{C}^n$. By a similar construction we can view contact toric manifolds as being obtained via reduction on the standard contact sphere, obtained as a hypersuface of contact type in $\mathbb{CP}^n$. Moreover there is a very nice relationship between contact reductions and with the symplectic reductions of both the symplectization and the orbifold base. The moment maps are directly related.

Theorem 3.2.5. *Suppose (X, ω) is a symplectic manifolds with the Hamiltonian action of a torus of maximal possible dimension. Let τ be a regular value of the moment map. Suppose moreover that T acts locally freely on $\mu^{-1}(\tau)$. Then the quotient*

$$X_\tau = \frac{\mu^{-1}(\tau)}{T}$$

is naturally a symplectic orbifold, called the symplectic reduction of X by μ.

There is a similar construction for contact manifolds. This can be done either via the symplectization with a symplectic reduction, or we can work directly with the contact case by using a *contact moment map.*

Theorem 3.2.6. *Any toric symplectic orbifold is the symplectic reduction of $\mathbb{C}^n$ by some torus action.*

Let us now state the main result on contact reduction from [BG08].

Theorem 3.2.7. *Let (M, ξ) be an oriented and co-oriented contact manifolds. Let G be a compact Lie group acting on M effectively via orientation and co-orientation preserving contactomorphisms. Let α be a G-invariant contact form for ξ and μ the moment map for this action and 1-form. Suppose that 0 is a regular value of μ and that G acts freely on $\mu^{-1}(0)$. α descends to a 1-form on the quotient. Then $M_0 = \mu^{-1}(0)/G$ is a contact manifold with contact structure $\xi_0 = ker\alpha_0$. Moreover if α is K-contact, then so is α_0, and if the invariant transverse almost complex structure on M is integrable, then so is the induced one on M_0.*

There is also a natural relationship between all the relevant symplectizations, and in the K-contact case, of the bases.

Theorem 3.2.8. *The symplectic reduction at a regular value of the moment map on a symplectization is the symplectization of the contact reduction of M. Moreover, if M is given a K-contact 1-form, then the base of the reduction of M is the reduction of the base of M.*

Proof. To show the first part we note that the torus action commutes with homotheties. This allows us to make the reduction of the the symplectization as a cone over

over the reduced space. Now we put the obvious symplectic structure on this cone, which can be done since the symplectic structure on ξ is invariant.

Now to see the fact about the quotient of the Reeb vector fields we just use that the torus acting on quotient of the contact manifold contains the torus acting on the symplectic reduction of the quotient space. We must be careful here to consider everything in the orbifold sense. □

Even better we have the following analogue of Delzant surjectivity.

Theorem 3.2.9. *Any toric contact manifold of Reeb type is the contact reduction of S^{2n-1}, with its standard contact form by some torus action.*

This discussion, of course gives another proof of the quasiregularity of all toric contact structures of Reeb type, and also of the integrability of their transverse almost complex structures.

3.3 Cohomology rings of Hamiltonian G-spaces

In this section we follow [GS99]. The thing that really makes our calculation possible in its simple form is the structure of the cohomology rings of symplectically reduced spaces, ie, they are all truncated polynomial rings in the Chern classes. Moreover in the simply connected case, we know that all of H_2 can be represented by spheres. Even better, we can always relate all of these homology and cohomology classes to the moment map.

First let's work out what we get in general. Let (M, ω) be a symplectic manifold of dimension $2n$. Let G be a compact connected Lie group of dimension d which acts via (strongly) Hamiltonian symplectomorphisms, and set $\mathfrak{g} = Lie(G)$. Let

$$\mu : M \to \mathfrak{g}^*$$

denote the corresponding moment map. Let τ be a regular value of μ, and set

$$X_\tau = \mu^{-1}(\tau),$$

Suppose that G acts locally freely on $\mu^{-1}(\tau)$. Then X_τ/G is a symplectic orbifold of dimension $2(n-d)$. Set $\mathcal{Z}_\tau = X_\tau/G$. Now suppose that G is a torus. Then the action defines a principle bundle. Let $c_1, \ldots, c_n$ be the Chern classes of the fibration $M \to \mathcal{Z}$. Suppose that ω_τ is the symplectic form on $\mathcal{Z}_\tau$. Then, in a neighborhood of $0 \in \mathfrak{g}^*$ we know that as a smooth manifold $\mathcal{Z}_\tau$ does not depend on τ. The symplectic form, however, does change in the following way:

$$[\omega_\tau] = [\omega_0] + \sum_i^d \tau_i c_i.$$

Let us now compute the symplectic volume in terms of τ, this is given by

$$v(\tau) = \int_{\mathcal{Z}_\tau} e^{[\omega_\tau]}.$$

The product in the exponential is the wedge product. This integral is equal to

$$\int_{\mathcal{Z}_0} ([\omega_0] + \sum_i^d \tau_i c_i).$$

This is a polynomial in in τ and the above discussion is a special case of the Duistermaat-Heckman theorem. We can use this to gain information about the cup products and pairings in the cohomology ring of $\mathcal{Z}$ as long as the c_i generate the cohomology. To do this we pick a multiindex, α, with $|\alpha| \leq d-n$ and consider

$$D_\alpha v|_{\tau=0} = \int_{\mathcal{Z}} \omega^{d-n-|\alpha|} c_1^{\alpha_1} \cdots c_d^{\alpha_d}.$$

This determines the cohomology pairings.

Theorem 3.3.1. *If the $c_1, \ldots, c_d$ generate $H^*(\mathcal{Z};\mathbb{C})$ then*

$$H^*(\mathcal{Z};\mathbb{C}) \simeq \mathbb{C}[x_1, \ldots, x_d]/ann(v_{top})$$

where $ann(v_{top})$ is the annhilator of the highest order homogeneous part of v.

Remark 3.3.1. *The ideal* $ann(v_{top})$ *is just the ideal generated by homogeneous polynomials, given by a multi-index* α *which act on a form* σ *by* $D_\alpha \sigma$*, where the differentiation is in the variables* τ_{α_j}.

To apply this to all homogeneous contact manifolds we need not only the case of flag manifolds but also of *generalized* flag manifolds. These are quotients of a complex semi-simple Lie group G by a *parabolic* subgroup P. These include the flag manifolds. We extend the result from [GS99] about flag manifolds to G/P. For more about generalized flag manifolds see [BE89] and [BGG82], the torus here is given by the relevant Cartan algebra contained in the defining Borel algebra.

Proposition 3.3.1. *Let* G/P *be a generalized flag manifold. Then the cohomology is generated by the Chern classes as above.*

Proof. Since P is parabolic, it contains a Borel subgroup. Each Schubert cell in G/P lifts to one in G/B. This gives an injective map

$$H^*(G/P;\mathbb{C}) \to H^*(G/B;\mathbb{C}).$$

Thus we need only to see that the Chern classes generate $H^*(G/B;\mathbb{C})$ which is known from [Bor53]. □

Again the following result is in [GS99]:

Proposition 3.3.2. *Let* $\mathcal{Z}$ *be a toric orbifold. Then the Chern classes as above generate* $H^*(\mathcal{Z};\mathbb{C})$.

3.3.1 Reduction and cohomology rings.

When we view these spaces as coming from symplectic rreduction there is a very nice formula for Chern classes, the author read about this particular isomorphism in [CS06]. We will build up the cohomology ring is from the Chern classes of the S^1 summands in the principal torus bundle defined by the reduction. We, of course, need to remove the assumption that T acts freely, and assume only that the action is locally free. In the following we relate the rings obtained in the previous section to Delzant or Lerman-Tolman polytopes.

To proceed let ρ be a diagonal homorphism $T^K \to T^n$

given by $(\rho_1, \ldots, \rho_n)$, where

$$\rho_j(exp(\zeta)) = e^{2\pi i \langle w_l, \zeta \rangle},$$

and the w_l are weight vectors, for $l = 0, \ldots, k$, $\zeta \in \mathfrak{t}^* = Lie(T^k)^*$. Since T^n acts on $\mathbb{C}^n$ composition of this action with ρ gives a new action with moment map

$$\mu : C^n \to \mathfrak{t}^*$$

given by

$$\mu(z_1, \ldots, z_n) = \sum_{j=1}^{k} (\sum_{l=1}^{n} w_{j,l} |z_l|^2) e_j^*.$$

Now, given a regular value, τ of the moment map, the action of T^k restricts to one on the level set $\mu^{-1}(\tau)$. Hence we look at the symplectic reduction

$$M_\tau := \mu^{-1}(\tau)/T^k.$$

Each weight vector gives rise to a 2-dimensional cohomology class in M_τ given by the Chern class of the bundle

$$\mathbb{C}^n \times_{\rho_j} \mu^{-1}(\tau).$$

These classes generate the cohomology, as in the proposition in the previous section. Moreover by [GS99] the symplectic volume is just the Euclidean volume of the Delzant polytope. Moreover these Chern classes are, for each toric symplectic structure weighted by the w_j. This gives a homomorphism between the integer lattice of T^k and the cohomology. The sum of the images under this homomorphism of the weight vectors gives the first Chern class of the reduced space.

Chapter 4

Index theory for Hamiltonian diffeomorphisms

4.1 The Conley-Zehnder, and Robbins-Salamon index

The Robbins-Salamon index associates to each path of symplectic matrices a rational number, it is a generalization of the Conley-Zehnder index to a more general class of paths of symplectic matrices. This particular definition originally appeared in [SR93]. This index determines the grading for the chain complex in contact homology. The Salamon-Robbins index should be thought of as analagous to the Morse index for a Morse function. The analogy isn't perfect, since the actual Morse theory we consider should give information about the loop space of the contact manifold. Also note that our action functional has an infinite dimensional kernel. It should be noted that we will describe three indices in the following. Two of them will be called the Maslov index. This is unfortunate, but it will always be clear which Maslov index we will use at any particular time.

Remark 4.1.1. *Historically, the Maslov index arose as an invariant of loops of Lagrangian subspaces in the Grassmanian of Lagrangian subspaces of a symplectic vector space V. In this setting the Maslov index is the intersection number of a path of Lagrangian subspaces with a certain algebraic variety called the Maslov cycle. This is of course related to the Robbins-Salamon and Conley-Zehnder indices of a path of symplectic matrices, since we can consider a path of Lagrangian subspaces given by the path of graphs of the desired path of symplectic matrices. For more information on this see [MS95], and [SR93].*

Remark 4.1.2. *For a symplectic vector bundle, E, over a Riemann surface, Σ there is symplectic definition of the first Chern number $\langle c_1(E), \Sigma \rangle$. It turns out that this Chern number is the loop Maslov index of a certain loop of symplectic matrices, obtained from local trivializations of Σ decomposed along a curve $\gamma \subset \Sigma$. This Chern number agrees with the usual definition, considering E as a complex vector bundle, and can be obtained via a curvature calculation.*

Let $\Phi(t)$, $t \in [0, T]$ be a path of symplectic matrices starting at the identity such that $det(I - \Phi(T)) \neq 0$ [1]. We call a number $t \in [0, T]$, a *crossing* if $det(\Phi(t) - I) = 0$. A crossing is called *regular* if the *crossing form* (defined below) is non-degenerate. One can always homotope a path of symplectic matrices to one with regular crossings, which, as we will see below, does not change the index.

For each crossing we define the *crossing form*

$$\Gamma(t)v = \omega(v, D\dot{\Phi}(t)).$$

Where ω is the standard symplectic form on $\mathbb{R}^{2n}$.

Definition 4.1.1. *The Conley-Zehnder index of the path Φt under the above as-*

[1] This is the *non-degeneracy* assumption. In the context of the Reeb vector field, this condition implies that all closed orbits are isolated

sumptions is given by:

$$\mu_{CZ}(\Phi) = \frac{1}{2} sign(\Gamma(0)) + \sum_{t \neq 0\,,\, t\, a\ crossing} sign(\Gamma(t))$$

The Conley-Zehnder index satisfies the following axioms:

i. (**Homotopy**) μ_{CZ} is invariant under homotopies which fix endpoints.

ii. (**Naturality**) μ_{CZ} is invariant under conjugation by paths in $Sp(n,\mathbb{R})$.

iii. (**Loop**) For any path, ψ in $Sp(n,\mathbb{R})$, and a loop ϕ,

$$\mu_{CZ}(\psi \cdot \phi) = \mu_{CZ}(\psi) + \mu_l(\phi).$$

Where μ_l is the Maslov index for loops of symplectic matrices.

iv. (**Direct Sum**) If $n = n' + n''$ and ψ_1 is a path in $Sp(n',\mathbb{R})$ and ψ_2 is a path in $Sp(n'',\mathbb{R})$ then for the path $\psi_1 \oplus \psi_2 \in Sp(n',\mathbb{R}) \bigoplus Sp(n'',\mathbb{R})$, we have

$$\mu(\psi_1 \oplus \psi_2) = \mu(\psi_1) + \mu(\psi_2).$$

v. (**Zero**) If a path has no eigenvalues on S^1, then its Conley-Zehnder index is 0.

vi. (**Signature**) Let S be symmetric and nondegenerate with

$$||S|| < 2\pi.$$

Let $\psi(t) = exp(JSt)$, then

$$\mu_{CZ}(\psi) = \frac{1}{2} sign(S).$$

The Conley-Zehnder index is still insufficient for our purposes since we need the assumption that at time $T = 1$ the symplectic matrix has no eigenvalue equal to 1. We introduce yet another index for arbitrary paths from [SR93]. We will call this index the Robbins-Salamon index and denote it μ.

For this new index we simply add half of the signature of the crossing form at the terminal time of the path to the formula for the Conley-Zehnder index.

$$\mu(\Phi(t)) = \frac{1}{2}sign(\Gamma(0)) + \sum_{t\neq 0\,,\,t\,a\ crossing} sign(\Gamma(t)) + \frac{1}{2}sign(\Gamma(T))$$

This index satisfies the same axioms as μ_{CZ} as well as the new property of catenation. This means that the index of the catenation of paths is the sum of the indices.

vii. (**Catenation axiom**) Suppose that Φ_1, Φ_2 are two paths of symplectic matrices which satisfy $\Phi_1(T) = \Phi_2(0)$. Then the new path Ψ defined by concatenation of Φ_1 with Φ_2 has index $\mu(\Phi_1) + \mu(\Phi_2)$.

4.1.1 Indices for homotopically trivial closed Reeb orbits

Let γ be a closed orbit of a Reeb vector field. Choose a symplectic trivialization of this orbit in M, i.e., take a map $u : D \to M$ from a disk into M, with the property that the boundary of the image of u is γ and a bundle isomorphism between $u^*\xi$ and standard symplectic $\mathbb{R}^{2n}$, $(\mathbb{R}^{2n}, \omega_0)$. Now we look at the Poincare time T return map of the associated flow (with respect to this trivialization, choosing a framing), where T is the period of γ. If the linearized flow has no eigenvalue equal to 1, we define the Conley-Zehnder index of γ to be the Conley-Zehnder index of the path of matrices given by the linearized Reeb flow. If there are eigenvalues equal to 1 we calculate the Maslov index of the path of matrices coming from the flow (in an appropriate symplectic trivialization.) Note that when there is no eigenvalue equal to 1, the two indices agree.

The Conley-Zehnder and Robbins-Salamon indices depends on the choice of spanning disk or Riemann surface used in the symplectic trivialization. Different choices of disks will change the index by twice the first Chern class[2] of ξ. Intuitively, given

[2]This is the reason that so often in the literature on contact homology authors insist

a periodic orbit of the Reeb vector field, this index reveals how many times nearby orbits "wrap around" the given orbit.

that $c_1(\xi) = 0$. This index defines the grading of contact homology so if this Chern class is non-zero we must be careful to keep track of which disks we use to cap orbits.

Chapter 5

J-holomorphic curves

In this chapter we define and collect properties of pseudoholomorphic curves in symplectic manifolds. This study was essentially initiated by Gromov in his groundbreaking paper [Gro85]. Also Witten noticed that one can do algebraic geometry on the moduli spaces of such curves with given "boundary conditions. This gave rise to the so-called Gromov-Witten invariants, which give a signed count of pseudoholomorphic curves intersecting specified geometric objects. Since then Floer discovered that one could interpret these curves as "flow lines" in a loop space, when, strictly speaking there is no global flow. In Floer's formulation the aforementioned boundary conditions correspond to periodic orbits of some Hamiltonian vector field. This was extended to symplectizations and to the dynamics of the Reeb vector field by Eliashberg, Hofer, and Givental see [EGH00]. There are, of course, far too many uses of these curves to even scratch the surface. A good comprehensive reference, though not completely general, to the uses of these curves to study compact symplectic manifolds is given in full detail in [MS04].

5.1 J-holomorphic curves in symplectic manifolds

Let (M, J) be an almost complex manifold. (Σ, j) a Riemann surface with j the standard complex structure.

Definition 5.1.1.

$$u \in C^\infty(\Sigma, M)$$

is called ***pseudoholomorphic*** *or* ***J-holomorphic*** *if*

$$Jdu = du \circ j.$$

In other words, u is J-holomorphic if the differential of u is complex linear with respect to J and j.

Though the study of J-holomorphic curves can be done in a general almost complex manifold one can vastly simplify their study if the target manifold has a symplectic structure which controls the almost complex structure J. This leads to the taming condition which among other things relates an appropriate energy functional to index theory.

Definition 5.1.2. *An almost complex structure J on a symplectic manifold (M, ω) is called* ω***-tame*** *if for every $p \in M$,*

$$\omega(v, Jv) > 0,$$

for each nonzero vector

$$v \in T_p(M).$$

Such an almost complex structure is called ω***-compatible*** *if in addition*

$$\omega(Jv, Jw) = \omega(v, w).$$

In this case

$$g(v, w) = \omega(v, Jw),$$

defines a Riemannian metric on TM.

This definition of energy will be crucial throughout this exposition. There will be several definitions of energy when we discuss holomorphic curves in the symplectization of a contact manifold, but they all come from this definition.

Definition 5.1.3 (Symplectic Energy)**.** *Let (M, ω) be a symplectic manifold, and let J be an ω-tame almost complex structure on M. Let (Σ, j) be a Riemann surface with complex structure given by j. Let $u : \Sigma \to M$ be J-holomorphic. Then the* symplectic energy *of u is given by*

$$E(u) = \int_{\Sigma} u^* \omega.$$

The various definitions of energy are very important to us since we always restrict to curves with finite (non-zero) energy. In this way we obtain compactness results on spaces of curves and constraints on their asymptotics when we move to the non-compact case of a symplectization.

5.1.1 Moduli spaces for compact M

In this section we introduce the analytic set-up for the case of a compact symplectic manifold for understanding the moduli spaces of J-holomorphic curves following [MS04]. Let us consider a symplectic manifold (M, ω) with a choice of compatible almost complex structure J. We would like to put some geometric structure on the moduli space of J-holomorphic curves representing $A \in H_2(M, \mathbb{Z})$. Let us consider only the genus 0 case. Let j be the standard complex structure on $\mathbb{CP}^1$. Then these are maps $u : \mathbb{CP}^1 \to M$ which satisfy

$$Jdu = du \circ j$$

which is equivalent to

$$\bar{\partial}_J = 0$$

where
$$\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j).$$

We can look now at the set of C^∞ maps from $\mathbb{CP}^1$ into M which represent the class A. We call this set $\mathcal{B}$. We think of the tangent space to a point $u \in \mathcal{B}$ as "vector fields along u," in other words

$$T_u(\mathcal{B}) = \Omega^0(\mathbb{CP}^1, u^*TM).$$

Then we can consider the infinite-dimensional vector bundle $\mathcal{E}$ over $\mathcal{B}$ whose fiber is given by
$$\mathcal{E}_u = \Omega^{0,1}(\mathbb{CP}^1, u^*TM).$$

Then we define the section S of $\mathcal{E}$ by

$$S(u) = (u, \bar{\partial}_J u).$$

Composing dS with the projection

$$\pi : T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$$

we get a map
$$D_u : \Omega^0(\mathbb{CP}^1, u^*TM) \to \Omega^{0,1}(\mathbb{CP}^1, u^*TM).$$

This is the linearized Cauchy-Riemann operator, and its zero set is the moduli space of curves
$$\mathcal{M}_0^A(M, J) = D_u^{-1}(0).$$

The operator D_u is Fredholm, hence as long as D_u is surjective, we know that the dimension of the kernel of D_u is the dimension of the moduli space, and it is given by the Fredholm index given by

$$ind(D_u) = 2n + 2c_1(u^*)(TM).$$

There are many cases when this can be done via a generic choice of J which perturbs the equation until we can achieve transversality. In the case of symplectizations this is a very difficult problem which still has to be overcome.

5.2 Moduli spaces of stable maps

For compact symplectic manifolds this discussion can be pieced together from the excellent book, [MS04]. It is well known that the space of J-holomorphic curves into a symplectic manifold need not be compact. However we have the notion of Gromov compactness, which is a symplectic analogue of the compactification of the moduli space of Riemann surfaces of genus g by adding the so-called stable curves. It is actually by studying the failure of compactness that many of the interesting phenomena happen in the study of J-holomorphic curves. We will consider only genus 0 curves here.

First we recall that given a sequence of J-holomorphic curves from a Riemann surface into a symplectic manifold with ω-tame almost complex structure J, with uniformly bounded first derivatives, then there is a uniformly convergent subsequence in C^∞ converging to a J-holomorphic curve. Hence, the only way for there to be loss of compactness is if each element in the sequence has at least one point where the first derivatives blow-up. By conformal rescaling we can produce a so-called cusp curve. This is the phenomenon of bubbling. Gromov compactness tells us exactly how this can happen. This leads to the symplectic version of stable curves. The stability condition ensures that the automorphism group of the moduli space is finite.

Definition 5.2.1. *An* ***n-labelled tree*** *is a triple (T, E, Λ), where T is the set of edges, E is a relation on $T \times T$ such that for α, $\beta \in T$, we have $\alpha E \beta$ if and only if there is an edge connecting them. T, E, are the sets of vertices (resp) edges of the tree, and Λ is a* ***labelling****, i.e., a map from T into an index set.*

We consider now trees whose edges represent copies of S^2, the vertices are intersection points of the various spheres. Our labels correspond to marked points which are not equal to the intersection points. We consider each sphere to be a separate component.

Definition 5.2.2. *Let (M, ω) be a compact symplectic manifold, with ω-compatible J. A stable J-holomorphic map of genus 0 modelled over the tree (T, E, Λ) is a tuple*

$$(\boldsymbol{u}, \boldsymbol{z}) = (\{u_\alpha\}, \{z_{\alpha\beta}\}, \{\alpha_i, z_i\})$$

where each u_α is a J-holomorphic sphere labelled by the vertices. We have the nodal points which are the intersection points of each component, and the n marked points which we demand are distinct and different from the nodal points. Together these points are all called special points. We impose the stability condition which forces components α with u_α constant to have at least 3 special points.

The stability condition forces the automorphism group of the curve to be finite. The point of all of this is that because of bubbling off of J-holomorphic spheres, we know that the moduli space of spheres is certainly *not* compact, but the stable maps that we have described here do serve as a compactification [MS04], [Gro85].

Theorem 5.2.1. *(Gromov Compactness) A sequence of stable maps has a subsequence converging in the sense of Gromov to stable map possibly having more components.*

5.2.1 J-holomorphic curves in Hamiltonian-T-manifolds

We will see in upcoming sections that J-holomorphic curves in the symplectization of M project in a nice way to curves in $\mathcal{Z}$ whenever $(M, \mathcal{Z})$ is a Boothby-Wang pair. When M admits an action of Reeb type of maximal possible dimension, then

$\mathcal{Z}$ is naturally a toric orbifold, and the count of J holomorphic curves in $\mathcal{Z}$ is tied closely to the toric symplectic structure. In the following we assume that $\mathcal{Z}$ is simply connected. By $\dot{S}^2$ we will mean a sphere with some marked points.

Lemma 5.2.1. *Let $\mathcal{Z}$ be a symplectic orbifold which admits the Hamiltonian action of a torus. Let $u : \dot{S}^2 \to \mathcal{Z}$ be a rigid J-holomorphic sphere representing the homology class $A \in H_2(\mathcal{Z})$. Then the image of u in the moment polytope $\triangle$ is completely contained in the set of edges, and the marked points must intersect the fixed points of the torus action, i.e., they map to the vertices of $\triangle$.*

Proof. Each 2 dimensional homology class is spherical since $\mathcal{Z}$ is simply connected. Since a rigid curve must be invariant under the S^1 action, the marked points must map to fixed points of the the circle action. The spheres mapped into the edges, by the moment map are the only ones which are invariant under any circle subgroups of the torus. □

Lemma 5.2.2. *T -invariant genus 0 curves as described above are completely determined by vertices of $\triangle$, or by the edges with multiplicity.*

Therefore, to understand holomorphic curves in a toric manifold is to understand the 1-skeleton of the Delzant polytope, labelled with multiplicities.

This will also allow us to compute the genus 0 Gromov-Witten potential for $\mathcal{Z}$. First let us recall that in a complex orbifold that orbicurves either intersect the orbifold singular set completely or in finitely many points. This already tells us a lot about what the potential should look like. It also tells us a lot about what J-holomorphic curves should look like in the symplectization of a toric contact manifold of Reeb type.

For curves whose image is entirely in the orbifold locus, then we may treat those which are completely contained in one stratum except at possibly finitely many points, as orbicurves in that toric Kähler orbifold. Therefore we may think of all of this stratum by stratum.

It is also very useful to characterize the invariant holomorphic curves in a symplectic toric manifold via Morse theory.

Proposition 5.2.1. *Let M be a symplectic toric manifold. Let S be a component of a critical submanifold of the moment map. Choose a T invariant complex structure on M, so that $-\nabla|\mu|^2$ is Morse-Smale with respect to a J-compatible metric. Let γ be a gradient trajectory. Then the integral surfaces of the distribution given by $\dot{\gamma}$, and $J\dot{\gamma}$ are the T-invariant J-holomorphic spheres.*

Proof. J-Holomorphicity follows from the definitions of the almost complex structures, and compatible metrics. We have chosen all of these to be T-invariant. To see that these are the only such curves, suppose that there is a curve u which is not made up of flow lines as above. We know that such a curve must be a complex submanifold, hence it must be J invariant, moreover if it fails to be tangent to some gradient trajectory of μ, then the flow perturbs the curve, hence it is not invariant. □

This immediately implies

Corollary 5.2.1. *In a simply connected toric symplectic manifold with compatible, invariant T-invariant metric, symplectic form and compatible almost complex structure J, the boundary of the moduli space of T-invariant J-holomorphic curves consists entirely of gradient spheres attached at poles.*

5.2.2 Symplectizations

When the target manifold is the symplectization of a contact manifold there are some important differences between the behavior of these curves and the behavior of J-holomorphic curves in compact symplectic manifolds. We still have a useable version of Gromov compactness, but we have the interesting relationship between finite energy curves and periodic orbits of the Reeb vector field on the contact manifold. Before we describe the Morse-Bott chain complex we need to describe the moduli spaces of pseudoholomorphic curves with which we will be working. So, as before let (M, ξ) be a contact manifold, α a contact 1-form, $(\mathbb{R} \times M, \omega = d(e^t\alpha)$ its symplectization, let J_0 be an almost complex structure on ξ, extend J_0 to an almost complex structure J on the symplectization by declaring R_α to be the imaginary part of the complex line defined by the trivial Reeb line bundle and the t direction in the symplectization. The curves that we are interested in are J-holomorphic maps from punctured S^2's into the the symplectization of our contact manifold. Such curves are said to be *asymptotically cylindrical* over closed Reeb orbits.

First we need some definitions. Let $\mathcal{P}(\alpha)$ be the set of periodic Reeb orbits.

Definition 5.2.3. *Let (M, ξ) be a contact manifold with contact form α. The **action spectrum**,*

$$\sigma(\alpha) = \{r \in \mathbb{R} | r = \mathcal{A}(\gamma), \gamma \in \mathcal{P}(\alpha)\}$$

.

Definition 5.2.4. *Let $T \in \sigma(\alpha)$. Let*

$$N_T = \{p \in M | \phi_p^T = p\},$$

$$S_T = N_T / S^1,$$

*where S^1 acts on M via the Reeb flow. Then S_T is called the **orbit space** for period T.*

When M is the total space of an S^1-orbibundle the orbit spaces are precisely the orbifold strata.

For our Morse-Bott set-up we assume that our contact form is of Morse-Bott type, i.e.

Definition 5.2.5. *A contact form is said to be of Morse-Bott type if*

i. The action spectrum:

$$\sigma(\alpha) := \{r \in \mathbb{R} : \mathcal{A}(\gamma) = r,\ for\, some\, periodic\, Reeb\, orbit\, \gamma.\}$$

is discrete.

ii. The sets N_T are closed submanifolds of M, such that the rank of $d\alpha|_{N_T}$ is locally constant and

$$T_p(N_T) = ker(d\phi_T - I).$$

Remark 5.2.1. *These conditions are the Morse-Bott analogues for the functional on the loop space of M. A contact form which is generic in the sense that Reeb orbits are isolated are the Morse analogue, in that the corresponding submanifolds N_T are all 0 dimensional. We will say such a form is of **Morse** type.*

Let Σ be a Riemann surface with a set of punctures

$$\Gamma = \{z_1, \ldots, z_k\}.$$

In the following s, t are to be thought of as cylidrical local coordinates centered at a puncture, s is the radial coordinate, t is the angular coordinate.

Definition 5.2.6. *A map*

$$\tilde{u} = (a(s,t), u(s,t)) : \Sigma \setminus Z \to \mathbb{R} \times M$$

*is called **asymptotically cylindrical** over the set of Reeb orbits*

$$\gamma_1, \ldots, \gamma_k$$

if for each $z_j \in \Gamma$ there are cylindrical coordinates centered at z_j such that

$$lim_{s\to\infty} u(s,t) = \gamma(Tt)$$

and

$$lim_{s\to\infty} \frac{a(s,t)}{s} = T$$

Let us define the Hofer energy, which is the energy that we are talking about when discussing holomorphic curves in symplectizations.

Definition 5.2.7. *Let $\phi : \mathbb{R} \to [0,1]$ be continuous and non-decreasing. Then we define the Hofer energy, or α energy to be*

$$E(\tilde{u}) = sup_{\phi} \int_{\Sigma} \tilde{u}^* d(\phi\alpha).$$

The Hofer energy is related to the symplectic area of a holomorphic curve

Definition 5.2.8 (Area of a J-holomorphic curve)**.**

$$A(\tilde{u}) = \int_{\Sigma} u^* d\alpha$$

These two notions are related:

Proposition 5.2.2. *The following are equivalent for J-holomorphic curves into a symplectization:*

i. $A(u(\tilde{s},t)) < \infty$ and $u(\tilde{s},t)$ is proper.

ii. $E(u(\tilde{s},t)) < \infty$ and $a(s,t)$ is not bounded in any neighborhood of a puncture of Σ.

The energy and area are easy to compute, the energy is given as the sum of the actions of positive puncture. The area is the difference of the actions of the orbits corresponding to positive punctures and the actions of the negative ones.

Here are some important facts from [BEH$^+$03]:

Proposition 5.2.3. *Suppose that α is of Morse, or Morse-Bott type. Let*

$$\tilde{u} = (a, u) : \mathbb{R} \times S^1 \to (\mathbb{R} \times M, J)$$

be a J-holomorphic curve of finite energy. Suppose that the image of u is unbounded in $\mathbb{R} \times M$. Then there exist a number $T \neq 0$ and a periodic orbit γ of R_α of period $|T|$ such that

$$lim_{s\to\infty} u(s,t) = \gamma(Tt)$$

and

$$lim_{s\to\infty} \frac{a(s,t)}{s} = T.$$

This immediately implies

Proposition 5.2.4. *Let (Σ, j) be a closed Riemann surface and let*

$$Z = \{z_1, \ldots, z_k\} \subset S$$

be a set of punctures. Every J-holomorphic curve

$$\tilde{u} = (a, u) : (\Sigma \setminus Z) \to \mathbb{R} \times M$$

of finite energy and without removable singularities is asymptotically cylindrical near each puncture z_i over a periodic orbit γ_i of R_α.

These propositions are extremely important to us because they show that it is reasonable to define gradient trajectories between Reeb orbits to be J-holomorphic curves in the symplectization. We have even more, i.e., a Gromov compactness theorem, which says that although the moduli spaces are not necessarily compact, we can compactify them by adding stable curves of height 2.

Let us now give the symplectization version of the definition for stable maps. We call such a map a level k holomorphic map, or a level k **holomorphic building**.

We will also call a tree of spheres or more generally of Riemann surfaces a nodal Riemann surface, where the nodes refer to the intersection points between the various components. Since we are mapping into a symplectization we specify a set of punctures, separate from the marked points and the nodal points. From the previous statements of this section we know what the asymptotics are like of a pseudoholomorphic map $\tilde{u} : \Sigma \to \mathbb{R} \times M$ near a puncture. We would now like to describe the compactification of the moduli spaces of such curves. As in the case of holomorphic maps into compact symplectic manifolds the moduli spaces are not in general compact, however we can give a good compactification by adding the analogue of nodal curves, i.e., holomorphic buildings of a bigger height.

Definition 5.2.9. *A level k holomorphic map from a nodal Riemann surface into $\mathbb{R} \times M$ is a collection, Σ_i, $i = 1, \ldots, k$ of disjoint unions of Riemann surfaces and a collection of J-holomorphic maps $\tilde{u} : \Sigma_i \to \mathbb{R} \times M$. These Riemann surfaces are obtained for a nodal Riemann surface by removing all nodes and labelling each connected component with an integer between 1 and k. This labelling is not necessarily distinct. Σ_i is the union of connected components with labelling k. If two components of the nodal Riemann surface share a node, then their labellings may differ by at most 1. For each u_i, we treat the nodes as punctures and, if two levels intersect at a node we must have that the positive asymptotics for the i-th level are the negative asymptotics for the $i+1$-st level. Such a map is called stable if for each component with 0 area and genus 0 has at least 3 special points.*

It is a theorem see [BEH$^+$03] or [Bou02], that every sequence of finite energy level k curves has a sequence which converges in an appropriate sense to one of level $k' > k$, hence the moduli space of curves of all levels is compact. We make a note that in order to set up Morse-Bott contact homology in full rigor, we need to introduce a different notion of holomorphic building, where we add auxillary Morse functions, whose gradient trajectories intersect the holomorphic curves near the limits of each level. For this further construction we direct the reader to [BEH+03] and [Bou02].

Now, we want to know what is the dimension of the moduli space. Let us first suppose that the linearized Poincaré return map about each periodic Reeb orbit has no eigenvalue equal to 1, i.e., that α is Morse. We then denote the moduli space of such finite energy genus 0 J-holomorphic curves with r marked points, 1 positive puncture and s negative punctures into the symplectization V representing the homology class A and asymptotically cylindrical over the closed Reeb orbits

$$\gamma_0, \gamma_1, \dots, \gamma_s$$

by

$$\mathcal{M}^A_{0,r}(s|V, \gamma_0, \gamma_1, \dots, \gamma_s).$$

The dimension of this moduli space is given by

$$\mu_{CZ}(\gamma_0) - \sum_{i=1}^{s}(\mu_{CZ}(\gamma_s)) + (n-3)(1-s) + 2c_1(A) + 2r.$$

For contact forms of Morse-Bott type we actually consider different types of moduli spaces. Here we look at holomorphic curves whose asymptotics are projected by the Reeb action into some S_{T_j} near punctures. We will write such moduli spaces as

$$\mathcal{M}^A_{0,r}(s|W, S_{T_1}, \dots, S_{T_s}).$$

This is to be understood as the space of J-holomorphic curves in the symplectization, W of M with asymptotics as decribed above with r marked points, s punctures, and which represent $A \in H_2(M) = H_2(M, \mathbb{Z})/torsion$). In this notation the first orbit space is from the *positive* puncture, all others are negative. These moduli spaces are the analogues of the gradient trajectories of Morse theory. We only count them when they come in zero dimensional families (after a quotient by the $\mathbb{R}$-translation). Thus we need a dimension formula for these spaces.

Proposition 5.2.5. *The virtual dimension for the moduli space of generalized genus 0 J-holomorphic curves asymptotic over the orbit spaces*

$$S_{T_0}, \ldots, S_{T_1}, \ldots, S_{T_s}$$

(with 1 positive, and s negative punctures) representing A is equal to

$$(n-3)(1-s) + \mu(S_{T_0}) + \frac{1}{2}dim(S_{T_0}) - \sum_{i=0}^{s}(\mu(S_{T_i}) + \frac{1}{2}dim(S_{T_i})) + 2c_1(\xi, \Sigma),$$

where Σ is a Riemann surface used to define the symplectic trivialization and homology class A.

In cylindrical contact homology, since we only are keeping track of cylinders, we take $s = 1$ and this formula reduces to

$$\mu(S_{T_+}) + \frac{1}{2}dim(S_{T_+}) - \mu(S_{T_-}) + \frac{1}{2}dim(S_{T^-}) + 2c_1(\xi, \Sigma).$$

Of course if ξ has a regular structure this boils down to

$$\mu(S_{T^+}) - \mu(S_{T^-}) + 2n - 2 + 2c_1(\xi, \Sigma).$$

For a proof of this formula see [Bou02]. Bourgeois' proof is of interest as traditionally these kinds of results come from a spectral flow analysis. Bourgeois, however, makes interesting use of the Riemann-Roch theorem.

We want to understand the structure of the moduli space since our Morse-like chain complex uses these curves to construct the differential. The reader should be aware that the formula for the dimension of the moduli space above is really a *virtual* dimension until some sort of transversality is achieved for some $\bar{\partial}_J$-operator. This formula is obtained via Fredholm analysis on the space of C^∞ maps from $S^2 \setminus \{z_1, z_2, \ldots, z_j\}$ into V. The $\bar{\partial}_J$ operator turns out to be a Fredholm section of a certain infinite dimensional bundle over this space whose kernel is precisely the set of J-holomorphic curves. The Fredholm index $\bar{\partial}_J$ is the dimension formula above.

The trouble is that a priori, we cannot rule out a non-zero cokernel of the linearized operator, hence our dimension formula could be *under counting* the relevant curves. There have been many attempts at transversality proofs, and it seems as though the new *polyfold* theory of Hofer, Zehnder and Wysocki [HWZ07] is a very strong candidate to solve the problem. There are also proofs using virtual cycle techniques (cf. [Bou02]); however, even here it seems that there may be potential gaps. Therefore we show how, in some cases, we can justify the validity of our curve counts through more elementary geometric considerations. Note that even with these abstract constructions it may still be that the moduli space fails to be a manifold or even an orbifold. In the cases that we are considering in this thesis, the almost complex structure will be integrable, thus we can use algebro-geometric techniques to find conditions for regularity of J.[1]

Now let us describe the relationship between moduli spaces of stable curves in a symplectic orbifold and the moduli space of curves into the symplectization of its Boothby-Wang manifold. Notice that the symplectization W is just the associated line (orbi)bundle to the principle S^1-(orbi)bundle[2], M, with the zero section removed. Given as many marked points as punctures we actually get a fibration, here curves upstairs are sections of L with zeroes of order k and poles of order l once we fix the phase of a section we actually get unique curves. This is described for the case of *regular* contact structures in [EGH00]. For S^1-bundles over $\mathbb{CP}^1$ with isolated cyclic singularities Rossi extended this result in [Ros]. We will actually need an extension of this to higher dimension. The point here is that we want to coordinate our curve counts upstairs with the "Gromov-Witten" curve count downstairs. In the case where the base is an orbifold we must make sure that we can get an appropriate curve in the sense of Gromov-Witten theory on orbifolds. It should be noted that in

[1]The word *regularity* is over used. Here we mean that for this J, the $\bar{\partial}_J$ operator is surjective, as a section in a suitable infinite dimensional vector bundle.

[2]Of course, in the situations we are dealing with in this thesis, M is a manifold even if it is the total space of an orbibundle.

symplectizations all moduli spaces come with an $\mathbb{R}$-action by translation. Whenever we talk about 0-dimensional moduli spaces, we really mean that we are considering 1-dimensional moduli spaces under the quotient by the $\mathbb{R}$-action giving 0-dimensional manifolds (or possibly something more general, like a branched orbifold) .

The following lemma comes from [CR02].

Lemma 5.2.3. *Suppose that u is a J-holomorphic curve into the symplectic orbifold $\mathcal{Z}$, then either u is completely contained in the orbifold singular locus or it intersects it in only finitely many points.*

We use this to prove:

Lemma 5.2.4. *Let $u : \Sigma \to \mathcal{Z}$ be a non-constant J-holomorphic map between a Riemann surface and a symplectic orbifold. Then there is a unique orbifold structure on Σ and a unique germ of a C^∞-lift $\tilde{u}$ of u such that u is an orbicurve.*

Proof. First let us assume that the marked points are all mapped into the singular locus, since otherwise by the lemma the curve only intersects the singular locus in a finite number of points and we put the obvious orbifold structure on the sphere. Now u_{z_i} corresponds to a closed Reeb orbit of non-generic period, i.e., a curve in S_{T_k}, say. Take an element from the moduli space of curves into W asymptotically cylindrical over S_{T_k} in some slot. We need only to take a local uniformizing chart equivariant with respect to $\mathbb{Z}_{T_k}$. □

From this we actually get a fibration.

Proposition 5.2.6. *There is a fibration*

$$pr : \mathcal{M}_{0,r}(k|S_{T_1}; S_{T_2}, \dots, S_{T_k}) \to \mathcal{M}_{0,k}(a_1, \dots, a_k).$$

Proof. Given a homotopy of curves with k marked points and given intersection configurations, we use the previous lemma to make these intersections map into the singular locus in these finitely many points. Each of these curves lifts to an element of $\mathcal{M}_{0,r}(S_{T_1}; S_{T_2})$. Given a homotopy of such curves in the base, with all of the marked points preserving the intersection constraints in S_{T_k} , we move along a cylinder in N_{T_k} transversely to the Reeb foliation. Hence given a choice of initial condition we may lift a homotopy, and we have a fibration. □

5.3 Transversality results

First we would like to treat an orbifold splitting principle for orbifold structures on $\mathbb{CP}^1$. The proof here is adapted from the smooth case in Griffiths and Harris. So let $\triangle = z_1, \ldots, z_N$ be a divisor on $\mathbb{CP}^1$ with local uniformizing groups $\mathbb{Z}_{m_i}$ for each i, defining an orbifold structure, $(\mathbb{CP}^1, \triangle)$. Let H^k denote the hyperplane bundle of rank k. In the following given a module or family of modules S, S_x denotes the skyscraper sheaf at the point x. Explicitly, let U be an open set in M, then $S_x(U) = S$ if $x \in U$, and the zero module otherwise.

Theorem 5.3.1. *Let E be a vector orbibundle over $(\mathbb{CP}^1, \triangle)$. Then E splits as a sum of line orbibundles.*

Proof. The main thing here is to use orbifold Riemann-Roch, for a particular line bundle built from orbisections of E. We use Riemann-Roch to get a lower bound on the number of zeroes of a section of E. Next prove for ranks 2 bundles, and get the general case by induction. First as in [GH78], E splits as a sum of line bundles if and only if $E \otimes H^k$ does for any k. Consider the exact sequence:

$$0 \to \mathcal{O}(E \otimes H^{k-1}) \to \mathcal{O}(E \otimes H^k) \to E_x \otimes H^k_x \to 0.$$

From the long exact sequence in cohomology we see that if $H^1(\mathbb{CP}^1, \mathcal{O}(E \otimes H^k)) = 0$ then the vector orbibundle $E \otimes H^k$ has a global holomorphic orbisection, σ. Let us now suppose that σ vanishes at n points, then there is another section with poles at those n points, (since this is $\mathbb{CP}^1$). We must be careful, however, that these points are not any of the divisors in $\triangle$. We multiply these two section to get a new one $\sigma^{'}$. Now, σ and $\sigma^{'}$ are never both 0, and they are always linearly dependent, hence they span a line orbibundle L. By Kawasaki-Riemann-Roch [Kaw79] we have

$$h^0 = c_1(L) + 1$$

so any global section of L has *at most* $h^0 - 1$ zeroes. With that bound set we now begin the induction argument. We assume that E has rank 2. Let

$$n = c_1(L)$$

where L is the orbibundle generated by a global section of E with the maximal number of zeroes. Let $L^{'} = E/L$. Then following [GH78] but being careful about strata we get:

$$c_1(L^{'}) \leq c_1(L).$$

Now we consider the exact sequence of orbibundles

$$0 \rightarrow Hom(L^{'}, L) \rightarrow Hom(L^{'}, E) \rightarrow Hom(L^{'}, L^{'}) \rightarrow 0.$$

Since we know that $c_1(L^{'}) \leq c_1(L)$ we have

$$H^1(\mathbb{CP}^1; \mathcal{O}(Hom(L^{'}, L))) = 0,$$

which implies that the global section of $Hom(L^{'}, E)$ map onto those of $Hom(L^{'}, L^{'})$. This means that L, and $L^{'}$ span E.

Now the induction proceeds as in [GH78]. □

Theorem 5.3.2 (Regularity criterion for genus 0 moduli spaces at a curve u when J is integrable)**.** *Suppose that J is integrable and that*

$$\langle c_1(L_j), A\rangle \geq -2+s-t$$

for every $A \in H_2(\mathcal{Z})$ which is represented by a 2-sphere. Then the linearized Cauchy-Riemann operator is surjective and the genus 0 *moduli space of curves with s positive punctures and t negative punctures is a smooth manifold of dimension given by the Fredholm index. In the case that $\mathcal{Z}$ is an orbifold, we require that*

$$c_1(L_j) \geq \sum_\alpha (1-\frac{1}{m_\alpha})c_1(\mathcal{O}(D_\alpha)) - 2 - s - t,$$

where D_α are branch divisors.

Proof. Let $z_1, \ldots, z_s, \ldots z_{s+t}$ be distinct points on S^2. Consider the divisor

$$D = k_1 z_1 + \ldots + k_s z_s - k_{s+1} z_{s+1} - \ldots - k_{s+t} z_{s+t}.$$

Then the Cauchy-Riemann operator is just the $\bar\partial$-operator of the Dolbeault complex for the line orbibundle $L_j \otimes \mathcal{O}(D)$.

$$\bar\partial : \Omega^0(\mathbb{CP}^1, L_j \otimes \mathcal{O}(D)) \to \Omega^{0,1}(\mathbb{CP}^1, L_j \otimes \mathcal{O}(D)).$$

The cokernel of $\bar\partial$ is just the $(0,1)$ cohomology of that complex. But we have the following isomorphisms:

$$H^{0,1}_{\bar\partial}(\mathbb{CP}^1, L_j \otimes \mathcal{O}(D)) \simeq H^{1,0}_{\bar\partial}(\mathbb{CP}^1, (L_j \otimes \mathcal{O}(D))^*)^* \simeq H^{0,1}_{\bar\partial}(\mathbb{CP}^1, (L_j \otimes \mathcal{O}(D))^* \otimes K).$$

For the last group to be 0, we must have

$$c_1(L_j \otimes \mathcal{O}(D))^* \otimes K) < 0.$$

This happens whenever

$$c_1(L_j) > -2 - deg(D).$$

□

Theorem 5.3.3.

$$\langle c_1(L), A\rangle \geq -2 + s - t$$

whenever L is a line (orbi)bundle obtained by the Boothby-Wang fibration whose total space is either of the following:

i. a homogeneous contact manifold

ii. a toric Fano contact manifold.

Proof. The proof of (i) is nearly the same Proposition 7.4.3 in [MS04] with u^*TM replaced with $u^*(\xi)$. Note that in case (i) we are always dealing with manifolds at each level rather than with orbifolds. For (ii), we apply the orbifold version of the splitting principle. Then, for a splitting of

$$u^*(\xi) = \bigoplus_j L_j$$

we get sections and positivity of Chern classes via the Fano condition. □

Corollary 5.3.1. *For homogeneous and toric contact manifolds the dimensions of all genus* 0 *moduli spaces into symplectizations are given by the Fredholm index as predicted.*

Remark 5.3.1. *These arguments actually fall short in the arguments for invariance of contact homology, since we must use the dimension formula for the moduli spaces in fairly general symplectic cobordisms. However, these arguments show that in the symplectizations,* or, *in a holomorphic filling with the assumptions given above we can use the dimension formulae effectively.*

5.4 A few words about Gromov-Witten invariants

In this section we layout some framework and definitions for Gromov-Witten invariants and the so-called Gromov-Witten potential for compact symplectic manifolds and orbifolds since it is used in the last chapter. In this thesis we only consider the genus 0 invariants. The Gromov-Witten invariants that we are interested in occur in the base orbifold $\mathcal{Z}$ of a Boothby-Wang $(M, \mathcal{Z})$ with $dim(M) = 5$. Hence we are in the semipositive case and we can define the Gromov-Witten invariants as in [MS04]. Our version of Gromov-Witten theory for symplectic orbifolds comes from [CR02]. The main difference here is that our marked points, and hence our cohomology classes taken as arguments for the invariant have constraints determining in which orbifold stratum the curves in question lie. This is an issue since some homology classes may live in several strata.

Roughly speaking a Gromov-Witten is a count of rigid J-holomorphic curves representing a homology class $A \in H_2(M) := H_2(M, \mathbb{Z})/torsion)$ in general position with marked points in a symplectic manifold M for which the marked points are mapped into the Poincaré duals of certain cohomology classes. For example we may ask how many spheres, (or lines), intersect 2 generic points in $\mathbb{CP}^n$. In this case we have 2 marked points, a top cohomology class, and for A the class of a line, $[L]$.

To make this precise let (M, ω) be a compact symplectic manifold, let J be an ω-compatible almost complex structure. Consider the moduli space

$$\mathcal{M}_{0,k}^{A}(M, J)$$

of genus 0 stable J-holomorphic curves into M representing the class A and assume here that we have regularity of the relevant linearized Cauchy-Riemann operator for the class A, either via circumstances as in section 5.3 or by some sort of abstract perturbation argument. Note also that when we discuss Gromov-Witten theory for compact symplectic manifolds we will consider only *somewhere injective* curves. We define maps

$$ev_j : \mathcal{M}_{0,k}^A(M, J) \to M$$

and

$$ev : \mathcal{M}_{0,k}^A(M, J) \to M^{\times k}$$

by evaluation at the marked points.

By semipositivity the evaluation map represents a submanifold of $M^{\times k}$ of dimension

$$2n + \langle 2c_1(M), A\rangle + 2k + 6.$$

Now we define the Gromov-Witten invariant as a homomorphism

$$GW_{A,k}^M : H^*(M)^{\otimes k} \otimes H_*(\mathcal{M}_{0,k}^A(M.J)) \to \mathbb{Q}$$

encoded formally as the integral

$$GW_{A,k}^M(\alpha_1, \dots, \alpha_k) := \int_{\mathcal{M}_{0,k}^A(M.J)} ev_1^*\alpha_1 \cup \cdots \cup ev_k^*\alpha_k \cup \pi^*[\mathcal{M}_{0,k}^A(M.J)].$$

This is the definition for manifolds. This definition can be used without the semipositivity condition as long as there is a construction of an appropriate object on which to integrate. Since we will be working in dimension 4 this will not be an issue.

To extend this definition to orbifolds, there are issues with the definitions of J-holomorphic curves, since the idea of a map between orbifolds can be a rather sticky issue. We content ourselves, here, to know that we have a notion of *good* map, and we will defer to [CR02] for the analytic set-up. With that said, we still must extend the definition above so that it makes sense in a stratified space. We should also note that the orbifold cohomology of Chen and Ruan is *not* the same as the cohomology that we set up in section 2.2. This cohomology is simply a way to organize how various classes interact with the stratification of the orbifold. As in the manifold case we start with a compact symplectic orbifold, $\mathcal{Z}$ and pick a compatible almost complex structure J. We then consider moduli spaces of (genus 0) J-holomorphic orbicurves into M representing a homology class $A \in H_2(\mathcal{Z}; \mathbb{Q})$. But we now need to consider a new piece of data which organizes the intersection data so that it is compatible with the stratification. The extra data will be defined by a tuple $\mathbf{x}$, of orbifold strata, $(\mathcal{Z}_1, \dots, \mathcal{Z}_k)$. The length k of this tuple should coincide with the number of marked points. We will write such a moduli space as

$$\mathcal{M}^A_{0,k}(\mathcal{Z}, J, \mathbf{x}),$$

and require that the evaluation takes the j-th marked point into $\mathcal{Z}_j$. The compactification is similar to the manifold case, and consists of stabe maps with the obvious adjustments, the caveat being that we must choose our lift to an orbicurve. After an appropriate construction of cycles as in the manifold case, Chen and Ruan use a virtual cycle construction we can define this invariant as in the smooth case above, but we integrate over (the compactification of) $\mathcal{M}^A_{0,k}(\mathcal{Z}, J, \mathbf{x})$. We will write these invariants

$$GW^{\mathcal{Z}}_{A,k,\mathbf{x}}(\alpha_1, \dots, \alpha_k).$$

Another key difference is that the difference of this moduli space differs from the predicted dimension in the smooth case by a factor of $-2\iota(\mathbf{x})$, the so-called *degree shifting number*, again for this definition see [CR02]. The Gromov-Witten invariants satisfy a list of axioms developed by Manin and Kontsevich. We will not list all of these but mention some which will be used later on. We will use the orbifold notation, for a manifold we would just delete $\mathbf{x}$ from the notation, setting $\iota(\mathbf{x}) = 0$.

i. **Effective**: $GW^{\mathcal{Z}}_{A,k,\mathbf{x}}(\alpha_1, \dots, \alpha_n) = 0$ as long as $\omega(A) < 0$.

ii. **Grading**: $GW^{\mathcal{Z}}_{A,k,\mathbf{x}}(\alpha_1, \dots, \alpha_n) \neq 0$ only if

$$\sum_j deg(\alpha_j) = dim(\mathcal{Z}) + 2c_1(A) + 2k - 6 - 2\iota(\mathbf{x}).$$

iii. **Divisor**: Let $\mathbf{x}^j = \mathbf{x}$ with the jth component removed. If $deg(\alpha_n) = 2$ then

$$GW^{\mathcal{Z}}_{A,k,\mathbf{x}}(\alpha_1, \ldots, \alpha_n) = (\int_A \alpha_n) GW^{\mathcal{Z}}_{A,k-1,\mathbf{x}^n}(\alpha_1, \ldots, \alpha_{n-1}).$$

We make no claim that these are the most important axioms for the Gromov-Witten invariants, they are just the ones which are used explicitly later in the thesis. Now we are in a position to define the Gromov-Witten potential. This is a generating function which gives a formal power series whose coeffecients give Gromov-Witten invariants. It is a way to organize all the information from these invariants into one big package.

We give the definition here for the manifold case. Pick a basis of $H^2(M)$, $a_1, \ldots, a_n$, for a vector t and a cohomology class a, write $a := a_t = \sum_i t_i a_i$.

Definition 5.4.1. *Let (M, ω) , J be as above. Define the genus 0* ***Gromov-Witten Potential*** *as*

$$\boldsymbol{f}(a_t) = \sum_A \sum_k \frac{1}{k!} GW^M_{A,k}(a_t, \ldots, a_t) z^{c_1(A)}.$$

The corresponding formula for orbifolds is obtained by accounting for the vector $\boldsymbol{x}$*.*

Chapter 6

Contact Homology

6.1 Cylindrical contact homology, a Floer type theory

Let (M, ξ) be a contact manifold, let α be a contact 1-form for α. Consider the action functional

$$\mathcal{A} : C^{\infty}(S^1, M) \to \mathbb{R},$$

where

$$\mathcal{A}(\gamma) = \int_{\gamma} \alpha.$$

We would like to do Morse theory for this functional. The first thing is to understand the critical points.

Proposition 6.1.1. *The critical points of $\mathcal{A}$ are periodic orbits of R_α.*

Proof. We consider a 1-paramater family of loops, $\gamma_t = \gamma + t\gamma'$ and differentiate with respect to t and evaluate at $t = 0$.

$$\frac{d}{dt}\int_{\gamma_t} \alpha_{|t=0} = \frac{d}{dt}\int_0^1 {\gamma_t}^{*}\alpha_{|t=0} = \int_0^1 \mathcal{L}_X \alpha = \int_0^1 d i_X \alpha + i_X d\alpha = \int_0^1 i_X d\alpha.$$

This quantity is zero if the Reeb vector field is tangent to the flow associated to the family of maps at $t = 0$ which means that γ was a periodic Reeb orbit to begin with. □

Unfortunately $\mathcal{A}$ has an infinite dimensional kernel and cokernel, so we will have to find a suitable Morse index. This will be taken care of by the Conley-Zehnder index defined earlier.

The question now is, what are the "gradient trajectories" of $\mathcal{A}$? After choosing a compatible J on the symplectization of M and hence on ξ we choose the natural metric,

$$g(v, w) = d\alpha(v, Jw)$$

on $T(M)$. From this metric we take

$$\int_0^1 g_{\gamma(t)}(X(t), Y(t))dt$$

on $T(\Lambda(M))$, where $T_\gamma(\Lambda) = C^\infty(S^1, \gamma^*TM)$. Then the gradient trajectories of $\mathcal{A}$ with respect to this metric are given by

$$\nabla\mathcal{A}(\gamma) = J\pi(\dot{\gamma}),$$

where

$$\pi(v) = \alpha(v)R_\alpha$$

is the projection onto the Reeb direction. Thus a gradient trajectory $u(t, s)$ satisfies

$$\frac{\partial u}{\partial s} = J\pi\frac{\partial u}{\partial t}.$$

Now we extend all of this to the symplectization applying the rule $J\frac{\partial}{\partial t} = R_\alpha$. Then setting $U = (u, \phi)$ where ϕ maps into the real direction of the symplectization we get

$$\frac{\partial U}{\partial s} = J\pi(\frac{\partial u}{\partial t}) + \frac{\partial \phi}{\partial t}R_\alpha(u)$$

$$\frac{\partial\phi}{\partial t} = -\langle\frac{\partial u}{\partial t}, R_\alpha(u)\rangle.$$

Solutions of these equations are cylinders connecting critical points of $\mathcal{A}$. Hence we can use these J-holomorphic cylinders as the differential of contact homology. In other words, these "gradient trajectories" will provide us with the differential for our "Morse" complex. Of course, as we shall see this only works in a fairly limited framework.

To define the grading we first perturb the contact 1-form so that its periodic orbits are isolated, or equivalently that the linearized Poincaré return map with respect to ξ and a periodic orbit γ of the Reeb vector field has no eigen-value equal to 1.

We then choose a Riemann surface Σ with boundary γ, trivialize ξ over Σ. We may then define the Conley-Zehnder index to be the Conley-Zehnder index, as defined before, of this path of symplectic matrices. Intuitively this index describes to what extent nearby orbits "wrap" around γ.

For technical reasons having to do with orientability of the relevant moduli spaces, we must exclude certain orbits.

Definition 6.1.1. *A closed Reeb orbit γ is called **bad** if it is a multiple cover of another Reeb orbit γ' and $\mu_{CZ}(\gamma) \neq \mu_{CZ}(\gamma') \mod 2$. If γ is not bad then it is called **good**.*

Now we try to take a chain complex to be the free complex generated by the good closed Reeb orbits with coeffecients in $\mathbb{Q}[H_2(M;\mathbb{Z})/torsion]/\mathcal{R}$ where $\mathcal{R} \subset ker(c_1(\xi))$. When c_1 vanishes on $\pi_2(M)$ then we can take rational coeffecients ignoring homology classes. We are concerned with various choices of the coefficient ring either to fix choices for grading issues, or to pick out the correct information.

We now define the differential, or, at least, we make an attempt at a definition of a differential which looks like the one from Morse theory. We will actually have to modify the differential depending on the situation. The issue is that as we shall see the Floer-like cylindrical homology is not always defined as a reasonble homology theory independent of choices of 1-forms and almost complex structures. Because of this we define the full homology which is always defined.

Remark 6.1.1. *Once this is done, we may perform a linearizartion procedure. If cylindrical homology were well defined to begin with then this "linearization" would give the same information. If the cylindrical homology was not defined then we can make sense of counting cylinders. Each of these theories is defined by which J-holomorphic curves we are counting in each case. Anstractly there are many linearizations, but the most popular one in use requires fillability of the contact manifold. One uses the filling to define a natural augmentation on the chain complex, which when composed with the differential gives a linearized part of the homology.*

With that said, let us consider the Floer type cylindrical homology. Let W denote the symplectization of M. As we mentioned before, we would like to look at the chain complex generated freely by closed Reeb orbits, graded by the Conley-Zehnder index, with coeffecients in the group algebra $\mathbb{C}[H_2(M;\mathbb{Z})]$ (modulo torsion). We choose a basis $A_1, \ldots, A_N$ of $H_2(M) := H_2(M;\mathbb{Z})/torsion$. Then a multi-degree vector $d = (d_1, \ldots, d_N)$ determines each two dimensional homology class A. Now we define the differential as follows, set

$$n^A_{\gamma,\gamma'} = \#\mathcal{M}^A_{0,0}(1|W\gamma,\gamma')$$

whenever

$$dim(\mathcal{M}^A(\gamma,\gamma')) = 1,$$

and set this number to 0 otherwise. Now we can define the differntial

$$\partial\gamma = \sum_{d,\gamma'} \frac{n_{\gamma,\gamma'}}{\kappa_{\gamma'}}\gamma' z^d.$$

Here κ_γ denotes the multiplicity of γ. This gives a reasonable definition of a differential, since once we quotient the moduli space out by the $\mathbb{R}$ action of the real direction in the symplectization by translation, we get a 0-dimensional moduli space, which we have compactified. Hence we get a finite number of things to count. Note that there are signs which appear in the differential which are akin to the choice of signs in Morse theory. These signs arise from defining certain coherent orientation on the moduli spaces. They are motivated by the orientations of stable and unstable manifolds of a pair of critical points in the finite dimensional situation.

Proposition 6.1.2 ([EGH00]). *As long as there are no contractible periodic orbits with Conley-Zehnder index* 1 *for any choice of spanning disc, then* $\partial^2 = 0$.

This proposition works as long as long as we can make sure that the moduli space of curves behaves properly. Suppose, first, that the all moduli spaces are smooth manifolds (this is not always the case.) To be more precise we can always prove that $\partial^2 = 0$ as long as we know that the boundary of the moduli spaces of dimension 2 are made up of "broken" trajectories of cylindrical curves. In other words, we need to know that all holomorphic curves on the boundary come from curves connecting the two Reeb orbits in question via an intermediate orbit. Then we know that all curves on the boundary are of the same type as those on the interior. Then we have the boundary of a 1 manifold after the quotient by translation, and the algebraic count of the boundary of a 1 manifold is always 0. In the case where the moduli space in a branched orbifold with corners (which happens), for each 1 dimensional component we can control the branches in such a way that we may choose certian weights, where the relevant sums cancel in pairs this is exactly why the κ term describing multiplicity comes into the differential, it is controlling the branching behavior that arises due to multiple covers.

To be able to see that the boundary behavior is as claimed, we must rule out certain behavior on the boundary. First we recall that each finite energy curve which we are considering has precisely one positive puncture. When we compactify the moduli spaces we must allow holomorphic buildings of higher level. Since we must have the index of top and bottom curves equal to 1, the only possibility for the top level is a pair of pants with two legs. Suppose that there were more than two legs, then to avoid picking up genus when we glue to the lower curve, the lower level curve can consist only of a single trivial cylinder and a single holomorphic plane, as long as $c_1(\xi)$ is nonnegative for index reasons. In particular, when $c_1(\xi)$ vanishes, the lower curve must have total index 1, cylinder with a holomorphic plane bubbling off. But now we have an holomorphic plane asymptotically cylindrical over an orbit of index 1, this cannot occur by assumption.

Now we need a condition to ensure that the homology is independent of all choices, including α and J.

Proposition 6.1.3. *As long as there are no contractible Reeb orbits of index* $-1, 0$ *the homology of this chain complex does not depend on the choice of almost complex structure,* 1*-form, or choice of orientation.*

Note that this second proposition really requires abstract transversality of the linearized $\bar{\partial}_J$ operator, since the isomorphism which is defined in the proof requires a dimension formula for a moduli space in a symplectic cobordism, hence even in really nice cases, i.e., toric Fano, or homogeneous, for example, we cannot use special circumstances to conclude regularity. These conditions are way too restrictive so we need something more. If we keep counting curves with as many Reeb orbits as we need, then we can build a new differential, which works even in the presence of index $0, 1$, and -1 curves. The idea here is to look at the graded supercommutative *algebra* generated by closed Reeb orbits, with coeffecients in a suitable ring, possibly keeping track of some information about homology classes or J-holomorphic curves. This just corresponds to counting curves with an arbitrary number of negative punctures. This forces the differential to be a polynomial in the q variables, each representing a negative puncture. In other words we consider an algebra generated by Reeb orbits with differential given by

$$dp = \sum_{\alpha} n_{p,q_{\alpha_1},\ldots,q_{\alpha_k}} q_{\alpha_1} \cdots q_{\alpha_k}$$

The coeffecient counts the number of elements in the moduli space of curves asymptotically cyindrical over the Reeb orbit γ correspinding to p at the positive puncture, and over Reeb orbits corresponding to the q variables at the negative punctures whenever the dimension of this moduli space is 1. The coeffecient is 0 otherwise. We will visit a version of this in the last chapter.

Chapter 7

Morse-Bott Contact Homology

In this chapter we give the definitions for the Morse-Bott version of contact homology [Bou02]. In its original formulation, contact homology requires that the Poincarè return map constructed about any periodic Reeb orbit has no eigenvalue equal to 1. This condition is generic, however many natural contact forms, especially those which arise from circle orbibundles are as far from generic as possible. In order to calculate contact homology for such manifolds one must make some sort of perturbation. It is only in very nice situations that this is not extremely difficult. The Morse-Bott version allows us to use the symmetries of nice contact structures and symmetric almost complex structures, by, rather than excluding non-isolated orbits, welcoming them. This is accomplished by considering Morse theory on the quotient space, and relating critical points, and gradient trajectories of a Morse function to pseudoholomorphic curves in the symplectization of the contact manifold. Since toric contact manifolds of Reeb type are always total spaces of circle orbibundles, *and* they admit nice Morse functions, the Morse-Bott formalism works quite well for us. For convenience we repeat the following definitions which appeared earlier.

Definition 7.0.2. *Let (M, ξ) be a contact manifold with contact form α. The* ***action***

spectrum,

$$\sigma(\alpha) = \{r \in \mathbb{R} | r = \mathcal{A}(\gamma)\}$$

for γ a periodic orbit of the Reeb vector field.

Definition 7.0.3. *Let $T \in \sigma(\alpha)$. Let*

$$N_T = \{p \in M | \phi_p^T = p\},$$

$$S_T = N_T / S^1,$$

*where S^1 acts on M via the Reeb flow. Then S_T is called the **orbit space** for period T.*

When M is the total space of an S^1-orbibundle the orbit spaces are precisely the orbifold strata.

For our Morse-Bott set-up we assume that our contact form is of Morse-Bott type, i.e.

Definition 7.0.4. *A contact form is said to be of Morse-Bott type if*

i. The action spectrum:

$$\sigma(\alpha) := \{r \in \mathbb{R} : \mathcal{A}(\gamma) = r,\ for\, some\, periodic\, Reeb\, orbit\, \gamma.\}$$

is discrete.

ii. The sets N_T are closed submanifolds of M, such that the rank of $d\alpha|_{N_T}$ is locally constant and

$$T_p(N_T) = ker(d\phi_T - I).$$

Remark 7.0.2. *These conditions are the Morse-Bott analogues for the functional on the loop space of M.*

Notice that in the case of S^1 orbi-bundles this is always satisfied. The key observation, as we soon shall see, is that we can relate J-holomorphic curves to Morse theory on the symplectic base. Since we consider only quasi-regular contact manifolds here, we can always approximate the contact structure by one with a dense open set of periodic orbits of period 1, say, and a finite collection of strata of orbits of smaller period, each such stratum has even dimension and has codimension at least 2 (other than the dense set of regular points, of course.)

What we would like to do is relate Reeb orbits to Morse theory in each orbit space. This works since we can study J-holomorphic curves with "degenerate" asymptotics, meaning holomorphic curves which are asymptotically cylindrical over *some* closed Reeb orbit, in a particular orbit space, S_T, for $T \in \sigma(\alpha)$.

7.1 Orbits, strata, and all that

We look at orbits and strata. The orbit types are given exactly by the orbifold stratification of the symplectic base. Given a contact manifold with a contact form of Morse-Bott type, we know the following from [Bou02]. Remember that there is an open dense set in $\mathcal{Z}$ which correspond to Reeb orbits of a single orbit type.

Proposition 7.1.1. *Let γ be a periodic orbit of the Reeb vector field in the orbit space S. Then any other orbit in S has the same Maslov index.*

Proof. To prove this we assume that some Reeb orbits come in a k-dimsensional family. Choose a 1-dimensional subfamily and we parametrize this family by a cylinder via a map

$$\Phi : S^1 \times [0, T] \to M$$

such that for each fixed $t \in [0, T]$, $\Phi(\cdot, t)$ is a periodic Reeb orbit γ_t. Now suppose that $s, t, \in [0, T]$ and suppose that the associated periodic Reeb orbits have periods

T_1, T_2, then

$$T_1 - T_2 = \int_{\gamma_s} \alpha - \int_{\gamma_t} \alpha$$

By Stokes theorem we have

$$\int_{\gamma_s} \alpha - \int_{\gamma_t} \alpha = \int_{\Phi(S^1 \times [s,t])} d\alpha = \int_{S^1 \times [s,t]} \Phi^* d\alpha = 0.$$

□

We would like now to set up the Morse-Bott chain complex. This was originally done in [Bou02] and discussed for circle bundles in [EGH00]. We have already discussed some of the basic setup, now, much like the case in Morse theory we would like to relate the Morse-Bott case to the generic case. The idea is to perturb our contact structure so it's periodic orbits are in 1-1 correspondence with the critical points of some Morse-function. In our case we would like to use our moment maps to get a perfect Morse or Morse-Bott function f. So first for appropriate ϵ and our Morse or Morse-Bott function$f$ we take the new contact form

$$\alpha_f = (1 + \epsilon f)\alpha.$$

Then critical points of f correspond to periodic orbits of α_f. If f is a perfect Morse function as in the toric case we take this as our contact form. Otherwise, suppose f is Morse-Bott, then we choose Morse functions on each critical submanifold. Note that in the case of a Hamiltonian action of a compact Lie group all such submanifolds have even index and even dimension. By G-invariance of the moment map this restricts to all orbit spaces. In this case the periodic orbits are in 1-1 correspondence with critical points of Morse functions on the critical submanifolds of f. We want to relate the Conley-Zehnder indices of the generic form with those of the original. Since we have a 1-1 correspondence between critical points and orbits, we will think of the chain complex associated to α_f as critical points of f. The index is the grading so we'll write the Conley-Zehnder index of the orbit corresponding to p, $\mu_{CZ}(\gamma_p)$ as $|p|$.

In the Morse case we have

$$|p| = \mu(S_{T_k}) - \frac{1}{2}dim(S_{T_k}) + ind_p f.$$

Note we must throw out certain *bad* critical points.

Definition 7.1.1. *A critical point is called **bad** if it corresponds to a multiple cover of a Reeb orbit corresponding to another critical point and the difference of the two indices is odd. If a critical point is not bad it is called **good**.*

In the case of a Morse-Bott function, we just proceed as in ordinary Morse theory to get a new Morse function on each critical submanifold, but now in this formula we use the Morse-Bott indices.

Now we just define the chain complex to be the one generated by the good critical points, where our choice of coeffecient ring can vary as in the generic case.

Now we can define the differential for Morse-Bott contact homology.

$$dp = \partial p + \sum_q n_{pq} q.$$

Where ∂p is just the Morse-Smale-Witten boundary operator, n_{pq} is similar to the coeffecient in the generic case, and ind_p is the Morse index for critical points. Bourgeois proves, in his thesis [Bou02], that this homology computes contact homology. For us, the particular form of the differential does not matter much since it will vanish since the moduli spaces all have dimension at least 2..

Theorem 7.1.1. *(Bourgeois) When the homology defined above exists it is isomorphic to the standard contact homology for non-degenerate contact forms.*

7.1.1 Dealing with non-zero Chern classes of the contact distribution in the circle orbibundle case.

It is well known that the Conley-Zehnder and Robbins-Salamon indices for a periodic orbit depend on the choice of symplectic trivialization, i.e., on a choice of spanning surface, unless the first Chern class of the contact distribution is zero. This, however is not the end of the game. We just have to make certain choices to get a well defined grading. In other words for each Reeb orbit, we choose and fix a homology class of disc(in the simply connected case), or more generally a Riemann surface with boundary equal to γ. We then pull back the contact bundle to this Riemann surface and choose a symplectic trivialization. Note that this works just as well in the case of orbifolds from the analysis in the section concerning symplectic vector bundles.

If we wish to label things, we may attach a variable to each Reeb orbit with exponent given by the element of $H_2(M)$ which it bounds. However, in our simple examples the following proposition will allow us just to fix homology classes in the base.

Proposition 7.1.2. *Suppose* (M, ξ), $(M, \xi^{'})$ *are contactomorphic contact manifolds with two different quasiregular contact forms such that the bases are the same when we forget the orbifold structure. Suppose that there are no contractible Reeb orbits of index* $0, 1, -1$ *for either contact form. Suppose that we fix discs over which to trivialize the contact bundles, by choosing specific spheres in the base space, then restricting to these classes only, the subalgebra of contact homology computed only using these trivialization are isomorphic.*

Proof. We need to check that the chain map counting degree 0 curves in a symplectic cobordism between the two contact forms, respects the coeffecient ring. However, this is simple. If we assume that there are no orbits of index 0 with either choice of contact form, then we see that the map counting index 0 curves in the symplectic cobordism connecting the two different contact manifolds gives an isomorphism on cylindrical contact homology. We do not need to worry about the index $1, -1$ since we always have a $\mathbb{C}^*$ -action on the moduli space. If the indices do not match up for two orbits trivialized over sections over the same spherical class, then their periods and the evaluation of $c_1(T(\mathcal{Z}))$ must differ since in this case all Maslov indices are determined by the first Chern class of the tangent bundle of the base evaluated on our chosen class as we shall see in the next chapter. In this case the total spaces cannot be contactomorphic. □

Remark 7.1.1. *In this proposition we make use of the dimension formulae for moduli spaces of genus* 0 *curves in symplectic cobordisms. Thus, we cannot necessarily use our transversality results for integrable almost complex structures. We are, for the moment, forced to take transversality of the linearized Cauchy-Riemann operator as a hypothesis.*

Notice that this proposition allows us to make certain choices in order to compare contact homology using only a very restriced part of the full algebra. This is, essentially, how we make the necessary choices in the next section, and how we are able to say much when the first Chern class is non-zero. We note that some of the ideas in the last section of this thesis use a different approach to compare contact manifolds with non-zero Chern classes.

Chapter 8

Calculations for some Hamiltonian G-spaces

8.1 Index calculations

Let us first set some notation. Suppose first that (M, ξ) is compact, simply connected, and admits a strongly Hamiltonian action of a Lie group as discussed in the introduction which is of Reeb type. Then we know that there is a quasi-regular contact form α for (M, ξ) equivariant with respect to the action. As above, let S_{T_k} denote the stratum in $\mathcal{Z} = M/(S^1)$ corresponding to Reeb orbits of period T_k. Let Γ_j denote the local uniformizing group for the stratum S_{T_k}. Recall that each stratum is a Kähler sub-orbifold of $\mathcal{Z}$. In what follows assume that $H^*(\mathcal{Z}; \mathbb{C})$ is a truncated polynomial ring generated by elements in $H^2(\mathcal{Z}; \mathbb{C})$, i.e., the Chern classes coming from the symplectic reduction defining $\mathcal{Z}$ as a symplectically reduced orbifold. Let us write such a basis of $H^2(\mathcal{Z}; \mathbb{C})$ as $\{c_1, \dots, c_k\}$. Now choose 1 forms $\tilde{c}_j$ representing the c_j's. Now we just consider circle bundles over $\mathcal{Z}$ by choosing connection 1-forms

α with curvature

$$d\alpha = \sum_j \pi^* w_j \tilde{c_j}.$$

Notice that for $\mathcal{Z}$ a toric orbifold, this construction yields all possible toric contact structures of Reeb type. Note that we implicitly choose a symplectic form $\omega = \sum w_i \tilde{c_i}$ on $\mathcal{Z}$ during this process. Then

$$c_1(T(\mathcal{Z})) = \sum \tilde{w_i}[\tilde{c_i}],$$

where $\tilde{w_i}$ is obtained via the spectral sequence for the Boothby-Wang fibration.

Remark 8.1.1. *In the case of contact reduction in $\mathbb{C}^n$ by a circle (where the action is of Reeb type) the coeffecients of $|z_j|^2$ in the (circle) moment map can be chosen to be the $\tilde{w_j}$'s.*

Now we choose elements of $H_2(\mathcal{Z};\mathbb{Z})$, $A_1, \ldots, A_n$, with

$$\langle [\tilde{c_i}], A_i \rangle = 1.$$

This is possible because the cohomology is a truncated polynomial ring generated by the $[c_j]$, all elements having even degree. Now let

$$A = \sum_j A_j.$$

Then for any Kähler suborbifold

$$i : S \hookrightarrow \mathcal{Z},$$

$$\sum_i \langle i^*[\tilde{c_i}], A \rangle$$

is nonzero. Thus we can also do this for each S_{T_j} by pulling the Chern classes back along the inclusion maps, then choosing homology classes in each stratum as above in terms of $i_j^*[\tilde{c_i}]$, where $i_j : S_{T_j} \to \mathcal{Z}$ is the inclusion, and $\{[c_i]\}$ are the Chern classes generating $H^*((Z);\mathbb{C})$. Call the corresponding homology class $A_{S_{T_j}}$.

The purpose here is to find a nice diagonally embedded sphere with which to make our calculations. Now let's use this set-up to do some index calculations. First we must find suitable trivializations and capping disks for Reeb orbits. The idea here is to find two trivializations for each Reeb orbit, then use the loop property of the Maslov index to calculate the index via integration of $c_1(T(\mathcal{Z}))$ over the sphere obtained by gluing the two disks (from the symplectic trivializations) along their boundaries. The author first encountered this idea in [Bou02] and [EGH00], however this was only for the regular[1] case. So let $\gamma_{S_{T_j}}$ be a Reeb orbit of period T_j, living, of course, in the stratum S_{T_j}. We now pull back ξ via the inclusion map over S_{T_j}, i_j. For the first disk we just cap off a tubular neighborhood of the Reeb orbit given by the product framing for M. In this framing the Maslov index is 0, since the return map is always the constant path in $Sp(2n-2,\mathbb{R})$ given by the identity. Now we need another disk to glue along the Reeb orbit to get a sphere. In order to do this consider a holomorphic sphere, i.e., a map $u : S^2 \to S_{T_j}$ passing through $p \in S_{T_j}$ such that $[u] = A_{S_{T_j}}$. This is always possible since the moment map is invariant and since we assume $\mathcal{Z}$ is simply connected, the Hurewicz homomorphism is surjective. Now consider a holomorphic (orbi)section of L over our sphere with a zero of order equal to the multiplicity of γ and no pole. Such a section exists since we are talking about line (orbi)bundles over $\mathbb{CP}^1$. With this set-up we prove:

Lemma 8.1.1. *Let M be an S^1-bundle over a symplectic orbifold admitting a Hamiltonian action of a compact Lie group, such that its cohomology is generated by the Chern classes associated to the action. Then the Maslov index of a Reeb orbit in the stratum S_{T_j} of multiplicity m is equal to*

$$2m|\Gamma_j| \int_{A_{S_{T_j}}} i^* c_1(T(S_{T_j})),$$

moreover this number is an integer.

[1]Regular in the sense of foliation theory.

Proof. By the loop property of the Maslov index, the Maslov index of the Reeb orbit is twice the Maslov index of the path of change of coordinate maps between the two disks glued along γ. Since the disk was obtained via an (orbi)section over a sphere representing $A_{S_{T_j}}$, we get

$$\mu(\gamma) = 2\langle c_1(\xi), \sigma(u)\rangle = 2\langle c_1(T(\mathcal{Z}), A_{S_{T_j}}\rangle.$$

This is exactly $c_1^{orb}(T(S_{T_j}))$ evaluated on A. Therefore the index of an orbit of multiplicity m is

$$2m \int_{A_{S_{T_j}}} c_1^{orb}(T(S_{T_j})).$$

Now going back to the work of Satake [Sat57] to compute the integral of an orbifold characteristic class over a homology class, we take intersections with all orbifold strata and divide out by the orders of the local uniformizing groups and sum:

$$2k \int_{A_{S_{T_j}}} c_1^{orb}(T(S_{T_j})) = 2m \sum_j \frac{1}{|\Gamma_j|} \int_{A_{S_{T_j}} \cap \Sigma_j} c_1(T(S_{T_j}))$$

where Γ_j is a local uniformizing group in the orbifold stratum $\Sigma_j = S_{T_j}$. Now, since each such spherical class is completely contained in S_{T_j}, we can just compute the integral

$$\frac{2}{|\Gamma_j|} \int_{A_{S_{t_j}}} c_1(T(S_{T_j}))|_{S_{T_j}} = \frac{2}{|\Gamma_j|} \int_{A_{S_{T_j}}} i^* c_1(T(S_{T_j}))$$

for simple orbits, multiplying by m for m-multiple orbits. Since the orbifold here is non-effective we multiply by the order of the local uniformizing group. Note however that, although we may compute the integral on $\mathcal{Z}$, this integral is equal to one which takes place as the evaluation of an integral form on the contact manifold, hence we always get an integer. □

Remark 8.1.2. *The idea above is that $A_{S_{T_j}}$ is a "sufficiently diagonal" sphere in S_{T_k}. This ensures that we pick up as much information as possible about the line bundle as possible during the integration. One should also note that in general $c_1(\xi) \neq 0$ so this* *grading scheme for contact homology is computed with respect to a* particular *choice of capping surface for each Reeb orbit. When comparing contact manifolds which are S^1-orbibundles over the same base, care must be taken to make the same choices each time, so that the weights are realized via the Chern classes of each* specific toric structure.

Remark 8.1.3. *The reader may wonder what role branch divisors play in the index calculation above. This is encoded in summing over the strata and dividing by the orders of local uniformizers.*

We want to use these calculations to compute cylindrical contact homology, however this is not well defined unless we can exclude Reeb orbits of degree 0, 1, -1. To ensure this we must assume that for all k

$$2(\sum_i i^* c_i \tilde{w}_i) - \frac{1}{2} dim(S_{T_k}) > 0.$$

For this it is sufficient to assume that

$$\sum_i \tilde{w}_i > 1.$$

We take this as a standing assumption in the following.

Now we notice that there are no rigid J-holomorphic cylinders other than the trivial ones. This follows from the fact that there is a $\mathbb{C}^*$-action on the moduli space of curves into the symplectization, hence the dimension of the moduli space is always at least 2. This means that the contact homology is given completely by the Morse-Smale-Witten complex of the moment map with degree shifts given by the Maslov indices. The discussion above yields theorem 1.1.2]. We obtain the following corollaries.

Corollary 8.1.1. *Let (M, ξ) be a simply connected compact homogeneous contact manifold. Then $CH_*(M)$ is generated by copies of $H_*(\mathcal{Z})$ with degree shifts given by*

$$2m \int_A c_1(T(\mathcal{Z})) = 2m \sum_i \tilde{w}_i - 2$$

.

Proof. In this case M is an S^1-bundle over a generalized flag manifold, (recall that in this case there is a *regular* contact 1-form, α for ξ). The cohomology of the base is a polynomial ring as per the discussion earlier, and all the relevant homology classes are spherical. By the regularity theorem for integrable J the dimension of the moduli space is the one predicted by the Fredholm index. The action of the group implies the existence of a circle action generated by the Reeb vector field which induces an action on the symplectization of M, this action in turn induces an action on the moduli space of curves, hence the dimension of the moduli space is at least 2 (there is also the $\mathbb{R}$ action thus there are no rigid J-holomorphic curves connecting orbit spaces. This contact homology is given completely in terms of the Morse-Smale-Witten differential, which vanishes since the moment map determines a perfect Morse function (this is always true for generalized flag manifolds), thus we get a generator for each critical point of the norm squared of the moment map in degree given by the Maslov indices as calculated in the previous discussion. □

Corollary 8.1.2. *Let (M, ξ) be a simply connected compact toric Fano contact manifold with a quasiregular contact form α. Then $CH_*(M)$ is generated by copies of $H_*(\mathcal{Z})$ with degree shifts given by the Maslov indices plus the dimension of the stratum containing the particular Reeb orbit as a point. If ξ has a regular contact form α then the degree shifts are given by*

$$2m \sum_j \tilde{w}_j - 2,$$

where the $\tilde{w}_j$ are defined as above.

Proof. The Fano condition gives transversality of the $\bar{\partial}_J$-operator via the Dolbeault complex. If we assume transversality we can drop the Fano assumption. Again, our cohomology ring is a truncated polynomial ring generated by all possible Chern classes, with spherical second homology because of simple connectivity. The indices are given by the even multiples of the sum of the weights. Again there are no non-trivial J-curves. So the homology is that given by the Morse-Smale-Witten complex (whose differential again vanishes by perfection of the Morse function) with the degree shifts given by the Maslov indices as calculated above. □

Chapter 9

Examples and Applications

In this chapter we apply the constructions and calculations from previous chapters to distinguishing contact structures on several interesting classes of examples of contact manifolds.

9.1 Case of circle reduction

In this section we generalize the calculation for the Wang-Ziller manifolds. In the case of contact reduction since we can always get the first Chern class of the tangent bundle of the base orbifold in terms of the reduction data we can always compute cylindrical contact homology as long as it is defined and as long as we have some sort of positivity on the relevant divisors and line bundles.

For this we first consider toric structures, and symplectic reduction of $\mathbb{C}^n$, or equivalently contact reduction of S^{2n-1} by a k-dimensional torus.

So let's consider circle reduction. This generalizes the example above of the

Wang-Ziller manifolds. Suppose S^1 acts on $\mathbb{C}^n$. This gives an exact sequence:

$$0 \to S^1 \stackrel{f_\Omega}{\to} T^n \to T^{n-1},$$

where

$$f_\Omega(\theta) = diag(w_1\theta, \ldots, w_n\theta)$$

and the w_i are integral weights.

This gives rise to a moment map

$$\mu(z_1, \ldots, z_n) = \sum_i w_i |z_i|^2.$$

Now we consider a regular value of the moment map, assume for simplicity that 0 is a regular value, for if it is not we just shift it by a constant. Suppose that S^1 acts locally freely on $\mu^{-1}(0)$. Then the quotient is Kähler, with a base whose first Chern class is given by

$$\sum_i \bar{w_i}.$$

Here $\bar{w_i}$ is given by the isomorphism given in the section on cohomology rings of reduced spaces.

Let us now study the 5 dimensional case. Here we are starting with $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. Let us quotient out by a circle action generated by the vector fields

$$p_i H_i = p_i(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{y_i})$$

and

$$-q_i L_i = -q_i(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{y_i}).$$

Where the H_i are defined on the first $\mathbb{C}^2$ and the L_i on the second, and $p_i, q_i > 0$. Then the moment map is given by

$$\sum_i p_i |z_i|^2 - q_1 |w_i|^2.$$

The total space is $S^2 \times S^3$, and the base is given by

$$\mathbb{CP}(p_1, p_2) \times \mathbb{CP}(q_1, q_2).$$

9.1.1 Wang-Ziller Manifolds

Now let's specialize to Wang-Ziller manifolds. These are toric manifolds either obtained from reduction in

$$\mathbb{C}^2 \times \mathbb{C}^2$$

via the moment map

$$\mu(z, w) = k|w|^2 - l|z|^2.$$

This manifold is also a homogeneous contact manifold. Note that as a toric manifold, this manifold is Fano, but we could also achieve transversality of the linearized Cauchy-Riemann operator via homogeneity (since this is a homogeneous contact manifold). We can also see this manifold as a Boothby-Wang manifold. Consider

$$\mathcal{Z} = \mathbb{CP}^1 \times \mathbb{CP}^1,$$

and we take the standard symplectic form on each summand and multiply each piece by relatively prime integers k and l. We take P to be the circle bundle with a connection form α satisfying

$$d\alpha = \pi^*(k\omega_1 + lomega_2),$$

where the $omega_j$ are the symplectic forms for each sphere. Then

$$c_1(\xi) = (2k - 2l)\beta,$$

for β a generator of $H_2(S^2 \times S^3; \mathbb{Z})$, and

$$c_1(T\mathcal{Z}) = (2\omega_1 + 2\omega_2),$$

here $\mathcal{Z}$ is topologically

$$\mathbb{CP}^1 \times \mathbb{CP}^1$$

with the toric structure obtained by with symplectic form determined by k, l. $\mathcal{Z}$ admits a perfect Morse function, and the Maslov indices in this case for orbits of multiplicity m are given by $4m(k+l)$. Here we choose our homology class $A = L_1 + L_2$, which is the class of a line on each sphere. This actually gives a fractional grading since we must divide out by $\omega(A)$ where ω is the form given above. Thus the grading of contact homology is given by

$$|p| = \frac{4m}{(k+l)} - 2 + d$$

where $m \in \mathbb{Z} \setminus \{0\}$, and d ranges over all possible degrees of homology classes in $\mathcal{Z}$, in this case $d = 0, 2, 4$. We must be a little careful, since the Chern class of the contact bundle is not zero, we must keep track of homology classes of curves. To do this we simply use a coefficient ring given by $H_2(M)$. Here we apply proposition 7.1.2.

This gives infinitely many distinct contact structure on $S^2 \times S^3$ since for each choice of relatively prime k and l, we get generators of contact homology in minimal dimension $\frac{4}{(k+l)} - 2$. Of course, for all pairs such that $k - l = c$ we get a single first Chern class for the contact bundle [WZ90]. Choosing now all pairs with $k - l = c$, we get infinitely many distinct contact structures in the same first Chern class. In [Ler03] Lerman showed that these contact structures are all pairwise non-equivalent as toric contact structures, but he asked whether or not they were pairwise contactomorphic. This answers that question in the negative. Via the above construction we get contact structures $\xi_{k,l}$ on $S^2 \times S^3$.

Corollary 9.1.1. *Fix $c \in \mathbb{Z}$, choose k, l such that $gcd(k, l) = 1$, and $k - l = c$ then the contact structures $\xi_{k,l}$ are pairwise non-contactomorphic all within the same first Chern class of 4-plane distribution.*

This example suggests a Kunneth-type formula for the **join** [BGO07] construction for quasiregular contact manifolds provided each summand has suitable contact homology. Suppose that $(\mathcal{Z}_1, \omega_1)$ and $(\mathcal{Z}_2, \omega_2)$ are both simply connected symplectic orbifolds which are reduced spaces so that their cohomology rings are polynomials in the Chern classes. Then we can build circle bundles over their product with curvature forms given as an integral linear combination of the ω_j. By choosing appropriate spheres "diagonally" embedded into the product we can evaluate the first Chern class of this bundle in order to get the Maslov indices as above. Assuming transversality of the $\bar{\partial}_J$-operator this always computes contact homology.

9.1.2 Circle bundles over weighted projective spaces.

Let $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^{n+1}$ and consider the weighted circle action for $\lambda \in S^1$:

$$(z_1, \ldots, z_n) \to (\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n).$$

Restricting to the sphere and quotienting out by this action we get the **weighted projective space**

$$\mathbb{CP}(\mathbf{w}) = S^{2n-1}/S^1_{\mathbf{w}}.$$

Notice that in this case we are just weighting the standard Reeb vector field on S^{2n-1} and modding out by its circle action. Here we can compare what is going on with these various contact structures corresponding to different choices of Reeb vector fields, and hence contact forms. In these examples first off, note that weighted projective spaces are toric Fano, (even in the orbifold sense.) Note that the cohomology ring is then just the standard one, and we just need to find the right spherical classes. Of course we just pullback k-multiples of the standard symplectic form to define our line bundles. Now to compute contact homology of the bundle we integrate

$$c_1(T(\mathcal{Z}))$$

over the class of a line. The base admits a perfect Morse function, so all we need to do is keep track of the strata. Integrating

$$i_j^* c_1(T(\mathcal{Z}))$$

over spheres representing the Kähler class for each stratum. Let G_j be the local uniformizing group for the stratum S_{T_j}. So the grading of contact homology for an orbit of multiplicity m will be

$$\frac{2m}{|G_j|}\sum_j(\langle i_j^* c_1^{orb}(T\mathbb{CP}(\mathbf{w}), [L]\rangle - \frac{1}{2}dimS_{T_j}) + d + n - 3$$

$$= (2km\sum_{j,k}\frac{1}{|\Gamma_j|}w_k) - \frac{1}{2}dimS_{T_j} + d + n - 3,$$

where the class $[S_j^2]$ is the class of a line in each stratum and d corresponds to the possible degree of a homology class on $\mathbb{CP}^n$, hence is an even number between 0 and $2n$ and $S \in \mathbb{Z}^+$. The dimension of the moduli space for genus 0 and 1 positive and 1 negative puncture is then never 1. Notice that $c_1(\xi) = 0$ in this case. So again we see that these contact manifolds can be distinguished by the bundle and orbifold data.

Remark 9.1.1. *One should be able to simplify the above formula when working with branch divisors. We choose to stick with our earlier notation, in which any information about such branch divisors is encoded in the calculation.*

These are all given by circle reduction of $\mathbb{C}^n$. The moment maps are given as above, and there are Chern classes for each stratum. These are given by sums of weights given by setting various terms to zero in the defining equations of $\mathbb{CP}(\mathbf{w})$. In all of these cases we see that each orbifold stratum given by a multiindex $I = i_1, \ldots, i_k$ has

$$c_1 = \sum_k w_{i_k}$$

This gives the grading for contact homology. Moreover since $c_1(\xi) = 0$ this gives an honest grading for the whole algebra.

9.2 Reading off indices from the moment polytope of the base.

Given a toric manifold we can use Morse theoretic facts about toric orbifolds along with the index calculations of the previous chapter to read off contact homology, from the LT-polytope.

Theorem 9.2.1. *Let (M, ξ) be simply connected a toric contact manifolds of Reeb type. Then the grading for contact homology can be read off from the LT polytope of $\mathcal{Z}$.*

Proof. This is actually simple and follows directly from the previous chapter. First note that if $c_1(\xi) = 0$ then the grading is independent of all choices. When we construct the polytope, the length of a side corresponds to the Chern number of $T(\mathcal{Z})$ evaluated on the sphere corresponding to that side. Hence, in the regular case, we simply take even positive integer multiples of the perimeter of the polytope offset by the possible dimensions of the stable and unstable manifolds of the critical points of the moment map.

In the quasiregular case, we simply note that the orbifold stratification is given by a labelling of the facets. We get the stratum of each lower dimensional face by considering the the product of the labels of higher dimensional faces intersecting in that lower dimensional face. Thus for each label and nontrivial intersection we get an orbit space, and we calculate as in the regualar case for that sub-polytope, except we must divide out by the order of the local uniformizing group for that stratum. □

Chapter 10

Further Examples and Applications

10.1 More invariants of Toric manifolds in dimension 5

In the previous discussion, everything boiled down to index calculations since, due to the $\mathbb{C}^*$-action on the moduli space of curves into the symplectization of a toric contact manifold. We would like to set up a situation where we can use the behavior of the holomorphic curves in a toric orbifold since in the toric case (of Reeb type) we know what is going on with holomorphic curves in the base. As we saw in section 5.2.1, the holomorphic curves in the base are quite well behaved and correspond to Morse theoretical objects which we can control. This allows us to sometimes compute the genus 0 Gromov-Witten potential for the base manifold or orbifold, even in the non-Fano case. The regular case is given in [Bou02], and [EGH00], so let us figure out what is going on when the contact form form is only assumed to be quasiregular. We choose variables as in the regular case, however, we must now have a slightly more complicated grading, and we must account for different orbit types. Just for simplicity let us restrict to dimension 5. We must first set up the Weyl formalism of rational symplectic field theory.

Let $(M, \mathcal{Z})$ be a Hamiltonian BW pair. Let us also assume that $\pi_1(\mathcal{Z}) = e$. Then we may view the symplectization of M as the associated line bundle to the S^1-bundle M, with the zero section removed. As in the regular case genus 0 holomorphic curves are then orbi-sections over embedded spheres in $\mathcal{Z}$ with prescribed zeros and poles ocurring in the various strata. As was mentioned before we get a fibration:

$$pr : \mathcal{M}_{0,r}(s|W, S_{T_1}, \dots, S_{T_s})/\mathbb{R} \to \mathcal{M}_{0,r+s}(\mathcal{Z}, J).$$

Let $T_1, \dots, T_N$, be the possible actions of *simple* closed Reeb orbits. For each $j < N$ we have the space of good periodic orbits $\mathcal{P}_j$ which we split in to positive and negative parts for each multiplicity m.

$$(\mathcal{P}_{j,m})^{\pm}.$$

Note that these spaces $\mathcal{P}^{\pm}_{j.m}$ are cyclic orbifolds, hence we may consider forms on them. Then we define evaluation maps

$$ev_0 : \mathcal{M}_{0,r}(s|W, J, \alpha)/\mathbb{R} \to M^{\times r}$$
$$ev^{\pm} : \mathcal{M}_{0,r}(s|W, J, \alpha) \to \bigsqcup_{j=0}^{\infty} \mathcal{P}^{\pm}_j.$$

These evaluations take place in the first case at marked points and in the second case at punctures. Also we really need to specify to which stratum does each puncture correspond. Now denote forms on $\mathcal{P}^+_j$ by p_j and those on $\mathcal{P}^-_{j,m}$ by q_j, the restrictions to the multiplicity m parts are denoted by $p_{j,m}$, $q_{j,m}$ respectively.

Now we organize these forms corresponding to periodic orbits into Fourier series

$$u = \sum_j \sum_{m=1}^{\infty} (p_{j,m} e^{imx} + q_{j,m} e^{-imx}).$$

Let us choose a basis,

$$\triangle_1, \dots, \triangle_a,$$

of $H^*(\mathcal{Z})$, which satisfies the condition that

$$i_j^*(\triangle_1, \dots, i_j^*(\triangle_N)$$

forms a basis for $H^*(S_{T_j})$. We write

$$p_{j,m} = \sum_{i=1}^{a} p_{j,i} i_j^* \triangle_i,$$

$$q_{j,m} = \sum_{i=1}^{a} q_{j,i} i_j^* \triangle_i,$$

and u_i the $\triangle_i$ component of u

$$u = \sum_i u_i \triangle_i.$$

Given such a closed form keeping track of strata and an element $A \in H_2(\mathcal{Z})$ we define a correlator

$$^{-1}\langle t, \dots, t, u, \dots, u \rangle_0^A$$

$$:= \int_{\overline{\mathcal{M}_{0,r}^A(s|W,J,\alpha)/\mathbb{R}}} ev_0^*(t \otimes \cdots \otimes) \wedge ev^{\pm *}(u \otimes \cdots \otimes u)|_{x=0}.$$

This integral counts J-holomorphic curves with s punctures and r marked points intersecting $PD(t)$ at the marked points and cylindrical over periodic orbits with non-zero coeffecients in the expression for u.

Recall that we consider homology classes as degree vectors $(d_1, \dots, d_N)$. We also write

$$t = \sum t_i \pi^* \triangle_i + \sum_j \tau_j \theta_j.$$

Here t is a form on M, and the θ_j's complete the pullbacks of basis elements in the cohomology of $\mathcal{Z}$ to a basis of $H^*(M)$. Let us now organize all possible correlators into a generating function, the so-called Hamiltonian:

$$\mathbf{h}(t,u) = \sum_d \sum_{r,s=0}^{\infty} {}^{-1}\langle t,\ldots,t;u,\ldots,u\rangle_0^d z^d$$

which counts all possible rigid genus 0 curves, each term is non-zero, only if the sums of the degrees of the appropriate parts of the t variables add up to the dimension of the moduli space of cylinders defined by the appropriate parts of the u variables. The z variable keeps track of curves in the class d.

These have the feel of Gromov-Witten invariants, indeed, they are, as we shall see, related to the Gromov-Witten invariants of $\mathcal{Z}$. As before the grading of the variables corresponding to Reeb orbits is as before. Because of the S^1 action, we know that the moduli space of J-holomprphic curves always has too big of a dimension. However we can still see differences in the contact homology *algebra* by imposing conditions on such curves such as marked points.

Notice that the above construction gives us a collection of DGA's parametrized by t. Specializing at 0, for genus 0, depending on which u's we allow gives the different incarnations of contact homology or rational SFT.

Since, in the case of S^1 orbibundles, the moduli spaces always admit a $\mathbb{C}^*$-action, we see that for $t = 0$ we recover the result from [EGH00] which they stated for a regular contact form.

Proposition 10.1.1. *For an S^1-orbibundle over a symplectic orbifold. The specializations at $t = 0$ of all contact homology algebras is freely generated by the p, q, variables.*

We can still try to find more interesting information by imposing marked point conditions. We will see how to use this in a moment. First let us state another theorem from [EGH00], this was extended to orbibundles over one dimensional complex projective spaces with orbifold singularities in [Ros]. The argument is the same in higher dimensions, one just must be careful about the definition of Gromov-Witten invariants for orbifolds, where one must keep track of strata since the same cohomology class could have a Poincaré dual intersecting several strata. Let us assume moreover that $\mathcal{Z}$ is simply connected.

Proposition 10.1.2. *Set*

$$\boldsymbol{h}^j_{W,J} = \frac{\partial \boldsymbol{h}}{\partial \tau_j}(\sum_{i=1}^{b} t_i pi^* \triangle_i + \tau_j \theta_j, q, p, z)|_{\tau_j=0}$$

and

$$\widetilde{\boldsymbol{f}}^j(t,z) = \frac{\partial f}{\partial s}(\sum_i t_i \triangle_i + s\pi_* \theta_j, z)|s = 0$$

where $\boldsymbol{f}$ *is the genus* 0 *Gromov-Witten potential of* $\mathcal{Z}$*. Then*

$$\boldsymbol{h}_{W,J}(t_1, \dots, t_b, q, p, z)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \widetilde{\boldsymbol{f}}^j(t_1 + u_1, \dots, t_b + u_b, u_{b+1}, \dots, u_a, e^{ix}, z)dx.$$

We would like to see more ways to distinguish toric contact manifolds with different bases. It is clear that if two contact manifolds are Boothby-Wang spaces for two toric symplectic orbifolds with a different number of faces in their Lerman-Tolman or Delzant polytope, then they cannot be contactomorphic. This is easy to see from the Gysin sequence of equivalently the Leray-Serre spectral sequence for the S^1-orbibundle. Therefore the following, adjusted from [EGH00], is useful for distinguishing toric contact structures.

Theorem 10.1.1. *Suppose we have two simply connected regular toric contact manifolds of Reeb type in dimension* 5*. Suppose that under the quotient of the Reeb action one of the base manifolds has an exceptional sphere while the other does not, and suppose that the two Delzant polytopes have the same number of facets. Then these two manifolds cannot be contactomorphic.*

Proof. We show that there is an odd element in the contact homology algebra of one manifold specialized at a class which is not in the other for any specialization. We assume here that all of the weights of the torus action are greater than 1 for the manifold containing no exceptional spheres. As in [EGH00] the potential specialized to the Poincaré dual of an exceptional divisor will give the potential for a standard S^3, but then for a chain which lifts to the volume form for this 3-form there is always a holomorphic curve to kill it as a generator for homology specialized at this 3 class. Hence this homology contains no odd elements. Let us look at the manifold containing no exceptional sphere. We must compute the Gromov-Witten potential. Unfortunately it does not vanish, but, for *any* 2-classes the potential always vanishes. This is because the Gromov-Witten invariant

$$GW^0_{A,k}(\alpha, \ldots, \alpha) \neq 0$$

for a 2-dimensional class α only if

$$2k = 4 + 2c_1(A) + 2k - 6 \Leftrightarrow c_1(A) = 1$$

But the weights make this impossible. Thus all coeffecients for such curves vanish, and the potential vanishes on $\mathcal{Z}$, hence on M. So for a 3 class in the contact manifold obtained from integration over the fiber of a two class, there is no holomorphic curve to kill it. Hence specialized at such a 3 class we have an odd generator which does not exist in the presence of exceptional spheres. □

Remark 10.1.1. *One would like to also make this work in the quasiregular case, indeed the Gromov-Witten potential should still vanish on* 2 *classes by the grading axiom, however there are problems with Gromov-Witten invariants of orbifolds. We only have the divisor axiom of the Gromov-Witten invariants when the relevant cohomology class has its Poincaré dual living outside of the orbifold singular locus. To prove the potential is as claimed for exceptional spheres we require the divisor axiom, with the relevant classes living* inside *the orbifold singular locus.*

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