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Design of strictly positive real, fixed-order dynamic compensators

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Abstract

This paper presents sufficient conditions for the design of strictly positive real (SPR), fixed-order dynamic compensators. The primary motivation for designing SPR compensators is for application to positive real (PR) plants. When an SPR compensator is connected to a PR plant in a negative feedback configuration, the closed loop is guaranteed stable for arbitrary plant variations as long as the plant remains PR. This paper gives equations that are a modified form of the optimal projection equations, with the separation principle not holding in either the full- or reduced-order case. A solution to the design equations is shown to exist when the plant is PR (or just stable). Finally, the closed loop system consisting of a PR plant and an SPR compensator is shown to be S-structured Lyapunov stable.

Introduction

This paper considers the design of SPR, fixed-order dynamic compensators. In previous work, we addressed the design of stable compensators [1,2]. The results of [1,2] are extended in this paper by taking advantage of the Kalman-Yakubovich Lemma to guarantee an SPR compensator in the state space setting. It is well known that if the compensator is SPR and the plant is PR, then the closed loop is stable for arbitrary variations in the plant parameters as long as the plant remains PR. Note that the plant must be square (number of inputs equal to number of outputs) if the compensator is to be designed SPR.

Problem Statement and Kalman-Yakubovich Lemma

The system to be controlled is given by:

\[ \dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \]
\[ y(t) = Cx(t) + w_2(t), \]

where the A, B, and C plant matrices may not be well known. The white noise processes w_1 and w_2 are zero mean with intensities \( V_1 \geq 0 \) and \( V_2 > 0 \), respectively. The problem is to design an SPR, fixed-order dynamic compensator

\[ \dot{x}_c(t) = A_c x_c(t) + B_c y(t), \]

where \( x_c(t) \) is the compensator state of order \( n_c \), which minimizes the performance objective

\[ J(A_c, B_c, C_c) = \lim_{t \to \infty} E[x(t)^T R_1 x(t) + u(t)^T R_2 u(t)], \]

where \( x_c \) is the compensator state of order \( n_c \); \( A_c, B_c, \) and \( C_c \) are the compensator matrices; \( R_1 \) and \( R_2 \) are the state and control weighting matrices and \( E[\cdot] \) denotes the expectation operator. Since the internal realization of the compensator does not affect the cost, the compensator will be limited to a minimal realizations, i.e. \( (A_c, B_c) \) controllable and \( (C_c, A_c) \) observable. The Kalman-Yakubovich Lemma is used to guarantee that the compensator is SPR.

Lemma 1 (Kalman-Yakubovich Lemma [3]): Given a stable matrix \( A \) and a minimal realization \( (A, B, C) \) of \( H(s) \), there exist positive definite matrices \( L \) and \( Q \) such that:

\[ AQ + QA^T = -L \]
\[ B = QC^T \]

if and only if

\[ H(s) = C(sI - A)^{-1}B \]

is SPR. This form of the lemma is actually the dual of that in [3].

Compensator Positive Realness and Upper Bound Minimization Problem

The results of [1] demonstrate that a stable compensator can be found by suitably overbounding the compensator covariance. The expected cost in equation (5) can easily be shown to be

\[ J = tr(Q), \]

where \( Q \) is the closed loop covariance defined as

\[ Q = \lim_{t \to \infty} E[x(t)^T x(t)]. \]

The closed loop system can be written as:

\[ \ddot{x} + \ddot{y} = 0, \]

where \( \ddot{A} = \ddot{A} \) is the closed loop matrix and the matrices in equation (12) may be partitioned as follows:

\[ \ddot{A} = \begin{bmatrix} A_{11} & B_{12} \\ B_{21} & A_{22} \end{bmatrix}, \quad \ddot{B} = \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \end{bmatrix}, \quad \ddot{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \]

The upper right block of equation (12) can be expanded as:

\[ 0 = A_{12} \ddot{x}_{12} + \ddot{B}_{12} \ddot{y} + B_{21} \ddot{y} + C_{21} \ddot{x}_{12} + C_{22} \ddot{x}_{12} + \ddot{C} \ddot{x}_{12}. \]

In general, \( A_{12} \) is not even guaranteed stable since the forcing terms in the Lyapunov equation (12) need not be nonnegative definite. By appending suitable terms to (12), however, the forcing terms can be made nonnegative definite, resulting in guaranteed stability of \( A_{12} \) when the modified equation (12) has a nonnegative definite solution. By appending only terms that are at least non-negative definite, the new
value of $Q$, becomes both a solution to the Lyapunov equation for $A_c$ and a covariance bound for $Q_2$ in equation (12). If the modified forcing term is positive definite and $Q > 0$, then $Q$ is a solution to the Lyapunov equation (6) of the Kalman Yakoobovich Lemma. Equation (7) of the Kalman-Yakubovich Lemma then requires simply that:

$$B_c = Q_2 C_c^T.$$  \hfill (15)

The following theorem formalizes the process.

**Theorem 1:** Let the symmetric matrix $\Phi > 0$ be such that

$$G(B_c, Q_2) > -B_c V_c B_c^T - B_c Q_2 B_c^T + Q_2 C_c^T B_c^T$$  \hfill (16)

and for given $A_c$, $B_c$, and $C_c$ satisfying equation (15), suppose that

$$(\tilde{A}, \tilde{V} + \tilde{B})$$ is stabilizable \hfill (17)

and that there exists $\tilde{Q} \geq 0$ satisfying

$$0 = \tilde{A} Q + \tilde{Q} \tilde{A}^T + \tilde{V} + \tilde{B},$$  \hfill (18)

where

$$\tilde{A} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \geq 0, \quad \tilde{Q} = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_2 \end{bmatrix}. \hfill (19)$$

Then,

$\tilde{A}$ is asymptotically stable, \hfill (20)

$$\tilde{Q} \leq \bar{Q},$$  \hfill (21)

$$J = \text{tr}(\bar{Q}) - J,$$  \hfill (22)

$A_c$ is stable, and \hfill (23)

$$(A_c, B_c, C_c)$$ has an SPR transfer function. \hfill (24)

**Proof:** That (17) implies (20) when a nonnegative definite solution $\bar{Q}$ exists to (18) is a standard consequence of Lyapunov equation solutions. Subtracting equation (12) from equation (18) gives:

$$\tilde{A} (\tilde{Q} - \bar{Q}) + (\bar{Q} - \tilde{Q}) \tilde{A}^T + \tilde{V} + \tilde{B} = 0$$  \hfill (25)

Since $\tilde{A}$ is stable and $\bar{Q} \geq 0$, $\tilde{Q} - \bar{Q} \geq 0$ which proves (21), from which (22) immediately follows. Rewriting the lower right block of equation (18) gives

$$A_c Q_2 + Q_2 A_c^T + B_c V_c B_c^T + B_c Q_2 B_c^T + Q_2 C_c^T B_c^T + \bar{Q} = 0.$$  \hfill (26)

Condition (16) guarantees that the forcing term of equation (26) is positive definite and the assumed existence of a nonnegative definite $Q$, (since $\bar{Q} \geq 0$) implies that $A_c$ is stable. In [1], it is further shown that $Q_2 > 0$ and thus satisfies equation (6). Since condition (15) is assumed to hold, (24) immediately follows. \hfill (24)

At this point we choose a form for $\tilde{Q}$ that satisfies equation (16).

**Proposition 1:** Let

$$\tilde{Q} = a B_c C_c^T + B_c C_c^T + \epsilon I > 0$$  \hfill (27)

where $T$ is an arbitrary positive definite matrix; then $\tilde{Q}$ satisfies (16) when $a = 1$ and $\beta = 1$ for any $\epsilon > 0$.

**Proof:** We know that

$$(B_c C_c^T + Q_2 C_c^T) (B_c C_c^T + Q_2 C_c^T) = 0$$  \hfill (28)

and adding the nonnegative definite term involving $V$, and the positive definite term involving $\epsilon$ to the left side gives:

$$B_c C_c^T + Q_2 C_c^T + B_c C_c^T + Q_2 C_c^T + \epsilon I > 0,$$  \hfill (29)

$$B_c C_c^T + Q_2 C_c^T + \epsilon I > 0,$$  \hfill (30)

Since the left side of (30) is precisely the chosen $\tilde{Q}$ when $\alpha - \beta - 1$, equation (16) is satisfied. Note that if $T$ is chosen such that $C_c^T C_c < V_c$, then an SPR compensator is guaranteed for $\alpha > 0$. Of course, if $\alpha$, $\beta > 0$, then equation (16) will still be satisfied; however, the objective is to keep $\alpha$ and $\beta$ as small as possible such that the deviation from the original problem is minimized. \hfill (30)

Conditions (15) and (18) are now incorporated into an upper bound minimization problem to guarantee an SPR compensator.

**Upper Bound Minimization Problem:** Determine $(A_c, B_c, C_c)$ and nonnegative definite $\tilde{Q}$ that minimize

$$J = \text{tr}(\tilde{Q})$$  \hfill (31)

subject to equations (15) and (18) with $\tilde{Q}$ given by equation (27). Solution of this problem gives an SPR compensator. The actual cost is guaranteed to be less than or equal to $J$ (and the actual covariance less than $\bar{Q}$).

**Sufficient Conditions for a Strictly Positive Real, Fixed-Order Dynamic Compensator**

Recall that a square matrix is nonnegative (positive) semisimple if it has a diagonal Jordan form and nonnegative (positive) eigenvalues. [4] A positive semisimple matrix may be thought of as a non-symmetric matrix that is similar to a (symmetric) positive definite matrix.

**Lemma 1:** Suppose $\hat{F}, \hat{Q}$ are n x n nonnegative definite matrices. Then $\hat{F}$ is nonnegative semisimple. Furthermore, if rank($\hat{Q}$) = $n_t$, then there exist n x n matrices $G$, $T$ and a positive semisimple n x n matrix $H$, unique except for a change of basis, such that: [4]

$$\hat{Q} = C H T,$$  \hfill (32)

$$G T = H T,$$  \hfill (33)

Any $G$, $M$, and $T$ satisfying Lemma 1 will be called a $G$, $M$, $T$-factorization of $\hat{Q}$. Since $\hat{Q}$ is nonnegative semisimple, the eigenvalues of $\hat{Q}$ are all nonnegative and $\hat{Q}$ has a generalized inverse $(\hat{Q})^*$ given by $G M^T$ [4]. The following simplified notation will be used in this section:
\[ \hat{\Sigma} = (B^TP - CQP)(\hat{A}_n + \hat{Q}P)^\dagger, \]  
(34)  
\[ V_2 = V_2 + aCTC^T, \]  
(35)

where \( R_2 = \tau V_2, \) with \( \tau \) a nonnegative design variable, has been assumed so that convenient, closed form expressions may be found for \( A_\alpha, B_\alpha, \) and \( C_\alpha. \) The generalized inverse in equation (34) may be replaced by a regular inverse if \( \tau > 0 \) since then \( \tau A_n + Q\hat{P} > 0. \)

**Theorem 6:** Assume that condition (17) holds and suppose there exist nonnegative definite matrices \( P, Q, \) and \( \hat{Q} \) satisfying

\[
0 = (A + \hat{Q}^T \hat{V}_2^{-1}C)\hat{Q} + Q(A + \hat{Q}^T \hat{V}_2^{-1}C)^T + V_1 \\
+ \hat{R} \hat{Q}^T \hat{Q} + \hat{Q}^T \hat{V}_2^{-1} \hat{R} \hat{Q} + \hat{Q} \hat{P},
\]  
(36)

\[
0 = A^T \hat{P} + PA + R_1 + \hat{R} \hat{Q} \hat{Q}^{-1} \\
+ \hat{Q} \hat{Q}^T \hat{V}_2^{-1} \hat{Q} + \hat{Q} \hat{P},
\]  
(37)

\[
0 = (A - (B + QC^T) \hat{V}_2^{-1} - \epsilon \hat{I})\hat{Q} \\
+ \hat{Q}(A - (B + QC^T) \hat{V}_2^{-1} - \epsilon \hat{I})^T \\
- \hat{R} \hat{Q} \hat{Q}^{-1} - \hat{Q} \hat{Q}^T \hat{V}_2^{-1} \hat{Q},
\]  
(38)

\[
0 = (A + \hat{Q}^T \hat{V}_2^{-1}C - \hat{R} \hat{Q}^{-1} - \epsilon \hat{I}) \hat{P} \\
+ \hat{P}(A + \hat{Q}^T \hat{V}_2^{-1}C - \hat{R} \hat{Q}^{-1} - \epsilon \hat{I}) + \hat{Q} \hat{Q}^T \hat{V}_2^{-1} \hat{Q},
\]  
(39)

\[
\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) = n_c,
\]  
(40)

with \( \alpha = \beta = 1 \) and \( \epsilon > 0. \) Then the compensator \( (A_\alpha, B_\alpha, C_\alpha) \) given by

\[
A_\alpha = \Gamma[A + \hat{Q}^T \hat{V}_2^{-1}C] \\
- (B + QC^T + \hat{Q}^T \hat{V}_2^{-1} \hat{C} - \hat{R} \hat{Q}^{-1} - \epsilon \hat{I}) \hat{C},
\]  
(41)

\[
B_\alpha = - \hat{Q} \hat{V}_2^{-1} \hat{C}
\]  
(42)

\[
C_\alpha = - \hat{V}_2^{-1} \hat{C}_T
\]  
(43)

using a \( G, M, \Gamma \)-factorization of \( \hat{Q}\hat{P}, \) assures that the triple \( (A_\alpha, B_\alpha, C_\alpha) \) has an SPR transfer function and that conditions (20)-(23) also hold.

Conversely if \( (A_\alpha, B_\alpha, C_\alpha) \) solves the upper bound minimization problem with \( (A_\alpha, B_\alpha, C_\alpha) \) having an SPR transfer function, then there exist real \( n \times n \) nonnegative definite matrices \( Q, P, \) and \( \hat{P} \) and \( 0 \leq \alpha, \beta \leq 1 \) and \( \epsilon \geq 0 \) that satisfy equations (36)-(40) with \( A_\alpha, B_\alpha, \) and \( C_\alpha \) given by equations (41)-(43).

**Proof:** See [1].

**Remark 1:** These optimal projection equations consist of four modified Riccati/Lyapunov equations that are coupled in both the full- and reduced-order cases. Thus, as expected, the separation principle is invalid in either case. Because the "binary" SPR condition has been imposed, one should not expect equations (36)-(40) to reduce to the usual separated equations of LQG theory because there is no simple way to relax the SPR requirement.

**Remark 2:** From an examination of the basic form of the optimal projection equations [4] specialized to the full-order case (i.e., LQG), it may be noted that there are other methods of attaining full-order SPR compensators (and perhaps reduced-order also) that are simpler than that presented here. In fact, any one of the full-order optimal projection equations could be modified to guarantee compensator stability after the optimization is complete, coupled with requiring a fixed relationship between \( B_\alpha \) and \( C_\alpha. \) With normal LQG, \( B_\alpha \) is dependent on \( V_2 \) and \( C_\alpha \) is dependent on \( R_2. \) Since \( B_\alpha \) and \( C_\alpha \) must be related by a fixed matrix \( \hat{Q}, \) the dependence on \( V_2 \) and \( R_2 \) is coupled. The method presented here considers this coupling in deriving the sufficient conditions for an SPR compensator. This is similar to the method employed in Ref. [5]. This alternative disregards the inherent coupling between \( B_\alpha \) and \( C_\alpha. \)

This section addresses the existence of solutions to equations (36)-(39) when the plant is open loop stable. That there exist SPR compensators of any order that stabilize a stable plant is trivial. The real questions are whether equations (36)-(39) are guaranteed to have a solution and whether a given algorithm can find that solution. Only the first question will be discussed here. The second question obviously depends on the algorithm chosen to solve the equations. The following discussion assumes that the infimum of the auxiliary cost is attained. Note that existence of a feasible solution to equations (15) and (18) using the chosen \( \hat{U} \) implies the existence of an optimal feasible solution to the upper bound minimization problem. Then the converse of Theorem 6 assures a nonnegative definite solution to equations (36)-(39). To show that a feasible solution exists to equations (15) and (18), choose \( B_\alpha = 0 \) and \( C_\alpha = 0, \) which clearly satisfies equation (15) for any \( Q. \) With these choices, expand equation (18) with \( \hat{U} \) given by equation (27) and \( \alpha, \beta, \) and \( T \) left variable to give

\[
\begin{bmatrix}
A & 0 \\
0 & A_c
\end{bmatrix}
\begin{bmatrix}
Q_1 & Q_{12} \\
Q_{12} & Q_2
\end{bmatrix}
+ \begin{bmatrix}
Q_1 & Q_{12} \\
Q_{12} & Q_2
\end{bmatrix}
\begin{bmatrix}
A^T & 0 \\
0 & A_c^T
\end{bmatrix}
\]

\[
\begin{bmatrix}
V_1 \\
0
\end{bmatrix}
+ \begin{bmatrix}
\beta \hat{Q}_{12}^T \hat{Q}_{12} + \epsilon I
\end{bmatrix}
= 0.
\]  
(44)

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Expansion of the cross term in equation (44) indicates that \( Q_1 \) is dependent only on \( A \) and \( A^\top \), and hence independent of \( Q_2 \) and \( Q_3 \). Thus, the forcing term in equation (44) is nonnegative definite and finite. Then for any stable \( A_0 \) of any order, \( Q \) exists and is nonnegative definite by Lyapunov's theorem. Note that this result holds for all \( 0 \leq \alpha, \beta \leq 1 \) and arbitrary \( \varepsilon \) \( \geq 0 \).

**S-Structured Lyapunov Stability of Positive Real/Strictly Positive Real Feedback Systems**

Boyd and Yang [6] have discussed the concept of S-Structured Lyapunov Stability (S-SLS). Let \( \mathbf{N} \) denote the set of real numbers and \( \mathbf{N}^{\infty} \) denote the set of real \( n \times n \) matrices.

**Definition 1** [6]: Let \( S \) be an \( n \times n \) matrix that is a subspace of \( \mathbf{N}^{\infty} \). \( A \in \mathbf{N}^{\infty} \) is S-structured, Lyapunov stable if there is a \( P \in S \) such that \( P - P^\top > 0 \) and \( A P + PA \leq 0 \).

**Proof**: See [2].

Theorem 1: Let \( S \) be given by:

\[
\mathbf{N} \otimes \mathbf{R}^n
\]

which denotes the set of block diagonal matrices with two diagonal blocks, the first a real \( n \times n \) matrix and the second a real \( n \times n \) matrix. Also let the triple \((A, B, C)\) be positive real and the triple \((A_0, B_0, C)\) be strictly positive real. Then the matrix

\[
\mathbf{\Lambda} = \begin{bmatrix}
A & -B C \\
B_0 C & A_0
\end{bmatrix}
\]

is S-structured Lyapunov stable if and only if \( A \) is stable (essentially unstructured Lyapunov stability).

The proof in [2] proceeds by showing that

\[
\bar{Q} = \begin{bmatrix}
Q & 0 \\
0 & \bar{Q}
\end{bmatrix}
\quad \text{and} \quad
\bar{L} = \begin{bmatrix}
L & 0 \\
0 & L
\end{bmatrix}
\]

satisfy the Lyapunov equation

\[
\bar{M} \bar{Q} + \bar{Q} \bar{A}^\top + \bar{L} = 0,
\]

indicating that the closed loop is S-SLS.

Theorem 1 may provide some insight into the mechanism of the maximum entropy modelling approach to control. Specifically, in [7], it is noted that the maximum entropy approach suppresses off-diagonal elements of the closed-loop covariance when high uncertainty exists. This leads to reduced position feedback of uncertain modes, while the velocity feedback of uncertain modes remains. The hypothesis in [7] is that the resultant control of highly uncertain modes is a dissipative rate feedback, similar to positive real feedback. Theorem 7 shows that positive real/strictly positive real negative feedback connections lead to complete suppression of the off-diagonal blocks of a solution to a certain block-structured Lyapunov equation.

**Conclusions**

This paper presents a method for designing SPR, dynamic compensators of order less than or equal to that of the plant. An overbounding technique on the state covariance combined with the Kalman-Yakubovich Lemma then guarantees that the compensator is SPR. If the plant is stable (or positive real), the design equations are guaranteed to possess a solution.

**References**


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