12-5-1990

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FP-11-2 4:20

Proceedings of the 29th Conference
on Decision and Control
Honolulu, Hawaii • December 1990

SAND90-2387C

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Abstract

This paper presents sufficient conditions for the design of strictly positive real (SPR), fixed-order dynamic compensators. The primary motivation for designing SPR compensators is for application to positive real (PR) plants. When an SPR compensator is connected to a PR plant in a negative feedback configuration, the closed loop is guaranteed stable for arbitrary plant variations as long as the plant remains PR. This paper gives equations that are a modified form of the optimal projection equations, with the separation principle not holding in either the full- or reduced-order case. A solution to the design equations is shown to exist when the plant is PR (or just stable). Finally, the closed loop system consisting of a PR plant and an SPR compensator is shown to be S-structured Lyapunov stable.

Introduction

This paper considers the design of SPR, fixed-order dynamic compensators. In previous work, we addressed the design of stable compensators [1,2]. The results of [1,2] are extended in this paper by taking advantage of the Kalman-Yakubovich Lemma to guarantee an SPR compensator in the state space setting. It is well known that if the compensator is SPR and the plant is PR, then the closed loop is stable for arbitrary variations in the plant parameters as long as the plant remains PR. Note that the plant must be square (number of inputs equal to number of outputs) if the compensator is to be designed SPR.

Problem Statement and Kalman-Yakubovich Lemma

The system to be controlled is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \tag{1}$$

$$y(t) = Cx(t) + w_2(t), \tag{2}$$

where the A, B, and C plant matrices may not be well known. The white noise processes w_1 and w_2 are zero mean with intensities $V_1 \geq 0$ and $V_2 > 0$, respectively. The problem is to design an SPR, fixed-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{3}$$

$$u(t) = C_c x_c(t) \tag{4}$$

of order n_c which minimizes the performance objective

$$J(A_c, B_c, C_c) = \lim_{t\to\infty} E[x(t)^T R_1 x(t) + u^T(t) R_2 u(t)], \tag{5}$$

where x_c is the compensator state of order n_c; A_c, B_c, and C_c are the compensator matrices; R_1 and R_2 are the state and control weighting matrices and $E(\cdot)$ denotes the expectation operator. Since the internal realization of the compensator does not affect the cost, the compensator will be limited to a minimal realizations, i.e. (A_c, B_c) controllable and (C_c, A_c) observable. The Kalman-Yakubovich Lemma is used to guarantee that the compensator is SPR.

<u>Lemma 1</u> (Kalman-Yakubovich Lemma [3]): Given a stable matrix A and a minimal realization (A, B, C) of H(s), there exist positive definite matrices L and Q such that:

$$AQ + QA^T = -L \tag{6}$$

and

$$B = QC^T \tag{7}$$

if and only if

$$H(s) = C(sI - A)^{-1}B \tag{8}$$

is SPR. This form of the lemma is actually the dual of that in [3].

Compensator Positive Realness and Upper Bound Minimization Problem

The results of [1] demonstrate that a stable compensator can be found by suitably overbounding the compensator covariance. The expected cost in equation (5) can easily be shown to be

$$J = \mathrm{tr}(\tilde{Q}\tilde{R}), \tag{9}$$

where $\tilde{Q}$ is the closed loop covariance defined as

$$\tilde{Q} = \lim_{t\to\infty} E(\tilde{x}\tilde{x}^T), \qquad \tilde{x} = \begin{bmatrix} x \\ - \\ x_c \end{bmatrix}. \tag{10}$$

and

$$\tilde{R} = \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}. \tag{11}$$

The closed loop system can be written as:

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0, \tag{12}$$

where $\tilde{A}$ is the closed loop matrix and the matrices in equation (12) may be partitioned as follows:

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix},$$

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_2 \end{bmatrix}. \tag{13a,b,c}$$

The lower right block of equation (12) can be expanded as:

$$0 = A_c\tilde{Q}_2 + \tilde{Q}_2 A_c^T + B_c V_2 B_c^T + B_c C\tilde{Q}_{12} + \tilde{Q}_{12}^T C^T B_c^T \tag{14}$$

In general, A_c is not even guaranteed stable since the forcing terms in the Lyapunov equation (12) need not be nonnegative definite. By appending suitable terms to (12), however, the forcing terms can be made nonnegative definite, resulting in guaranteed stability of A_c when the modified equation (12) has a nonnegative definite solution. By appending only terms that are at least non-negative definite, the new value of Q_2 becomes both a solution to the Lyapunov equation for A_c and a covariance bound for $\tilde{Q}_2$ in equation (12). If the modified forcing term is positive definite and $Q_2 > 0$, then Q_2 is a solution to the Lyapunov equation (6) of the Kalman Yakubovich *Lemma*. Equation (7) of the Kalman-Yakubovich Lemma then requires simply that:

$$B_c = Q_2 C_c^T. \tag{15}$$

The following theorem formalizes the process.

Theorem 1: Let the symmetric matrix $\Omega > 0$ be such that

$$\Omega(B_c, Q_{12}) > - B_c V_2 B_c^T - B_c C Q_{12} - Q_{12}^T C^T B_c^T \tag{16}$$

and for given A_c, B_c, and C_c satisfying equation (15), suppose that

$$(\tilde{A},\ [\tilde{V} + \tilde{\Omega}]^{1/2}) \text{ is stabilizable} \tag{17}$$

and that there exists $\bar{Q} \geq 0$ satisfying

$$0 = \tilde{A}\bar{Q} + \bar{Q}\tilde{A}^T + \tilde{V} + \tilde{\Omega}, \tag{18}$$

where

$$\tilde{\Omega} = \begin{bmatrix} 0 & 0 \\ 0 & \Omega \end{bmatrix} \geq 0, \qquad \bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}. \tag{19}$$

Then,

$$\tilde{A} \text{ is asymptotically stable,} \tag{20}$$

$$\tilde{Q} \leq \bar{Q}, \tag{21}$$

$$J \leq tr(\bar{Q}\tilde{R}) = \bar{J}, \tag{22}$$

$$A_c \text{ is stable, and} \tag{23}$$

$$(A_c,\ B_c,\ C_c) \text{ has an SPR transfer function.} \tag{24}$$

Proof: That (17) implies (20) when a nonnegative definite solution $\bar{Q}$ exists to (18) is a standard consequence of Lyapunov equation solutions.

Subtracting equation (12) from equation (18) gives:

$$\tilde{A}\ (\bar{Q} - \tilde{Q}) + (\bar{Q} - \tilde{Q})\ \tilde{A}^T + \tilde{\Omega} = 0 \tag{25}$$

Since $\tilde{A}$ is stable and $\tilde{\Omega} \geq 0$, $\bar{Q} - \tilde{Q} \geq 0$ which proves (21), from which (22) immediately follows. Rewriting the lower right block of equation (18) gives

$$A_c Q_2 + Q_2 A_c^T + B_c V_2 B_c^T + B_c C Q_{12} + Q_{12}^T C^T B_c^T + \Omega = 0. \tag{26}$$

Condition (16) guarantees that the forcing term of equation (26) is positive definite and the assumed existence of a nonnegative definite Q_2 (since $\bar{Q} \geq 0$) implies that A_c is stable. In [1], it is further shown that $Q_2 > 0$ and thus satisfies equation (6). Since condition (15) is assumed to hold, (24) immediately follows. ■

At this point we choose a form for Ω that satisfies equation (16).

Proposition 1: Let

$$\Omega = \alpha B_c C T C^T B_c^T + \beta Q_{12}^T T^{-1} Q_{12} + \varepsilon I > 0 \tag{27}$$

where T is an arbitrary positive definite matrix; then Ω satisfies (16) when $\alpha = 1$ and $\beta = 1$ for any $\varepsilon > 0$.

Proof: We know that

$$(B_c C T^{1/2} + Q_{12}^T T^{-1/2})\ (B_c C T^{1/2} + Q_{12}^T T^{-1/2})^T \geq 0 \tag{28}$$

and adding the nonnegative definite term involving V_2 and the positive definite term involving ε to the left side gives:

$$B_c C Q_{12} + Q_{12}^T C^T B_c^T + B_c C T C^T B_c^T + Q_{12}^T T^{-1} Q_{12} + B_c V_2 B_c^T + \varepsilon I > 0, \tag{29}$$

$$B_c C T C^T B_c^T + Q_{12}^T T^{-1} Q_{12} + \varepsilon I > - B_c V_2 B_c^T - B_c C Q_{12} - Q_{12}^T C^T B_c^T. \tag{30}$$

Since the left side of (30) is precisely the chosen Ω when $\alpha = \beta = 1$, equation (16) is satisfied. Note that if T is chosen such that $CTC^T \leq V_2$, then an SPR compensator is guaranteed for $\alpha = 0$. Of course, if α, $\beta > 1$, then equation (16) will still be satisfied; however, the objective is to keep α and β as small as possible such that the deviation from the original problem is minimized. ■

Conditions (15) and (18) are now incorporated into an upper bound minimization problem to guarantee an SPR compensator.

Upper Bound Minimization Problem: Determine (A_c, B_c, C_c) and nonnegative definite $\bar{Q}$ that minimize

$$\bar{J} = tr(\bar{Q}\tilde{R}) \tag{31}$$

subject to equations (15) and (18) with Ω given by equation (27). Solution of this problem gives an SPR compensator. The actual cost is guaranteed to be less than or equal to $\bar{J}$ (and the actual covariance less than $\bar{Q}$).

Sufficient Conditions for a Strictly Positive Real, Fixed-Order Dynamic Compensator

Recall that a square matrix is nonnegative (positive) semisimple if it has a diagonal Jordan form and nonnegative (positive) eigenvalues. [4] A positive *semisimple matrix* may be thought of as a non-symmetric matrix that is similar to a (symmetric) positive definite matrix.

Lemma 2: Suppose $\hat{P}$, $\hat{Q}$ are nxn nonnegative definite matrices. Then $\hat{P}\hat{Q}$ is nonnegative semisimple. Furthermore, if rank$(\hat{Q}\hat{P}) = n_c$, then there exist $n_c$xn matrices G, Γ and a positive semisimple n_cxn_c matrix M, unique except for a change of basis, such that: [4]

$$\hat{Q}\hat{P} = G^T M \Gamma, \tag{32}$$

$$\Gamma G^T = I_{\hat{n}_c} \tag{33}$$

Any G, M, and Γ satisfying Lemma 2 will be called a G, M, Γ-factorization of $\hat{Q}\hat{P}$. Since $\hat{Q}\hat{P}$ is nonnegative semisimple, the eigenvalues of $\hat{Q}\hat{P}$ are all nonnegative and $\hat{Q}\hat{P}$ has a generalized inverse $(\hat{Q}\hat{P})^+$ given by $G^T M^{-1} \Gamma$ [4]. The following simplified notation will be used in this section:

$$\hat{\Sigma} = (B^T P - C\hat{Q}P)(\zeta I_n + \hat{Q}\hat{P})^+, \tag{34}$$

$$V_2' = V_2 + \alpha C T C^T, \tag{35}$$

where $R_2 = \zeta V_2'$, with ζ a nonnegative design variable, has been assumed so that convenient, closed form expressions may be found for A_c, B_c, and C_c. The generalized inverse in equation (34) may be replaced by a regular inverse if $\zeta > 0$ since then $\zeta I_n + \hat{Q}\hat{P} > 0$.

<u>Theorem 6</u>: Assume that condition (17) holds and suppose there exist nonnegative definite matrices P, Q, $\hat{P}$, and $\hat{Q}$ satisfying

$$0 = (A + \hat{Q}\hat{\Sigma}^T V_2'^{-1} C)Q + Q(A + \hat{Q}\hat{\Sigma}^T V_2'^{-1} C)^T + V_1 + \beta Q T^{-1}\hat{Q} + \hat{Q}\hat{\Sigma}^T V_2'^{-1}\hat{\Sigma}\hat{Q} + \varepsilon\hat{Q}, \tag{36}$$

$$0 = A^T P + PA + R_1 + \beta T^{-1}\hat{Q}\hat{P} + \beta\hat{P}\hat{Q}T^{-1} - \zeta\hat{\Sigma}^T V_2'^{-1}\hat{\Sigma} + \varepsilon\hat{P}, \tag{37}$$

$$0 = [A - (B + QC^T)V_2'^{-1}\hat{\Sigma} - \tfrac{1}{2}\varepsilon I]\hat{Q} + \hat{Q}[A - (B + QC^T)V_2'^{-1}\hat{\Sigma} - \tfrac{1}{2}\varepsilon I]^T - \beta\hat{Q}T^{-1}\hat{Q} - \hat{Q}\hat{\Sigma}^T V_2'^{-1}\hat{\Sigma}\hat{Q}, \tag{38}$$

$$0 = [A + \hat{Q}\hat{\Sigma}^T V_2'^{-1} C - \beta\hat{Q}T^{-1} - \tfrac{1}{2}\varepsilon I]^T\hat{P} + \hat{P}[A + \hat{Q}\hat{\Sigma}^T V_2'^{-1} C - \beta\hat{Q}T^{-1} - \tfrac{1}{2}\varepsilon I] + \zeta\hat{\Sigma}^T V_2'^{-1}\hat{\Sigma}, \tag{39}$$

$$\mathrm{rank}(\hat{Q}) = \mathrm{rank}(\hat{P}) = \mathrm{rank}(\hat{Q}\hat{P}) = n_c, \tag{40}$$

with $\alpha = \beta = 1$ and $\varepsilon > 0$. Then the compensator (A_c, B_c, C_c) given by

$$A_c = \Gamma[A + \hat{Q}\hat{\Sigma}^T V_2'^{-1} C - (B + QC^T + \hat{Q}\hat{\Sigma}^T)V_2'^{-1}\hat{\Sigma} - \beta\hat{Q}T^{-1} - \varepsilon I]G^T, \tag{41}$$

$$B_c = -\Gamma\hat{Q}\hat{\Sigma}^T V_2'^{-1}, \tag{42}$$

$$C_c = -V_2'^{-1}\hat{\Sigma}G^T \tag{43}$$

using a G, M, Γ-factorization of $\hat{Q}\hat{P}$, assures that the triple (A_c, B_c, C_c) has an SPR transfer function and that conditions (20)-(23) also hold.

Conversely if (A_c, B_c, C_c) solves the upper bound minimization problem with (A_c, B_c, C_c) having an SPR transfer function, then there exist real n x n nonnegative definite matrices Q, P, $\hat{Q}$, and $\hat{P}$ and $0 \le \alpha$, $\beta \le 1$ and $\varepsilon \ge 0$ that satisfy equations (36)-(40) with A_c, B_c, and C_c given by equations (41)-(43).

<u>Proof</u>: See [1].

<u>Remark 1</u>: These optimal projection equations consist of 4 modified Ricatti/Lyapunov equations that are coupled in both the full- and reduced-order cases. Thus, as expected, the separation principle is not valid in either case. Because the "binary" SPR condition has been imposed, one should not expect equations (36)-(40) to reduce to the usual separated equations of LQG theory because there is no simple way to relax the SPR requirement.

<u>Remark 2</u>: From an examination of the basic form of the optimal projection equations [4] specialized to the full-order case (i.e., LQG), it may be noted that there are other methods of attaining full-order SPR compensators (and perhaps reduced-order also) that are simpler than that presented here. In fact, any one of the full-order optimal projection equations could be modified to guarantee compensator stability after the optimization is complete, coupled with requiring a fixed relationship between B_c and C_c by deleting the normal expression for either B_c or C_c. This is similar to the method employed in Ref. [5]. This alternative disregards the inherent coupling between B_c and C_c. With normal LQG, B_c is dependent on V_2 and C_c is dependent on R_2. Since B_c and C_c must be related by a fixed matrix (Q_2 in the method presented here), the dependence on V_2 and R_2 is coupled. The method presented here considers this coupling in deriving the sufficient conditions for an SPR compensator. This is not meant to imply that a compensator designed using equations (36)-(43) is *the* optimal SPR compensator.

<u>Remark 3</u>: When solving equations (36)-(39) to get an SPR compensator, α and β less than one may yield an SPR compensator. In fact, $\alpha = \beta = 0$ may give an SPR compensator. The condition $\alpha = \beta = 1$ is the limiting case that *guarantees* an SPR compensator.

<u>Application of the Positive Real Design Equations to Stable Plants</u>

This section addresses the existence of solutions to equations (36)-(39) when the plant is open loop stable. That there exist SPR compensators of any order that stabilize a stable plant is trivial. The real questions are whether equations (36)-(39) are guaranteed to have a solution and whether a given algorithm can find that solution. Only the first question will be discussed here. The second question obviously depends on the algorithm chosen to solve the equations. The following discussion assumes that the infimum of the auxiliary cost is attained. Note that existence of a feasible solution to equations (15) and (18) using the chosen $\bar{\Omega}$ implies the existence of an optimal feasible solution to the upper bound minimization problem. Then the converse of Theorem 6 assures a nonnegative definite solution to equations (36)-(39). To show that a feasible solution exists to equations (15) and (18), choose $B_c = 0$ and $C_c = 0$, which clearly satisfies equation (15) for any Q_2. With these choices, expand equation (18) with Ω given by equation (27) and α, β, and T left variable to give

$$\begin{bmatrix} A & 0 \\ 0 & A_c \end{bmatrix}\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} + \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}\begin{bmatrix} A^T & 0 \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} V_1 & 0 \\ 0 & \beta Q_{12}^T T^{-1} Q_{12} + \varepsilon I \end{bmatrix} = 0. \tag{44}$$

Expansion of the cross term in equation (44) indicates that Q_{12} is dependent only on A and A_c, and hence is independent of Q_1 and Q_2. Thus, the forcing term in equation (44) is nonnegative definite and finite. Then for any stable A_c of any order, $\bar{Q}$ exists and is nonnegative definite by Lyapunov's theorem. Note that this result holds for all $0 \le \alpha, \beta \le 1$ and arbitrary $\epsilon \ge 0$.

S-Structured Lyapunov Stability of Positive Real/Strictly Positive Real Feedback Systems

Boyd and Yang [6] have discussed the concept of S-Structured Lyapunov Stability (S-SLS). Let $\Re$ denote the set of real numbers and $\Re^{m\times n}$ denote the set of real m x n matrices.

Definition 1 [6]: Let S be an n x n matrix that is a subspace of $\Re^{n\times n}$. $A \in \Re^{n\times n}$ is S-structured Lyapunov stable if there is a $P \in S$ such that $P = P^T > 0$ and $A^TP + PA \le 0$.

S is referred to as a structure and may, for example, consist of diagonal or block-diagonal matrices. Note that if $S = \Re^{n\times n}$ then A is S-SLS if and only if A is stable (essentially unstructured Lyapunov stability). Thus, in general S-SLS is a stronger condition than just Lyapunov stability. The following theorem concerns the interconnection of a positive real plant with a strictly positive real compensator in a negative feedback system.

Theorem 7: Let S be given by:

$$\Re^{n\times n} \oplus \Re^{n_c\times n_c}, \tag{45}$$

which denotes the set of block diagonal matrices with two diagonal blocks, the first a real n x n matrix and the second a real n_c x n_c matrix. Also let the triple (A, B, C) be positive real and the triple (A_c, B_c, C_c) be strictly positive real. Then the matrix

$$\tilde{A} = \begin{bmatrix} A & -BC_c \\ B_cC & A_c \end{bmatrix}, \tag{46}$$

i.e., the negative feedback connection of (A, B, C) and (A_c, B_c, C_c) is S-SLS.

Proof: See [2]. ■

The proof in [2] proceeds by showing that

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q_c \end{bmatrix} \text{ and } \tilde{L} = \begin{bmatrix} L & 0 \\ 0 & L_c \end{bmatrix} \tag{47}$$

satisfy the Lyapunov equation

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{L} = 0, \tag{48}$$

indicating that the closed loop is S-SLS.

Theorem 7 may provide some insight into the mechanism of the maximum entropy modelling approach to control. Specifically, in [7], it is noted that the maximum entropy approach suppresses off-diagonal elements of the closed-loop covariance when high uncertainty exists. This leads to reduced position feedback of uncertain modes, while the velocity feedback of uncertain modes remains. The hypothesis in [7] is that the resultant control of highly uncertain modes is a dissipative rate feedback, similar to positive real feedback. Theorem 7 shows that positive real/strictly positive real negative feedback connections lead to *complete suppression of the off-diagonal blocks* of a solution to a certain block-structured Lyapunov equation.

Conclusions

This paper presents a method for designing SPR, dynamic compensators of order less than or equal to that of the plant. An overbounding technique on the state covariance combined with the Kalman-Yakubovich Lemma then guarantees that the compensator is SPR. If the plant is stable (or positive real), the design equations are guaranteed to possess a solution.

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This work was supported by Sandia National Laboratories, which is operated for the U.S. Department of Energy under contract DE-AC04-76DP00789.